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Singular Elliptic Equations with Variable Exponents

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Abstract

This paper deals with study of the nonlinear singular elliptic equations in a bounded domain $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$:

$$Au = \frac{f}{u^{\gamma(\cdot)}} + \mu,$$

Where $A := -\mathrm{div}\,(\widehat{a}(\cdot, Du))$ is a Leray-Lions type operator which maps continuously $W_0^{1,p(\cdot)}(\Omega)$ into its dual $W^{-1,p'(\cdot)}(\Omega)$ whose simplest model is the $p(\cdot)$-laplacian type operator (i.e. $\widehat{a}(\cdot, \xi) = |\xi|^{p(\cdot)-2}\xi$) such that f is a nonnegative function belonging to the Lebesgue space with variable exponents $L^{m(\cdot)}(\Omega)$ with $m(\cdot)$ being small (or $L^1(\Omega)$) and μ is a nonnegative function belongs to $L^1(\Omega)$ as nonhomogeneous datum while $m : \overline{\Omega} \to (1, +\infty)$, $\gamma : \overline{\Omega} \to (0, 1)$ are continuous functions satisfying certain conditions depend on $p(\cdot)$. We prove the existence, uniqueness, and regularity of nonnegative weak solutions for this class of problems with $p(\cdot)$-growth conditions. More precisely, we will discuss that the nonlinear singular term has some regularizing effects on the solutions of the problem which depends on the summability of f, $m(\cdot)$, and the value of $\gamma(\cdot)$. The functional framework involves Sobolev spaces with variable exponents as well as Lebesgue spaces with variable exponents. Our results can be seen as a generalization of some results given in the constant exponents case.

Keywords: Singular elliptic equations; Nonlinear elliptic equations; Weak solutions; Nonnegative solutions; Uniqueness; Regularity; Leray-Lions operator; Nonlinear singular term; Variable exponents; $L^1(\Omega)$ data.

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1. Introduction

This paper is devoted to studying the existence, uniqueness, and regularity of nonnegative weak solutions for a class of nonlinear singular elliptic equations with variable exponents. A prototype

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example is

$$\begin{cases} -\Delta_{p(\cdot)}(u) = \frac{f}{u^{\gamma(\cdot)}} + \mu, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$) with Lipchitz boundary $\partial\Omega$, $0 \leq \mu \in L^1(\Omega)$, $0 \leq f \in L^{m(\cdot)}(\Omega)$, $m(\cdot)$ as in (7), and γ as in (8). The problem (1) is called $p(\cdot)$-Laplacian problem with singular nonlinearity having variable exponents while the operator $-\text{div}\,(\widehat{a}(\cdot, Du))$ is called $p(\cdot)$-Laplacian type operator and is a generalization of the $p(x)$-Laplace operator

$$-\Delta_{p(\cdot)}(u) := -\operatorname{div}(|Du|^{p(\cdot)-2}Du)$$

which appears in problem (1). Instead of (1) we will consider more general nonlinear singular elliptic equations with variable exponents of the form

$$\begin{cases} -\text{div}\,(\widehat{a}(x, Du)) = \frac{f}{u^{\gamma(\cdot)}} + \mu, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

With $0 \leq \mu \in L^1(\Omega)$ and $0 \leq f \in L^{m(\cdot)}(\Omega)$. Recall that a Leray-Lions type operator is a Caratheodory function $\widehat{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying: a.e $x \in \Omega$ and for all $\xi, \xi' \in \mathbb{R}^N$ the following:

$$\widehat{a}(x,\xi)\xi \geq \alpha|\xi|^{p(x)}, \quad \widehat{a}(x,\xi) = (a_1, \ldots, a_N) \tag{3}$$

$$|\widehat{a}(x,\xi)| \leq \beta \left(h + |\xi|^{p(x)-1}\right) \tag{4}$$

$$(\widehat{a}(x,\xi) - \widehat{a}(x,\xi'))(\xi - \xi') > 0, \; \xi \neq \xi', \tag{5}$$

where α, β are strictly nonnegative real numbers, h is a given nonnegative function in $L^{p'(\cdot)}(\Omega)$, where, $p'(\cdot) := \frac{p(\cdot)}{p(\cdot)-1}$ while $\gamma : \overline{\Omega} \to (0,1)$, $m : \overline{\Omega} \to (1, +\infty)$, and the variable exponents $p : \overline{\Omega} \to (1, +\infty)$ are continuous functions such that: for all $x \in \overline{\Omega}$

$$1 - \gamma(x) + \frac{1}{m(x)} - \frac{1 - \gamma(x)}{N} < p(x) < N, \quad |\nabla m| \in L^\infty(\Omega), \quad |\nabla \gamma| \in L^\infty(\Omega), \tag{6}$$

$$1 < m(x) < \widehat{m}_1(x), \quad |\nabla m| \in L^\infty(\Omega), \tag{7}$$

where

$$\widehat{m}_1(x) = \frac{Np(x)}{Np(x) - (N - p(x))(1 - \gamma(x))} = \Big(\frac{p^\star(x)}{1 - \gamma(x)}\Big)',$$

$$|\nabla \gamma| \in L^\infty(\Omega), \quad p^\star(x) = \frac{Np(x)}{N - p(x)},$$

with $\gamma : \overline{\Omega} \longrightarrow (0,1)$ is a continuous function satisfies:

$$0 < \gamma^- := \min_{x\in\overline{\Omega}} \gamma(x) \le \gamma^+ := \max_{x\in\overline{\Omega}} \gamma(x) < 1, \quad |\nabla\gamma| \in L^\infty(\Omega), \text{ for all } x \in \overline{\Omega}. \tag{8}$$

Closely observe that, the assumptions (7) and (8) guarantee that (8) is clear-cut. More elaborate, we consider the simplest model

$$\begin{cases} -\Delta_{p(\cdot)} u = \frac{f}{u^{\gamma(\cdot)}} + \mu, & \text{in } B, \\ u > 0, & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases} \tag{9}$$

where $0 \le f \in L^1(\Omega)$, $0 \le \mu \in L^1(\Omega)$, $p(\cdot)$ as in (6), γ as in (8), and

$$B = \{x \in \mathbb{R}^N \mid |x| < 1\}.$$

In recent years, boundary value problems with variable exponents has received a lot of attention, reader can look at [1], the nice surveys books [17,19,23,40], and the references therein. While the study of the elliptic and parabolic problems involving singular nonlinearities with variable exponents is still developing with slowness because there are a very limited number of results exist on this topic, reader can have the opportunity to look at the new marvellous works [9,25,29,42]. We mention also that much attention has been devoted to nonlinear elliptic equations with singularities because of their wide application to physical models such as Non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogenous, etc.(we refer to [10]).

Problem (2) has been extensively studied in the past. In the constant exponent cases. A remarkable paper on the topic published in 1987, is due to the authors in [18], they investigated a problem whose prototype like (2) with $p(\cdot) = p = 2$, $\mu = 0$, $\gamma(\cdot) = \gamma > 0$, and $f \equiv 1$. A classical solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ has been obtained via shifting method. Also, a priori estimates and regularity results have been established, as $u \in C^{0,\frac{2}{1+\gamma}}(\Omega)$ provided $\gamma > 1$ which is called strongly singular case. In their work [24], the authors obtained existence and nonexistence results for problem (2) when $p(\cdot) = p = 2$, $\mu \in L^1(\Omega)$, $\mu \ge 0$ in Ω, $0 < \gamma(\cdot) = \gamma < 1$, and $f \equiv -1$. They emphasized the role of variational techniques in this context and also delineated the connection between the solutions of (2) and ϕ_1, (ϕ_1 denotes the first eigenfunction of $(-\Delta; W_0^{1,2}(\Omega))$). The latter consideration will turn out to be very useful in the analysis of behavior of solutions to (2) near the boundary: indeed, few years later, the authors in [30] have been discussed the case when $p(\cdot) = p = 2$, $\mu = 0$, $\gamma(\cdot) = \gamma > 0$ and f is a nonnegative regular function in $\overline{\Omega}$. They proved that the solution u of problem (2) in $H_0^1(\Omega)$ if and only if $\gamma(\cdot) = \gamma < 3$ and if $\gamma(\cdot) = \gamma > 1$ then u is not in $\mathcal{C}^1(\overline{\Omega})$. All results of [30] has been extended in [31], the authors have been proved that the problem (2) when $p(\cdot) = p = 2$, $\mu = 0$ with $0 < \gamma(\cdot) = \gamma < 1$ has a unique nonnegative weak solution in $H_0^1(\Omega)$ if f is a nonnegative nontrivial function in $L^2(\Omega)$.

Furthermore, in the case where $p(\cdot) = p = 2$, $\mu = 0$, $\gamma(\cdot) = \gamma > 0$, and $m(\cdot) = m \geq 1$, the existence and nonexistence of solutions for problem (2) has been proved in [7]. Similar results, with different proofs, were obtained in [6,13,14,21,22,34,38,45]. In [12] the authors studied problem (2) when $p(\cdot) = p = 2$, $\mu = 0$, $m(\cdot) = m > 1$. They proved that there exists a solution to problem (2) in the natural energy space $H_0^1(\Omega)$ when $\gamma(\cdot) \leq 1$ in a strip around the boundary, also for another case, belongs to $H_{loc}^1(\Omega)$. Moreover, the authors in [15] generalized the results in [16] to the case when $p(\cdot) = p$, $\gamma(\cdot)$ as in (8), $\mu = 0$, $m(\cdot) = m \geq 1$ and the left-hand side is a p-Laplace operator. They proved the existence of a nonnegative solution u of problem (2) and discussed the relationship among the regularity of solutions, the summability of f and the value of $\gamma(\cdot)$. If $p(\cdot) = p$, $\gamma(\cdot) = \gamma > 0$, f is a nonnegative function which has some Lebesgue regularity and μ is a nonnegative bounded Radon measure, then problem (2) has been considered and extensively studied in [37], the authors proved the existence of nonnegative weak solutions for problem (2) in the general case through an approximation argument.

Recently, in the variable exponents cases, the authors in [25] improved the results of [7,31] and studied the problem (2) in general form depending on $\gamma(\cdot) > 0$, $\mu = 0$, $m(\cdot) = m \geq 1$, $2 < p(\cdot) \leq N$, and $p(\cdot)$-Laplace operator. They proved the existence and regularity results of problem (2) by adapting some techniques used in [15], nevertheless it was a new spirit of idea different from that used in [13], (when $\gamma(\cdot) = \gamma \geq 1$, see [26] for a similar paper). In [29], the authors generalized the work [3] by considering nonlinear elliptic equations with a singular nonlinearity, lower order terms and L^1 datum in the setting of Sobolev spaces with variable exponents. They proved that the lower order term has some regularizing effects on the solutions. For more and different aspects concerning singular problems see [36]), whereas the elliptic operator A depends on u and Du with the degenerate coercivity, we refer to [41] for the existence and regularity results of the solutions.

In this paper, we consider a new condition on $m(\cdot)$ which depends on $\gamma(\cdot)$, $p(\cdot)$ and a nonnegative measure as nonhomogeneous datum which allows us to show that the nonlinear singular term has regularizing effects on the solutions of the problem (2) that depends on the summability of f, $m(\cdot)$ and the value of $\gamma(\cdot)$. We will further assume greater regularity on f in problem (2) to guarantee the existence of a very weak solution. Here we use a new fresh ideas different from that used in [9,25,42] to generalize the previously obtained some results in [7,11,12,15,20,37,39] to the setting of variable exponents p, m and γ which appear in (6)-(7)-(8) respectively depends on the variable x. Obviously, the nonlinearity of (1) (resp. (2)) is more complicated than nonlinearity of the p-Laplacian (resp. the operator appears in (2)) as the exponent, which appears in (1) (resp. (2)), depends on the variable x, due to the nonlinearity of a $p(\cdot)$-Laplace operator (resp. the operator appears in (2)) and the assumptions (6)-(7)-(8) (see Remark 3.15), some classical methods for elliptic operators which investigated in [33] may not directly be applied to solve the problem (1) (resp. (2)). Distinctly, this motivated us to propose the study of the problem (2) and get new uniquely results which were not considered in the literature. Inspired by [35], we will prove the existence, uniqueness and regularity of nonnegative weak solution for the problem (2) by applying the method of approximations, Schauder fixed point theorem and adopting some techniques used in [29], make sure we will show that how the nonlinear singular term has regularizing effects on the solutions of the problem (2). More precisely, we will prove (Theorems 3.3-3.6-3.11 below). In particular, Theorems 3.3-3.6-3.11 below can be viewed as a generalization of the existence results in [7, Theorem 5.2; Theorem 5.6], [15, Theorem 3.1; Theorem 3.2], [25, Theorem 3.1; Theorem 3.2], [37, Theorem 2.6; Theorem 3.2.(i)], [39, Theorem 2.6]) respectively.

The main difficulties are the facts that the elliptic operator A depends on Du and the nonlinear singular term has regularizing effects on solutions. To overcome these difficulties we will work by approximation and truncating the singular term $\frac{1}{u^{\gamma(\cdot)}}$, so that it becomes not singular at the origin. Furthermore, we will get some an uniform estimates on the nonnegative weak solutions u_n of the approximating problems which will allow us to pass to the limit and find a unique nonnegative weak solution to problem (2). To be more precise, we will be able to prove that the model problem (20) has a suitable solution u for every f in L^m, $m \geq 1$ and for every γ and how the regularity of u depends on the summability of f, on p and on γ which are restricted as in (6)-(7)-(8) respectively.

To analyse the problem more clearly, we point out that our estimates on the gradients of u (see Lemma 5.3) are new compared to [15,25]. In fact, it follows from Remarks 3.8-3.10 that the regularity given in Theorem 3.6 is better than ([15, Theorem 3.2]; [25, Theorem 3.2]). To prove Theorem 3.6, a key result (Lemma 5.3) about an $L^{p(\cdot)}$ estimate for the gradients of solution to problem (2). For the sake of exposition, we assumed that the elliptic operator A was chosen to be independent of u. In any way whatever one can easily realize that the same proof of Theorems 3.3-3.6-3.11 can be forthrightly extended to more general case when the operator A depends of u and Du with a general nonlinear singular term under the above assumptions (6)-(7)-(8) (see Remark 6.2).

Throughout this paper, C will indicate any nonnegative constant which depends only on data and whose value may change from line to line.

2. Mathematical Background and Auxiliary Results

This section is devoted to preliminary ideas which will be helpful to advance towards establishing our principle results. We first recall some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. We refer to [2], [17], [19], [23] and [40] for further properties of variable exponents Lebesgue-Sobolev spaces. Second, we briefly recall some auxiliary results, reader can easily look at the nice surveys [27] and [28].

Let $p : \overline{\Omega} \to [1, \infty)$ be a continuous function. We denote by $L^{p(\cdot)}(\Omega)$ the space of measurable function $u(x)$ on Ω such that

$$\rho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx < +\infty.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 \mid \rho_{p(\cdot)}(u/\lambda) \leq 1 \right\}$$

becomes a Banach space. Moreover, if $p^- := \min_{x\in\overline{\Omega}} p(x) > 1$, then $L^{p(\cdot)}(\Omega)$ is reflexive and the dual of $L^{p(\cdot)}(\Omega)$ can be identified with $L^{p'(\cdot)}(\Omega)$, where $p'(x) := \frac{p(x)}{p(x)-1}$.
For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the Hölder type inequality:

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \le 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

holds true.
We define also the Banach space $W_0^{1,p(x)}(\Omega)$ by

$$W_0^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega),\, |Du| \in L^{p(\cdot)}(\Omega) \text{ and } u = 0 \text{ on } \partial\Omega\right\}$$

endowed with the norm $\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|Du\|_{p(\cdot)}$.
The space $W_0^{1,p(\cdot)}(\Omega)$ is separable and reflexive provided that with $1 < p^- \le p^+ < \infty$. The smooth functions are in general not dense in $W_0^{1,p(\cdot)}(\Omega)$, but if the exponent variable $p(x) > 1$ is logarithmic Hölder continuous, that is

$$|p(x) - p(y)| \le -\frac{M}{\ln(|x-y|)} \quad \forall x, y \in \Omega \text{ such that } |x-y| \le 1/2, \tag{10}$$

then the smooth functions are dense in $W_0^{1,p(\cdot)}(\Omega)$. For $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C(\overline{\Omega}, [1, +\infty))$, the Poincaré inequality holds

$$\|u\|_{p(\cdot)} \le C\|Du\|_{p(\cdot)}, \tag{11}$$

for some constant C which depends on Ω and the function p. An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$ was showed in the following result

Lemma 2.1 ([28]). *If (u_n), $u \in L^{p(\cdot)}(\Omega)$, then the following relations hold*

- $\|u\|_{p(\cdot)} < 1(> 1; = 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1(> 1; = 1)$,
- $\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^-}}; \rho_{p(\cdot)}(u)^{\frac{1}{p^+}}\right) < \|u\|_{p(\cdot)} < \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^-}}; \rho_{p(\cdot)}(u)^{\frac{1}{p^+}}\right)$,
- $\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \le \rho_{p(\cdot)}(u) \le \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right)$,
- $\|u\|_{p(\cdot)} \le \rho_{p(\cdot)}(u) + 1$,
- $\|u_n - u\|_{p(\cdot)} \to 0 \Leftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0$,
  since $p^+ < \infty$.

Remark 2.2. *As in [28], the following inequality*

$$\int_\Omega |u|^{p(x)}\,dx \le C\int_\Omega |Du|^{p(x)}\,dx,$$

in general does not hold. So, thanks to Lemma 2.1 and (11)*, we get the following inequality which will be used later*

$$\min\left\{\|Du\|_{p(\cdot)}^{p^-};\|Du\|_{p(\cdot)}^{p^+}\right\}\le\int_\Omega|u(x)|^{p(x)}\,dx\le\max\left\{\|Du\|_{p(\cdot)}^{p^-};\|Du\|_{p(\cdot)}^{p^+}\right\}. \tag{12}$$

An important embedding as follows:

Lemma 2.3 (Sobolev embedding,[27])**.** *Let $\Omega\subset\mathbb{R}^N$ be an open bounded set, with Lipschitz boundary, and let $p:\overline{\Omega}\to(1,N)$ satisfy the log-Hölder continuity condition* (10)*. Then we have the following continuous embedding:*

$$W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{p^\star(\cdot)}(\Omega),$$

where $p^\star(\cdot)=\frac{Np(\cdot)}{N-p(\cdot)}$.

Henceforth, we will denote by Ω a bounded open subset of $\mathbb{R}^N$ $(N\ge 2)$ with Lipchitz boundary $\partial\Omega$, we will use through this paper, the truncation function T_k at height k $(k>0)$ which denoted by, for every $t\in\mathbb{R}$

$$T_k(t)=\max\Big\{-k,\min\{k,t\}\Big\}, \tag{13}$$

It is obvious that T_k is Lipschitz functions satisfying $|T_k(t)|\le k$.

We will often use the Young inequality in the following form, for every $\eta>0$, $p\in(1,+\infty)$ and for all nonnegative real numbers U,V:

$$U\cdot V=\left((\eta p)^{\frac{1}{p}}U\cdot(\eta p)^{-\frac{1}{p}}V\right)\le\eta U^p+\frac{(p-1)\eta^{-\frac{1}{p-1}}}{p^{p'}}V^{p'},\quad\text{where } p'=\frac{p}{p-1}. \tag{14}$$

3. Main Results and Some Remarks

Our aim is to prove the existence of nonnegative weak solutions to problem (2). Here we give two important definitions which are essential to our study of the problem (2).

Definition 3.1. *Let $(\mu_n)_n$ be the sequence of nonnegative measurable functions in $L^1(\Omega)$. We say $(\mu_n)_n$ converges weakly to the nonnegative function μ in $L^1(\Omega)$, if*

$$\int_\Omega\mu_n\psi\,dx\to\int_\Omega\mu\psi\,dx\quad\textit{for all } \psi\in L^\infty(\Omega). \tag{15}$$

i.e. $\mu_n\rightharpoonup\mu$ in $L^1(\Omega)$.

Definition 3.2. If $0<\gamma^+<1$ and $0\le\mu\in L^1(\Omega)$ then we say A nonnegative function $u\in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution for problem (2) if the following conditions are satisfied:

1. $u\in W_0^{1,1}(\Omega)$, $\widehat{a}(x,Du)\in\Big(L^1(\Omega)\Big)^N$, $\frac{f}{u^{\gamma(\cdot)}}\in L^1_{loc}(\Omega)$,
2. $$\int_\Omega\widehat{a}(x,Du)\cdot D\varphi\,dx=\int_\Omega\frac{f}{u^{\gamma(\cdot)}}\cdot\varphi\,dx+\int_\Omega\mu\cdot\varphi\,dx, \tag{16}$$

for every $\varphi \in C_0^\infty(\Omega)$.

The main results of the paper are the following theorems:

Theorem 3.3. *Under the assumptions* (3)-(6) *and* (8). *If* $0 \le \mu \in L^1(\Omega)$ *and* $f \in L^{m(\cdot)}(\Omega)$ *is a nonnegative function* ($f \neq 0$) *such that* $m(\cdot) = \widehat{m}_1(\cdot)$ *with* $\widehat{m}_1(\cdot)$ *as in* (7). *Then the problem* (2) *has a nonnegative solution* $u \in W_0^{1,p(\cdot)}(\Omega)$ *satisfying* (16).

In Theorem 3.3 a point that is worth paying attention are the following remarks,

Remark 3.4. *The result of Theorem 3.3 coincides with regularity result of [7, Theorem 5.2], if and only if* $\mu = 0$, $p(\cdot) = p = 2$, $\gamma(\cdot) = \gamma$ *and* $m(\cdot) = m$.
The result of Theorem 3.3 coincides with regularity result of [11, Theorem 1.3] as long as $\mu = 0$, $p(\cdot) = p$, $\gamma(\cdot) = \gamma$, *and* $m(\cdot) = m$.

Remark 3.5. *In Theorem 3.3, in the case where* $\mu = 0$, $p(\cdot) = p = 2$ *and* $\gamma(\cdot)$ *tends to be* 0, *gives us* $m(\cdot) = m = \frac{2N}{N+2}$. *So, the regularity result of Theorem 3.3 has been obtained in [12, Theorem 1.1]. If* $\mu = 0$ *and* $\gamma(\cdot)$ *tends to be* γ^-, *then the regularity result of Theorem 3.3 has been obtained in [25, Theorem 3.1] and [16, Theorem 7].*

The regularity of u depends on the summability of f and on $\gamma(\cdot)$, which is presented in the next Theorem.

Theorem 3.6. *Under the assumptions* (3)-(6) *and* (8). *If* $0 \le \mu \in L^1(\Omega)$ *and* $f \in L^{m(\cdot)}(\Omega)$ *is a nonnegative function with* $m(\cdot) = m$ *as in* (7). *Then problem* (2) *has at least one nonnegative solution* $u \in W_0^{1,q(\cdot)}(\Omega)$ *satisfying* (16), *where* $q(\cdot)$ *is a continuous function on* $\overline{\Omega}$ *satisfying*

$$1 \le q(x) < \frac{Nm\Big(p(x) - 1 + \gamma^-\Big)}{\Big(N - m(1-\gamma^-)\Big)}, \quad \text{for all } x \in \overline{\Omega}, \quad \gamma^- := \min_{x\in\overline{\Omega}} \gamma(x), \tag{17}$$

The regularity result of this Theorem leads to make some remarks as below:

Remark 3.7. *Observe that in Theorem 3.6, the assumption* (6) *guarantees that*

$$1 < \frac{Nm\Big(p(\cdot) - 1 + \gamma^-\Big)}{\Big(N - m(1-\gamma^-)\Big)},$$

So, the assumption (17) *is well defined. Remark that the condition* (17) *is the key to prove the Lemma 5.3.*

Remark 3.8. *The regularity result of Theorem 3.6 has been treated in [7, Theorem 5.6], in the case where* $p(\cdot) = p = 2$, $\mu = 0$, $m(\cdot) = m$ *and* $\gamma(\cdot) = \gamma$. *The result of Theorem 3.6 coincides with regularity result of [15, Theorem 3.2] provided that* $\mu = 0$, $p(\cdot) = p$ *and* $m(\cdot) = m$.

Remark 3.9. *If $\mu = 0$ and $p(\cdot)$ tends to be p^-, then $\frac{Nm\left(p(x)-1+\gamma^-\right)}{\left(N-m(1-\gamma^-)\right)}$ tends to be $\frac{Nm(p^--1+\gamma^-)}{N-m(1-\gamma^-)}$, which is bound on $q(\cdot)$ obtained in [25, Theorem 3.2]. In the case where $p(x) = p = 2$, m tends to be 1 and γ^- tends to be 0, we have $\frac{Nm\left(p(x)-1+\gamma^-\right)}{\left(N-m(1-\gamma^-)\right)}$ tends to be $\frac{N}{N-1}$, which is bound on $q(\cdot)$ obtained in [37, Theorem 2.6]. The case $p(x) = p$, m tends to be 1 and γ^- tends to be 0, then $\frac{Nm\left(p(x)-1+\gamma^-\right)}{\left(N-m(1-\gamma^-)\right)}$ tends to be $\frac{N(p-1)}{N-1}$, which is bound on $q(\cdot)$ obtained in [37, Theorem 3.2-(i)].*

Remark 3.10. *Note that the condition on $p(\cdot)$ in* (6) *is stronger than in [15,25], but is an outcome of the condition on $q(\cdot)$ in* (17)*. This condition is important to prove Lemma 5.3 below.*

Theorem 3.11. *Under the assumptions* (3)-(6) *and* (8)*. If $0 \leq \mu \in L^1(\Omega)$ and $f \in L^1(\Omega)$ is a nonnegative function. Then problem* (2) *has at least one nonnegative solution $u \in W_0^{1,q(\cdot)}(\Omega)$ satisfying* (16)*, where $q(\cdot)$ is a continuous function on $\overline{\Omega}$ satisfying*

$$1 \leq q(x) < \frac{N\left(p(x)-1+\gamma^-\right)}{\left(N-1+\gamma^-\right)}, \quad \text{for all } x \in \overline{\Omega}, \quad \gamma^- := \min_{x\in\overline{\Omega}} \gamma(x), \tag{18}$$

Remark 3.12. *In Theorem 3.11. If $p(\cdot) = p$ and γ^- tends to be 0, then $\frac{N\left(p(x)-1+\gamma^-\right)}{\left(N-1+\gamma^-\right)}$ tends to be $\frac{N(p-1)}{N-1}$ which is bound on $q(\cdot)$ obtained in [37].*

Remark 3.13. *The result of Theorem 3.11 coincides with regularity result of [37, Theorem 2.6] and [39, Theorem 2.6] on condition that $p(\cdot) = p = 2$ and $\gamma^- \to 0$. Also, the result of Theorem 3.11 coincides with regularity results of [37, Theorem 3.2 (i)] providing that $p(\cdot) = p$ and $\gamma^- \to 0$.*

Remark 3.14. *In Theorem 3.11, it is clear that the assumption* (6) *imply that* (18) *holds since we have*

$$1 < \frac{N\left(p(x)-1+\gamma^-\right)}{\left(N-1+\gamma^-\right)} < \frac{Nm\left(p(\cdot)-1+\gamma^-\right)}{\left(N-m(1-\gamma^-)\right)}, \ \forall x \in \overline{\Omega}.$$

So, Theorem 3.6 improves Theorem 3.11 (and [37, Theorem 2.6;Theorem 3.2 (i)], [39, Theorem 2.6]).

Remark 3.15. *Under the assumptions $\mu \in L^1(\Omega)$ and $f \in L^{m(\cdot)}(\Omega)$ in Theorems 3.3-3.6, we can deduce that $\frac{f}{u^{\gamma(\cdot)}} + \mu$ is never in the dual space $\left(W_0^{1,p(\cdot)}(\Omega)\right)'$, so that the result of this paper deals with nonlinear singular term that has regularizing effects on the solutions of the problem* (2)*.*

4. Approximation of Problem (2)

In order to prove the previous results, we will work by truncating the singular term $\frac{1}{u^{\gamma(\cdot)}}$ so that, it becomes not singular at the origin and we study the behaviour of a sequence u_n of solutions of the approximated problems. Due to (15) in definition (3.1), we can suppose that $\{f_n\}_{n\in\mathbb{N}}$, $\{\mu_n\}_{n\in\mathbb{N}}$ are sequences of functions satisfying

$$\begin{cases} f_n := T_n(f), \quad \forall n \geq 1, \\ 0 < \{f_n\}_{n\in\mathbb{N}} \in C_0^\infty(\Omega), \quad \|f_n\|_{L^m(\Omega)} \leq \|f\|_{L^m(\Omega)}, (m \geq 1), \\ f_n \longrightarrow f \text{ strongly in } L^m(\Omega), \quad \text{as } n \to +\infty, \\ 0 \leq \{\mu_n\}_{n\in\mathbb{N}} \in L^1(\Omega), \quad \|\mu_n\|_{L^1(\Omega)} \leq \|\mu\|_{L^1(\Omega)}, \quad \forall n \geq 1, \\ \int_\Omega \psi \, d\mu_n \to \int_\Omega \psi \, d\mu, \quad \forall \psi \in L^\infty(\Omega), \quad \text{as } n \to +\infty, \end{cases} \tag{19}$$

Where f_n is the truncation at level n of f as in (13). Here let us consider the following scheme of approximation

$$\begin{aligned} -\text{div}\,(\widehat{a}(x, Du_n)) &= \frac{f_n}{(u_n + \frac{1}{n})^{\gamma(x)}} + \mu_n \quad \text{in } \Omega, \\ u_n &> 0 \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{20}$$

First of all we need to show the existence of a nonnegative weak solution to (20). The proof which is based on the Schauder fixed point theorem.

Lemma 4.1. *For every $n \in \mathbb{N}^\star$, there exists a unique nonnegative solution $u_n \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ to problem* (20) *in the sense that*

$$\int_\Omega \widehat{a}(x, Du_n) D\phi \, dx = \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^{\gamma(x)}} \phi \, dx + \int_\Omega \mu_n \phi \, dx, \tag{21}$$

for every $\phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. Moreover, the sequence $(u_n)_n$ is increasing with respect to n, $u_n > 0$ in Ω, and for every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ (independent of n) satisfied

$$u_n(x) \geq C_\omega > 0, \textit{ for every } x \in \omega, \textit{ for every } n \in \mathbb{N}^\star. \tag{22}$$

In particular, there exists the pointwise limit u of the sequence u_n, with u that satisfies (22)*. Furthermore, for all γ as in* (8) *and for all $\phi \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega \frac{f_n \phi}{(u_n + \frac{1}{n})^{\gamma(x)}} \, dx \to \int_\Omega \frac{f\phi}{u^{\gamma(x)}} \, dx, \quad \textit{as} \quad n \to +\infty. \tag{23}$$

Proof. Let $n \in \mathbb{N}$ be fixed, $v \in L^{p(\cdot)}(\Omega)$ and consider the following non singular problem

$$\begin{aligned} -\text{div}\,(\widehat{a}(x, Dw)) &= \frac{f_n}{(|v| + \frac{1}{n})^{\gamma(x)}} + \mu_n \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{24}$$

It follows from [37] (see also [4,8,32]) that the problem (24) has a unique solution $w \in W_0^{1,p(\cdot)}(\Omega)$.

Furthermore, since the datum $\frac{f_n}{(|v|+\frac{1}{n})^{\gamma(\cdot)}} + \mu_n$ is bounded, we have that $w \in L^\infty(\Omega)$ and there exists a positive constant ν, independents of v and w (but possibly depending in n), such that $\|w\|_{L^\infty(\Omega)} \leq \nu$. Our aim is to prove the existence of fixed point of the map

$$\widetilde{G} : L^{p(\cdot)}(\Omega) \longrightarrow L^{p(\cdot)}(\Omega)$$

where $\widetilde{G}(v) = w \in W_0^{1,p(\cdot)}(\Omega)$ and w the weak solution of problem (24). We know that $\widetilde{G}$ is compact if and only of it is continuous and it maps every bounded subset of $L^{p(\cdot)}(\Omega)$ into a relatively compact set. We will show that $\widetilde{G}$ maps the ball $B_c(0) \subset L^{p(\cdot)}(\Omega)$ of radius C into itself, the constant C is independent of v. Thanks to the regularity of the datum $\frac{f_n}{(|v|+\frac{1}{n})^{\gamma(\cdot)}} + \mu_n$, allows us to take w as test function in the weak formulation of (24) which gives

$$\int_\Omega \widehat{a}(x, Dw) \cdot Dw \, dx = \int_\Omega \frac{f_n}{(|v| + \frac{1}{n})^{\gamma(x)}} \cdot w \, dx + \int_\Omega \mu_n w \, dx,$$

By using (3) and (19), we find

$$\frac{\alpha}{(n^{\gamma^+ +1} + \|\mu\|_{L^1(\Omega)})} \int_\Omega |Dw|^{p(x)} \, dx \leq \int_\Omega |w| \, dx,$$

Using Young's inequality (14) for all $\eta > 0$, Poincaré inequality and the fact that $|Du_n|^{p^-} \leq |Du_n|^{p(\cdot)} + 1$ on the left hand side, we obtain

$$\begin{aligned}
\frac{\alpha}{(n^{\gamma^+ +1} + \|\mu\|_{L^1(\Omega)})} \int_\Omega |Dw|^{p(x)} \, dx &\leq \eta \int_\Omega |w|^{p^-} \, dx + \frac{(p^- - 1)\eta^{-\frac{1}{p^- -1}}}{(p^-)^{(p^-)'}} |\Omega| \\
&\leq \eta C_{p^-}^{p^-} \int_\Omega |Dw|^{p^-} \, dx + \frac{(p^- - 1)\eta^{-\frac{1}{p^- -1}}}{(p^-)^{(p^-)'}} |\Omega| \\
&\leq \eta C_{p^-}^{p^-} \int_\Omega |Dw|^{p(x)} \, dx + \Big(\eta C_{p^-}^{p^-} + \frac{(p^- - 1)\eta^{-\frac{1}{p^- -1}}}{(p^-)^{(p^-)'}}\Big) |\Omega|,
\end{aligned}$$

Which implies

$$\begin{aligned}
\int_\Omega |Dw|^{p(x)} \, dx \leq \eta \frac{C_{p^-}^{p^-} (n^{\gamma^+ +1} + \|\mu\|_{L^1(\Omega)})}{\alpha} \int_\Omega |Dw|^{p(x)} \, dx \\
+ \frac{(n^{\gamma^+ +1} + \|\mu\|_{L^1(\Omega)})}{\alpha} \cdot \Big(\eta C_{p^-}^{p^-} + \frac{(p^- - 1)\eta^{-\frac{1}{p^- -1}}}{(p^-)^{(p^-)'}}\Big) |\Omega|,
\end{aligned}$$

So,

$$\int_\Omega |Dw|^{p(x)} \, dx \leq \eta C_1 \int_\Omega |Dw|^{p(x)} \, dx + C_2,$$

where $C_1 = \frac{C_{p^-}^{p^-}(n^{\gamma^+ +1}+\|\mu\|_{L^1(\Omega)})}{\alpha}$,

$$C_2 = \frac{(n^{\gamma^+ +1} + \|\mu\|_{L^1(\Omega)})}{\alpha} \cdot \Big(\eta C_{p^-}^{p^-} + \frac{(p^- - 1)\eta^{-\frac{1}{p^- -1}}}{(p^-)^{(p^-)'}}\Big)|\Omega|$$

are nonnegative constants and C_{p^-} is the constant of Poincaré.

Now, we can choose $\eta = \frac{1}{2C_1}$, we obtain

$$\int_\Omega |Dw|^{p(x)}\,dx \leq C_3,$$

with C_3 is independent from v. Thanks to Lemma 2.1, we have

$$\|Dw\|_{L^{p(\cdot)}(\Omega)} \leq C_4,$$

for some constant C_4 independent on v. By another application of Poincaré inequality (11), we obtain

$$\|w\|_{L^{p(\cdot)}(\Omega)} \leq C_{n,p}, \tag{25}$$

where $C_{n,p}$ is a nonnegative constant independent form v.

In particular, we have that the ball $B := B_{C_n}(0)$ of $L^{p(\cdot)}(\Omega)$ of large enough radius $C_{n,p}$ is invariant for the map $\widetilde{G}$. Moreover, from the compact Sobolev embedding, we deduce that $\widetilde{G}$ is continuous and compact on $L^{p(\cdot)}(\Omega)$.

Indeed, first we prove that the map $\widetilde{G}$ is continuous in B. Let us choose a sequence (v_k) that converges strongly to v in $L^{p(\cdot)}(\Omega)$, implies that $v_k \to v$ a.e in Ω, hence $\frac{f_n}{(|v_k|+\frac{1}{n})^{\gamma(x)}} + \mu_n \to \frac{f_n}{(|v|+\frac{1}{n})^{\gamma(x)}} + \mu_n$ a.e in Ω, going back to (19), the dominated convergence theorem gives

$$\frac{f_n}{(|v_k| + \frac{1}{n})^{\gamma(x)}} + \mu_n \to \frac{f_n}{(|v| + \frac{1}{n})^{\gamma(x)}} + \mu_n \text{ strongly in } L^{p(\cdot)}(\Omega),$$

then we need to prove that $\widetilde{G}(v_k)$ converge to $\widetilde{G}(v)$ in $L^{p(\cdot)}(\Omega)$. By compactness we already know that the sequence $w_k = \widetilde{G}(v_k)$ converge to some function w in $L^{p(\cdot)}(\Omega)$. So we only need to prove that $w = \widetilde{G}(v)$. Since the sequence w_k is bounded in $W_0^{1,p(\cdot)}(\Omega)$ and by uniqueness of the weak solution for the problem (24), we deduce the desired.

Second we need to check that the set $\widetilde{G}(B)$ is relatively compact. Let v_k be a bounded sequence in B and let $w_k = \widetilde{G}(v_k)$. Analogously to (25), for any $v_k \in L^{p(\cdot)}(\Omega)$, we get

$$\|w_k\|_{L^{p(\cdot)}(\Omega)} = \|\widetilde{G}(v_k)\|_{L^{p(\cdot)}(\Omega)} \leq C_{n,p},$$

for some constant $C_{n,p}$ independent on v_k. So that, $w_k = \widetilde{G}(v_k)$ is relatively compact in $L^{p(\cdot)}(\Omega)$. Thus, we can use Schauder's fixed point theorem to prove the existence of $u_n \in W_0^{1,p(\cdot)}(\Omega)$ solving the problem (20). Here $\frac{f_n}{(u_n+\frac{1}{n})^{\gamma(x)}} + \mu_n \geq 0$, using as a test function $u_n^- = \min\{u_n, 0\}$ in problem (21), one has $u_n \geq 0$. Moreover, by proceeding as in [5,43], we deduce $u_n \in L^\infty(\Omega)$ because the righthand side of (20) is in $L^\infty(\Omega)$.

Now, we will closely follow the proof of (Lemma 2.1, Lemma 2.2, [7]), (Lemma 2.1, [21]), (Lemma 4.2, [25]) and of (Lemma 2, [29]) hence we will omit the details, giving only a sketch of the passages. By (5), (8) and the fact that $0 \leq f_n \leq f_{n+1}$, we can prove that the sequence u_n is increasing with respect to n. knowing that, for $n \in \mathbb{N}^\star$ fixed, $u_n \in L^\infty(\Omega)$. So, in particular for $n = 1$, we have that

$$-\mathrm{div}\,(\widehat{a}(x, Du_1)) = \frac{f_1}{(u_1+1)^{\gamma(x)}} + \mu_1 \geq \frac{f_1}{(\|u_1\|_{L^\infty(\Omega)}+1)^{\gamma(x)}} + \mu_1 \geq 0,$$

Since $\frac{f_1}{(\|u_1\|_{L^\infty(\Omega)}+1)^{\gamma(x)}} + \mu_1$ is not identically zero, we apply the strong maximum principle (As in [44]), which ensures that, for all $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ (independent of n) such that

$$u_1(x) \geq C_\omega \text{ in } \omega,$$

Thus, (22) holds, because $u_n \geq u_1$ for all $n \in \mathbb{N}^\star$. Since u_n is increasing in n, we can define u as the pointwise limit of u_n. It follows that $u \geq u_n$ and by (22) we get, for every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ (independent of n) satisfied

$$u(x) \geq C_\omega > 0, \text{ for every } x \in \omega, \text{ for every } n \in \mathbb{N}^\star.$$

Observe that for all γ as in (8) and for all $\phi \in C_0^\infty(\Omega)$, if $\omega = \left\{x \in \Omega : |\phi| > 0\right\}$, we get

$$\left|\frac{f_n\phi}{(u_n+\frac{1}{n})^{\gamma(x)}}\right| \leq \frac{f\|\phi\|_{L^\infty(\Omega)}}{\min\left\{C_\omega^{\gamma^-}; C_\omega^{\gamma^+}\right\}} \in L^1(\Omega),$$

If $u = +\infty$ then $\frac{f\phi}{u^{\gamma(x)}} = 0$ and that, for $n \to +\infty$, we have

$$\frac{f_n\phi}{(u_n+\frac{1}{n})^{\gamma(x)}} \to \frac{f\phi}{u^{\gamma(x)}}, \quad \text{a.e. in} \quad \Omega,$$

Therefore, by Lebesgue dominated convergence theorem, it follows that (23) holds.

Finally we shall prove the uniqueness of the solution u_n of (20), let us consider u_n and v_n two different solutions of problem (20). For every fixed $n \in \mathbb{N}^\star$, we have

$$-\mathrm{div}\,(\widehat{a}(x, Du_n)) = \frac{f_n}{(u_n+\frac{1}{n})^{\gamma(x)}} + \mu_n,$$

and

$$-\mathrm{div}\,(\widehat{a}(x, Dv_n)) = \frac{f_n}{(v_n+\frac{1}{n})^{\gamma(x)}} + \mu_n,$$

By subtracting the two previous equality, we obtain that

$$-\text{div}\,(\widehat{a}(x,Du_n)-\widehat{a}(x,Dv_n))=f_n\frac{(v_n+\frac{1}{n})^{\gamma(x)}-(u_n+\frac{1}{n})^{\gamma(x)}}{(u_n+\frac{1}{n})^{\gamma(x)}(v_n+\frac{1}{n})^{\gamma(x)}},$$

We take $\phi=(u_n-v_n)^+:=\max\Big\{(u_n-v_n);0\Big\}$ as test function in the weak formulation of the previous equality and using that

$$\Big(\widehat{a}(x,Du_n)-\widehat{a}(x,Dv_n)\Big)\cdot D(u_n-v_n)^+\geq 0,$$
$$\Big((v_n+\frac{1}{n})^{\gamma(x)}-(u_n+\frac{1}{n})^{\gamma(x)}\Big)\cdot(u_n-v_n)^+\leq 0,$$

We find

$$0\leq\int_\Omega\Big(\widehat{a}(x,Du_n)-\widehat{a}(x,Dv_n)\Big)\cdot D(u_n-v_n)^+\,dx\leq 0,$$

From (3), we get

$$0\leq\alpha\int_\Omega|D(u_n-v_n)^+|^{p(x)}\,dx\leq 0,$$

Thus, for every $n\in\mathbb{N}^\star$, we have $(u_n-v_n)^+=0,$ a.e. in Ω, so $u_n\leq v_n$. By substituting u_n with v_n and repeating the same proof, we get that $v_n\leq u_n$, so that $u_n\equiv v_n$. This concludes the proof. □

Lemma 4.2. *The solution u_1 to problem* (20) *with $n=1$ satisfies, for all γ as in* (8)*, there exists a constant $C>0$ such that*

$$\int_\Omega\frac{1}{u_1^{\gamma(x)}}\,dx\leq C, \tag{26}$$

Proof of Lemma 4.2. The proof based on adapting the approach of ([15], Lemma 2.3) with similar arguments as in the proof of Lemma 4.3 in [25], we deduce (26). □

5. Uniform Estimates

In this section, we state and prove an uniform estimates for the solutions u_n of the problem (20).

Lemma 5.1. *Suppose that the assumptions of Theorem 3.3 are satisfied and assume that $m<p'(\cdot)$. Then there exists a constant $C>0$ such that*

$$\|u_n\|_{W_0^{1,p(\cdot)}(\Omega)}\leq C, \tag{27}$$

Proof of Lemma 5.1. Choosing u_n as test function in the weak formulation of (20), by (3), Hölder's inequality, Young's inequality with $\eta>0$, Poincaré's inequality and the fact that $f_n\leq f$, we get

$$\begin{aligned}\alpha\int_\Omega|Du_n|^{p(x)}\,dx&\leq\int_\Omega fu_n^{1-\gamma(x)}\,dx+\int_\Omega\mu_nu_n\,dx,\\&\leq\|f\|_{L^m(\Omega)}\Big(\int_\Omega u_n^{(1-\gamma(x))m'}\,dx\Big)^{\frac{1}{m'}}+\eta\int_\Omega|u_n|^{p^-}\,dx\end{aligned}$$

$$
\begin{aligned}
&+\frac{(p^- -1)\eta^{-\frac{1}{p^- -1}}}{(p^-)^{(p^-)'}}\int_\Omega (\mu_n)^{(p^-)'}\,dx\\
&\leq \|f\|_{L^m(\Omega)}\Big(\int_\Omega u_n^{(1-\gamma(x))m'}\,dx\Big)^{\frac{1}{m'}}+\eta C_{p^-}^{p^-}\int_\Omega |Du_n|^{p^-}\,dx\\
&+\frac{(p^- -1)\eta^{-\frac{1}{p^- -1}}}{(p^-)^{(p^-)'}}\int_\Omega (\mu_n)^{(p^-)'}\,dx\\
&\leq \|f\|_{L^m(\Omega)}\Big(\int_\Omega u_n^{(1-\gamma(x))m'}\,dx\Big)^{\frac{1}{m'}}+\eta C_{p^-}^{p^-}\int_\Omega |Du_n|^{p(x)}\,dx+C_1
\end{aligned}
$$

where C_{p^-} is the constant of Poincaré and

$$
C_1=\frac{(p^- -1)\eta^{-\frac{1}{p^- -1}}}{(p^-)^{(p^-)'}}\|\mu\|_{L^1(\Omega)}^{(p^-)'}+\eta C_{p^-}^{p^-}|\Omega|
$$

Choosing $\eta=\frac{\alpha}{2C_{p^-}^{p^-}}$, we have

$$
\frac{\alpha}{2}\int_\Omega |Du_n|^{p(x)}\,dx\leq \|f\|_{L^m(\Omega)}\Big(\int_\Omega u_n^{(1-\gamma(x))m'}\,dx\Big)^{\frac{1}{m'}}+C_1,
$$

By (7), we have

$$
(1-\gamma(\cdot))m'=p^\star(\cdot),
$$

According to Lemma 2.3 (Sobolev Embedding applied on the right hand side) and Lemma 2.1, we obtain

$$
\frac{\alpha}{2}\delta\Big(\int_\Omega u_n^{p^\star(x)}\,dx\Big)^{\frac{p^-}{p^\star{}_-}}\leq \|f\|_{L^m(\Omega)}\Big(\int_\Omega u_n^{p^\star(x)}\,dx\Big)^{\frac{1-\gamma(x)}{p^\star(x)}}+C_1,
$$

where $p^\star{}_- := \min_{x\in\overline{\Omega}} p^\star(x)$ and δ is the best constant of the Sobolev embedding. Going back to (8), since $1-\gamma(\cdot)<p^-$, we have $\frac{1-\gamma(\cdot)}{p^\star(\cdot)}<\frac{p^-}{p^\star{}_-}$, by Lemma 2.1, we obtain that u_n is bounded in $L^{p^\star(\cdot)}(\Omega)$ which finishes the proof. □

It is convenient to mention also here the following lemma.

Lemma 5.2. *Suppose that the assumptions of Theorem 3.6 are satisfied. Then, there exists a constant C_{γ^-} independent of n such that*

$$
\int_\Omega \frac{|Du_n|^{p(x)}}{(1+|u_n|)^{(1-\gamma^-)}}\,dx\leq C_{\gamma^-}\left(1+\Big(\int_\Omega (1+|u_n|)^{\frac{m\gamma^-}{m-1}}\,dx\Big)^{1-\frac{1}{m}}\right),\quad \gamma^- := \min_{x\in\overline{\Omega}}\gamma(x). \tag{28}
$$

Proof of Lemma 2.1. We define the function $\phi_{\gamma^-}:\mathbb{R}\to\mathbb{R}$ by

$$
\phi_{\gamma^-}(z)=\int_0^z \frac{dx}{(1+|x|)^{(1-\gamma^-)}}.
$$

Since ϕ_{γ^-} is a continuous function, it is easy to see that

$$\phi_{\gamma^-}(z) = \frac{-1}{\gamma^-}\left(1-(1+|z|)^{\gamma^-}\right)\text{sign}(z),$$

$\phi_{\gamma^-}(0) = 0$ and $|\phi'_{\gamma^-}(z)| \leq 1$. We take $\phi_{\gamma^-}(u_n)$ as a test function in (20), by (3), (19), Hölder's inequality, we obtain

$$\alpha\int_\Omega \frac{|Du_n|^{p(x)}}{(1+|u_n|)^{(1-\gamma^-)}}\,dx \leq \left(\left\|\frac{f}{u_n^{\gamma(x)}}\right\|_{L^m(\Omega)} + \|\mu\|_{L^1(\Omega)}\right)\|\phi_{\gamma^-}(u_n)\|_{L^{m'}(\Omega)},$$

In view of Lemma 4.1, we know that $u_n \geq u_1$ and there exists a constant $M > 0$ such that $u_1 \leq M$, we have $(\frac{M}{u_1})^{\gamma(\cdot)} \leq (\frac{M}{u_1})^{\gamma^+}$. Hence, it follows from (26), the sequence $(u_n)_n$ is increasing with respect to n and Hölder's inequality that

$$\left\|\frac{f}{u_n^{\gamma(x)}}\right\|_{L^m(\Omega)} \leq C_1\|f\|_{L^m(\Omega)},$$

It follows that,

$$\int_\Omega \frac{|Du_n|^{p(x)}}{(1+|u_n|)^{(1-\gamma^-)}}\,dx \leq C_2\Big(\int_\Omega |\phi_{\gamma^-}(u_n)|^{\frac{m}{m-1}}\,dx\Big)^{1-\frac{1}{m}}, \tag{29}$$

Since $|\phi_{\gamma^-}(z)| = \frac{1}{\gamma^-}\left((1+|z|)^{\gamma^-}-1\right)$, we can write

$$|\phi_{\gamma^-}(u_n)|^{\frac{m}{m-1}} \leq C_3\left(1+(1+|u_n|)^{\frac{m\gamma^-}{m-1}}\right)$$

Thanks to this estimate and (29), we deduce that (28) hold. □

The estimates on the gradients of u_n in $L^{q(\cdot)}(\Omega)$ which is proved in the next Lemma.

Lemma 5.3. *Suppose that the assumptions of Theorem 3.6 are satisfied. Then there exists a constant $C > 0$ such that*

$$\|u_n\|_{W_0^{1,q(\cdot)}(\Omega)} \leq C, \tag{30}$$

for all continuous functions $q(\cdot)$ as in (17).

Proof. In the previous Remark 3.7, we have seen that

$$1 < \frac{Nm\Big(p(x)-1+\gamma^-\Big)}{\Big(N-m(1-\gamma^-)\Big)}, \quad \text{for all} \quad x\in\overline{\Omega}.$$

Now, we discuss two cases:

Case (a): In a first step, let q^+ be a constant satisfying

$$q^+ < \frac{Nm\Big(p^- - 1+\gamma^-\Big)}{\Big(N-m(1-\gamma^-)\Big)}, \tag{31}$$

We set $\theta = \frac{q^+}{p^-}$, due to (7), we see that

$$\frac{Nm\Big(p^- - 1 + \gamma^-\Big)}{\Big(N - m(1-\gamma^-)\Big)} < p^-, \tag{32}$$

So, (31) and (32) imply

$$\theta \in (0,1), \quad \text{and} \quad \theta < m, \tag{33}$$

We can apply Hölder's inequality and using (28), we obtain

$$\begin{aligned}
\int_\Omega |Du_n|^{q^+}\,dx &= \int_\Omega \frac{|Du_n|^{q^+}}{(1+|u_n|)^{\theta(1-\gamma^-)}}(1+|u_n|)^{\theta(1-\gamma^-)}\,dx \\
&\le \left(\int_\Omega \frac{|Du_n|^{p^-}}{(1+|u_n|)^{(1-\gamma^-)}}\,dx\right)^{\theta} \cdot \left(\int_\Omega (1+|u_n|)^{\frac{\theta}{1-\theta}(1-\gamma^-)}\,dx\right)^{1-\theta} \\
&\le \left(C_{\gamma^-}\left(1 + \Big(\int_\Omega (1+|u_n|)^{\frac{m\gamma^-}{m-1}}\,dx\Big)^{1-\frac{1}{m}}\right)\right)^{\theta} \cdot \left(\int_\Omega (1+|u_n|)^{\frac{\theta}{1-\theta}(1-\gamma^-)}\,dx\right)^{1-\theta} \\
&\le C_\gamma^\theta\left(1 + \Big(\int_\Omega (1+|u_n|)^{\frac{m\gamma^-}{m-1}}\,dx\Big)\right)^{(1-\frac{1}{m})\theta} \\
&\times \left(1 + \Big(\int_\Omega (1+|u_n|)^{\frac{\theta}{1-\theta}(1-\gamma^-)}\,dx\Big)\right)^{1-\theta}
\end{aligned} \tag{34}$$

By (31) and (32), we get

$$0 < \frac{(m-1)q^+}{(m-1)q^+ + m(p^- - q^+)} < 1,$$

The inequalities (31), (32), (33) and the assumption (8) guarantee that

$$\frac{\theta}{1-\theta}(1-\gamma^-) < \frac{m\gamma^-}{m-1} < q^{+\star} := \frac{Nq^+}{N-q^+}, \tag{35}$$

Indeed, since $q^+ < \frac{N\left(p^- - 1 + \gamma^-\right)}{\left(N - 1 + \gamma^-\right)}$, we have

$$\frac{\theta}{1-\theta}(1-\gamma^-) < q^{+\star} := \frac{Nq^+}{N-q^+},$$

From (31), (32) and (8), we obtain

$$\frac{(m-1)q^+}{(m-1)q^+ + m(p^- - q^+)} < \gamma^- < 1 - q^{+\star}\Big(\frac{m-1}{m}\Big),$$

On the other hand, since $m < \frac{Np^-}{Np^- - (N-p^-)(1-\gamma^-)} = \left(\frac{p^{-\star}}{1-\gamma^-}\right)'$ we have

$$p^- < \frac{mN(1-\gamma^-)}{N(m-1) + m(1-\gamma^-)},$$

This inequality equivalent to (32), which gives

$$\gamma^{-} < 1 - q^{+\star}\Big(\frac{m-1}{m}\Big)$$

So, by (8), we have $p^{-} < \frac{mN}{N(m-1)+m}$ and using (33) we find $q^{+} < \frac{mN}{N(m-1)+m}$, which is equivalent to

$$0 < 1 - q^{+\star}\Big(\frac{m-1}{m}\Big).$$

Therefore, by (34), (35), we can write

$$\begin{aligned}
\int_{\Omega} |Du_n|^{q^{+}}\, dx &\leq C_1 \left(1 + \int_{\Omega} (1+|u_n|)^{\frac{m\gamma^{-}}{m-1}}\, dx\right)^{1+(1-\frac{1}{m})\theta-\theta} \\
&\leq C_2 \left(1 + \int_{\Omega} |u_n|^{\frac{m\gamma^{-}}{m-1}}\, dx\right)^{1-\frac{\theta}{m}} \\
&\leq C_3 \left(1 + \int_{\Omega} |u_n|^{q^{+\star}}\, dx\right)^{1-\frac{\theta}{m}},
\end{aligned} \tag{36}$$

Due to the Sobolev inequality with $q^{+\star}$, we see that

$$\int_{\Omega} |Du_n|^{q^{+}}\, dx \leq C_4 \left(1 + \int_{\Omega} |Du_n|^{q^{+}}\, dx\right)^{(\frac{N}{N-q^{+}})(1-\frac{\theta}{m})},$$

Consequently

$$\int_{\Omega} |Du_n|^{q^{+}}\, dx \leq C_5 + C_6 \left(\int_{\Omega} |Du_n|^{q^{+}}\, dx\right)^{\eta}, \quad \eta = \Big(\frac{N}{N-q^{+}}\Big)\Big(1-\frac{\theta}{m}\Big), \tag{37}$$

Thanks to (7) and (8), which gives

$$m < \frac{Np^{-}}{Np^{-} - (N-p^{-})(1-\gamma^{-})} < \frac{Np^{-}}{Np^{-} - N + p^{-}} < \frac{N}{p^{-}}, \tag{38}$$

together with the assumption (31) and (33), this implies that $\eta \in (0,1)$. Hence, by (37) and applying $|Du_n|^{q(\cdot)} \leq |Du_n|^{q^{+}} + 1$, we deduce that (30) hold. This completes the proof in case (a).

Case (b): In a second step, we suppose that (17) hold and

$$q^{+} \geq \frac{Nm\Big(p^{-} - 1 + \gamma^{-}\Big)}{\Big(N - m(1-\gamma^{-})\Big)},$$

By the continuity of $p(\cdot)$ and $q(\cdot)$ on $\overline{\Omega}$, there exists a constant $\delta > 0$ such that

$$\max_{z\in B(x,\delta)\cap\Omega} q(z) < \min_{z\in B(x,\delta)\cap\Omega} \frac{Nm(p(z) - 1 + \gamma^{-})}{(N - m(1-\gamma^{-}))}, \quad \text{for all } x \in \overline{\Omega}, \tag{39}$$

Where $B(x,\delta)$ is a ball with center x and diameter δ.

Note that $\overline{\Omega}$ is compact and therefore we can cover it with a finite number of balls $(B_i)_{i=1,\ldots,k}$. Moreover, there exists a constant $\rho > 0$ such that

$$|\Omega_i| = \text{meas}(\Omega_i) > \rho,\ \Omega_i := B_i \cap \Omega, \ \text{ for all } i = 1, \ldots, k. \tag{40}$$

We denote by q_i^+ the local maximum of q on $\overline{\Omega_i}$ (respectively p_i^- the local minimum of p on $\overline{\Omega_i}$) i.e. $q_i^+ := \max_{t\in\overline{\Omega}} q(t)$, $p_i^- := \min_{t\in\overline{\Omega}} p(t)$, such that

$$q_i^+ < \frac{Nm\Big(p_i^- - 1 + \gamma^-\Big)}{\Big(N - m(1-\gamma^-)\Big)}, \quad \text{for all} \quad i = 1, \ldots, k. \tag{41}$$

Arguing locally as in (36), we obtain

$$\int_{\Omega_i} |Du_n|^{q_i^+}\, dx \le C_7 \left(1 + \int_{\Omega_i} |u_n|^{q_i^{+\star}}\, dx\right)^{(1-\frac{\theta_i}{m})}, \quad \text{for all} \quad i = 1, \ldots, k. \tag{42}$$

Where $q_i^{+\star} := \frac{Nq_i^+}{N-q_i^+}$ and $\theta_i = \frac{q_i^+}{p_i^-}$. Thanks to the Poincaré-Wirtinger inequality, it results

$$\|u_n - \widetilde{u_n}\|_{L^{q_i^{+\star}}(\Omega_i)} \le C_8 \|Du_n\|_{L^{q_i^+}(\Omega_i)}, \tag{43}$$

Where

$$\widetilde{u_n} = \frac{1}{|\Omega_i|} \int_{\Omega_i} u_n(x)\, dx,$$

Since $(u_n)_n$ is bounded in $L^1(\Omega)$. So, in view of (40), we find

$$\|\widetilde{u_n}\|_{L^1(\Omega)} \le C_9,$$

Moreover it follows from (43) that

$$\begin{aligned} \|u_n\|_{L^{q_i^{+\star}}(\Omega_i)} &\le \|u_n - \widetilde{u_n}\|_{L^{q_i^{+\star}}(\Omega_i)} + \|\widetilde{u_n}\|_{L^{q_i^{+\star}}(\Omega_i)} \\ &\le C_8 \|Du_n\|_{L^{q_i^+}(\Omega_i)} + C_9, \quad \text{for all} \quad i = 1, \cdots, k. \end{aligned}$$

Therefore, from (42), we derive

$$\int_{\Omega_i} |Du_n|^{q_i^+}\, dx \le C_{10} + C_{11} \left(\int_{\Omega_i} |Du_n|^{q_i^+}\, dx\right)^{\left(\frac{N}{N-q_i^+}\right)\left(1-\frac{\theta_i}{m}\right)},$$

Going back to (41) and we reason locally as in (38), we conclude that

$$0 < \Big(\frac{N}{N-q_i^+}\Big)\Big(1 - \frac{\theta_i}{m}\Big) < 1,$$

Hence, we have

$$\int_{\Omega_i} |Du_n|^{q_i^+} \, dx \leq C_{12}, \text{ for all } i = 1, \ldots, k.$$

Knowing that

$$q(x) \leq q_i^+, \quad \text{ for all } x \in \Omega_i \text{ and for all } i = 1, \ldots, k.$$

We conclude that

$$\int_{\Omega_i} |Du_n|^{q(x)} \, dx \leq \int_{\Omega_i} |Du_n|^{q_i^+} \, dx + |\Omega_i| \leq C_{13}.$$

Since $\Omega \subset \bigcup_{i=1}^{N} \Omega_i$, for all $i = 1, \ldots, k$. So that

$$\int_{\Omega} |Du_n|^{q(x)} \, dx \leq \sum_{i=1}^{k} \int_{\Omega_i} |Du_n|^{q(x)} \, dx \leq C_{13}.$$

where C_{13} is a constant independent of u. This finishes the proof of the Lemma 5.3. □

Remark 5.4. *Remark that the result given in Lemma 5.3 also holds for any measurable function $q : \overline{\Omega} \to \mathbb{R}$ such that*

$$ess \inf_{x \in \overline{\Omega}} \left(\frac{Nm(p(x) - 1 + \gamma^-)}{N - m(1 - \gamma^-)} - q(x) \right) > 0, \quad \gamma^- := \min_{x \in \overline{\Omega}} \gamma(x),$$

Indeed, there exists a continuous function $r : \overline{\Omega} \to \mathbb{R}$ such that for almost every $x \in \overline{\Omega}$:

$$q(x) \leq r(x) \leq \frac{Nm(p(x) - 1 + \gamma^-)}{N - m(1 - \gamma^-)}.$$

Moreover, from Lemma 5.3, we deduce that $(u_n)_n$ is bounded in $W_0^{1,r(\cdot)}(\Omega)$. So, thanks to the continuous embedding $W_0^{1,r(\cdot)}(\Omega) \hookrightarrow W_0^{1,q(\cdot)}(\Omega)$, we get the desert result.

To continue we need the following lemma.

Lemma 5.5. *Suppose that the assumptions of Theorem 3.11 are satisfied. Then there exists a constant $C > 0$ such that*

$$\|u_n\|_{W_0^{1,q(\cdot)}(\Omega)} \leq C, \tag{44}$$

for all continuous functions $q(\cdot)$ as in (18).

Proof. Let $\nu > 1$, we define the function $\varrho_\nu(\cdot) : \mathbb{R} \longmapsto \mathbb{R}$ given by

$$\varrho_\nu(t) = \int_0^t \frac{dx}{(1 + |x|)^{(\nu - \gamma^-)}}, \quad \gamma^- := \min_{x \in \overline{\Omega}} \gamma(x),$$

It is clear that

$$\varrho_\nu(t) = \frac{1}{1+\gamma^- - \nu}\Big((1+|t|)^{1+\gamma^- - \nu} - 1\Big)\,\mathrm{sign}(t),$$

Note that ϱ_ν is a continuous function satisfies $\varrho_\nu(0) = 0$ and $|\varrho_\nu'(\cdot)| \leq 1$. We take $\varrho_\nu(u_n)$ as a test function in weak formulation of (20), using the assumption (3), (19) and the fact that

$$\|\varrho_\nu(t)\|_{L^\infty(\Omega)} \leq \int_{-\infty}^{+\infty} \frac{dx}{(1+|x|)^{(\nu-\gamma^-)}} < +\infty,$$

we obtain

$$\int_\Omega \frac{|Du_n|^{p(x)}}{(1+|u_n|)^{(\nu-\gamma^-)}}\,dx \leq \frac{1}{\alpha}\Big(C_\nu\|f\|_{L^1(\Omega)} + \|\mu\|_{L^1(\Omega)}\Big), \tag{45}$$

In particular, there exists $C_1 > 0$ such that

$$\int_\Omega \frac{|Du_n|^{p^-}}{(1+|u_n|)^{(\nu-\gamma^-)}}\,dx \leq C_1, \tag{46}$$

Observe that (18), (6) and (7) imply that

$$q(x) < \frac{N\Big(p(x) - 1 + \gamma^-\Big)}{\Big(N - 1 + \gamma^-\Big)} < p(x), \quad \text{for all } x \in \overline{\Omega}, \quad \gamma^- := \min_{x\in\overline{\Omega}} \gamma(x), \tag{47}$$

Let us write

$$\int_\Omega |Du_n|^{q(x)}\,dx = \int_\Omega \frac{|Du_n|^{q(x)}}{(1+|u_n|)^{\frac{q(x)}{p(x)}(\nu-\gamma^-)}}(1+|u_n|)^{\frac{q(x)}{p(x)}(\nu-\gamma^-)}\,dx.$$

Then, by (45) and Young's inequality, we obtain

$$\int_\Omega |Du_n|^{q(x)}\,dx \leq C_2 + C_3\int_\Omega (1+|u_n|)^{\frac{q(x)(\nu-\gamma^-)}{p(x)-q(x)}}\,dx, \tag{48}$$

The assumption (18) guarantees that $\frac{q^\star(x)(p(x)-q(x))}{q(x)} + \gamma^- > 1$,, where

$$q^\star(\cdot) = \frac{Nq(\cdot)}{N - q(\cdot)},$$

Choosing

$$\nu = \min_{x\in\overline{\Omega}}\Big(\frac{q^\star(x)(p(x)-q(x))}{q(x)} + \gamma^-\Big) > 1,$$

Again, thanks to the choice of ν and (18), we find

$$\frac{q(x)(\nu-\gamma^-)}{p(x)-q(x)} \leq q^\star(x), \quad \forall x \in \overline{\Omega}, \tag{49}$$

Hence, it follows from (48), (49) that

$$\int_\Omega |Du_n|^{q(x)}\,dx \leq C_2 + C_4\int_\Omega u_n^{q^\star(x)}\,dx, \tag{50}$$

From Lemma 2.3 (Sobolev Embedding applied on the right hand side) and Lemma 2.1, we get

$$\delta\Big(\int_\Omega u_n^{q^\star(x)}\,dx\Big) \leq C_2 + C_4\Big(\int_\Omega u_n^{q^\star(x)}\,dx\Big)^{1-\frac{q^-}{N}},$$

where $q^\star{}_- := \min_{x\in\overline{\Omega}} p^\star(x)$ and δ is the best constant of the Sobolev embedding. Using Lemma 2.1, we obtain that u_n is bounded in $L^{p^\star(\cdot)}(\Omega)$, hence (44) holds. $\square$

Remark 5.6. *By the same manner as the proof of Lemma 5.3 with similar reasoning, we can easily prove lemma 5.5.*

6. Proof of Main Results

In this section, using the uniform estimates of Section 5, we prove Theorems 3.3-3.6-3.11.
Remark that the proof of Theorem 3.3 by using Lemma 5.1 is similar to that of Theorem 3.6, here we only give the proof of Theorem 3.6 and Theorem 3.11.

Proof of Theorem 3.6. According to Lemma 5.3, the sequence (u_n) is bounded in $W_0^{1,p(\cdot)}(\Omega)$, where $p(\cdot)$ is defined as (6). we can therefore deduce that

$$\begin{aligned} u_n \rightharpoonup u \quad &\text{weakly in} \quad W_0^{1,p(\cdot)}(\Omega),\\ u_n \to u \quad &\text{strongly in} \quad L^{p(\cdot)}(\Omega),\\ u_n \to u \quad &\text{a.e. in} \quad \Omega,\\ Du_n \rightharpoonup Du \quad &\text{weakly in} \quad L^{p(\cdot)}(\Omega), \end{aligned} \tag{51}$$

By the assumption (5), Lebesgue's dominated convergence theorem, with the help of techniques used in [35] and adapting the approach of [33], we obtain, there exists a subsequence (still denoted (u_n)) such that

$$Du_n \to Du \quad \text{a.e in} \quad \Omega, \tag{52}$$

From (52), we get

$$\widehat{a}(x, Du_n) \to \widehat{a}(x, Du) \quad \text{a.e in} \quad \Omega, \tag{53}$$

Now, we prove that

$$\widehat{a}(x, Du_n) \to \widehat{a}(x, Du) \quad \text{strongly in } L^{r(\cdot)}(\Omega), \tag{54}$$

where r is a continuous function on $\overline{\Omega}$ such that

$$1 < r(x) < \frac{Nm}{\Big(N - m(1-\gamma^-)\Big)}, \qquad \text{for all } x \in \overline{\Omega}, \quad \gamma^- := \min_{x\in\overline{\Omega}} \gamma(x), \tag{55}$$

Note that the choice of $r(\cdot) > 1$ is possible since we have (6), (7) and (8). By (55), we can choose σ a continuous function on $\overline{\Omega}$ such that

$$1 < r(x) < \sigma(x) < \frac{Nm}{\Big(N - m(1-\gamma^-)\Big)}, \quad \text{for all } x \in \overline{\Omega},$$

and

$$\frac{1}{p(\cdot) - 1 + \gamma^-} < \sigma(x) < \frac{Nm}{\Big(N - m(1-\gamma^-)\Big)}, \quad \text{for all } x \in \overline{\Omega},$$

Then

$$1 < \sigma(x)\Big(p(\cdot) - 1 + \gamma^-\Big) < \frac{Nm\Big(p(\cdot) - 1 + \gamma^-\Big)}{\Big(N - m(1-\gamma^-)\Big)}, \quad \text{and} \quad \sigma(\cdot) < \frac{Nm}{\Big(N - m(1-\gamma^-)\Big)} < p'(\cdot), \tag{56}$$

Using the assumption (4), we get

$$|\widehat{a}(\cdot, Du_n)|^{\sigma(\cdot)} \leq C_1\Big(h^{\sigma(\cdot)} + |Du_n|^{\sigma(\cdot)(p(\cdot)-1)}\Big),$$

Therefore, by the last estimate together with (56), Lemma 5.3, Lemma 2.1, (12) and the Poincaré inequality, we conclude that $(\widehat{a}(\cdot, Du_n))_n$ is bounded in $L^{r(\cdot)}(\Omega)$.
To establish the equi-integrability of $(\widehat{a}(\cdot, Du_n))_n$ on Ω, We can apply Hölder's inequality and using Lemma 2.1, we obtain

$$\begin{aligned}\int_E |\widehat{a}(x, Du_n)|^{r(x)}\, dx &\leq \| |\widehat{a}(x, Du_n)|^{r(x)} \|_{L^{\frac{\sigma(\cdot)}{r(\cdot)}}(\Omega)} \cdot \|1\|_{L^{(\frac{\sigma(\cdot)}{r(\cdot)})'}(\Omega)} \\ &\leq C_2 \max\Big\{|E|^{\frac{1}{\nu^+}}; |E|^{\frac{1}{\nu^-}}\Big\}, \quad \nu = \frac{\sigma(\cdot)}{\sigma(\cdot) - r(\cdot)},\end{aligned}$$

Hence, thanks to (53) and Vitali's theorem, we derive (54).
Finally, for $\phi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \widehat{a}(x, Du_n) D\phi\, dx = \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^{\gamma(x)}} \phi\, dx + \int_\Omega \mu_n \phi\, dx, \tag{57}$$

So, using (19), (23) and (54), we can easily pass to the limit in (57) for all $\phi \in C_0^\infty(\Omega)$. This proves Theorem 3.6. □

By proceeding as in Theorem 3.6 and using Lemma 5.5, we have Theorem 3.11. We are now ready to prove Theorem 3.11.

Proof of Theorem 3.11. Suppose that (19) hold true, then in an analogous way, it is possible to prove that

$$\widehat{a}(x, Du_n) \to \widehat{a}(x, Du) \quad \text{strongly in } L^{\tau(\cdot)}(\Omega), \tag{58}$$

where τ is a continuous function on $\overline{\Omega}$ such that

$$1 < \tau(x) < \frac{N}{\Big(N - 1 + \gamma^-\Big)}, \quad \text{for all } x \in \overline{\Omega}, \quad \gamma^- := \min_{x\in\overline{\Omega}} \gamma(x), \tag{59}$$

Remark that the choice of $\tau(\cdot) > 1$ is possible since we have (8). Arguing as the proof of Theorem 3.6, by taking into account (19), (23), (59) and (57), we conclude the proof of the Theorem 3.11. □

Remark 6.1. *Observe that in the constant case and $f \in L^m(\Omega)$, according to* (37) *the problem* (2) *has a unique nonnegative weak solution $u \in W_0^{1,q}(\Omega)$. Moreover, u possesses the regularity $q = \frac{Nm\big(p-1+\nu\big)}{\big(N-m(1-\nu)\big)}$ provided that $\nu \in (0,1)$. For the nonconstant case, it remains an open problem to show that $u \in W_0^{1,q(\cdot)}(\Omega)$, where $q(\cdot)$ is a continuous function on $\overline{\Omega}$ satisfying*

$$q(x) = \frac{Nm(x)\Big(p(x) - 1 + \gamma^-\Big)}{\Big(N - m(x)(1-\gamma^-)\Big)}, \quad \textit{for all } x \in \overline{\Omega}, \quad \gamma^- := \min_{x\in\overline{\Omega}} \gamma(x),$$

with $m(\cdot)$ recorded as in (7).

Remark 6.2. *We point out that all our previous results (Theorems 3.3-3.6-3.11) hold true as long as the problem* (2) *is exchanged by a more general one,*

$$\begin{cases} -\mathrm{div}\,(\widehat{a}(x,u,Du)) = H(u)f + \mu, & \textit{in } \Omega, \\ u > 0, & \textit{in } \Omega, \\ u = 0, & \textit{on } \partial\Omega, \end{cases}$$

Where $\widehat{a}(x,t,\xi) = \{a_i(x,t,\xi)\}_{i=1,\ldots,N} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-valued function such that for a.e. $x \in \Omega$ and for every $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$, the following assumptions hold:

$$\begin{aligned} &\widehat{a}(x,t,\xi)\xi \geq \alpha|\xi|^{p(x)}, \quad \widehat{a}(x,t,\xi) = (a_1,\ldots,a_N), \quad \alpha > 0, \\ &|\widehat{a}(x,t,\xi)| \leq \beta\left(h + |t|^{p(x)-1} + |\xi|^{p(x)-1}\right), \quad \beta > 0, \quad h \in L^{p'(\cdot)}(\Omega), \\ &(\widehat{a}(x,t,\xi) - \widehat{a}(x,t,\xi'))(\xi - \xi') > 0, \ \xi \neq \xi', \end{aligned}$$

With Ω is a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, $0 \leq \mu \in L^1(\Omega)$, and f is a nonnegative function of $L^1(\Omega)$ (or $L^{m(\cdot)}(\Omega)$) with $m(\cdot) > 1$ being small.
We assume that the nonlinearity term $H : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and possibly singular function, such that

$$\begin{aligned} &H(0) \neq 0, \quad \lim_{u\to+\infty} H(u) := H(+\infty) < +\infty, \\ &\textit{There exists } M > 0, \textit{ for all } u \in (0,+\infty), \textit{ such that } : H(u) \leq M, \\ &\textit{There exists } \widehat{M} > 0, \textit{ such that } : H(u) \leq \frac{\widehat{M}}{u^{\gamma(x)}}, \textit{ for all } u \in (0,+\infty), \textit{ for all } x \in \overline{\Omega}, \end{aligned}$$

where $\gamma : \overline{\Omega} \longrightarrow (0,1)$ *is continuous functions satisfies:*

$$0 < \gamma^- := \min_{x\in\overline{\Omega}} \gamma(x) \le \gamma^+ := \max_{x\in\overline{\Omega}} \gamma(x) < 1, \quad |\nabla\gamma| \in L^\infty(\Omega), \text{ for all } x \in \overline{\Omega}.$$

while $m : \overline{\Omega} \to (1, +\infty)$ *and the variable exponent* $p : \overline{\Omega} \to (1, +\infty)$ *are continuous functions such that, for all* $x \in \overline{\Omega}$,

$$1 - \gamma(x) + \frac{1}{m(x)} - \frac{1-\gamma(x)}{N} < p(x) < N, \quad |\nabla m| \in L^\infty(\Omega), \quad |\nabla\gamma| \in L^\infty(\Omega),$$

$$1 < m(x) < \widehat{m}_1(x), \quad \widehat{m}_1(x) = \frac{Np(x)}{Np(x) - (N - p(x))(1-\gamma(x))} = \Big(\frac{p^\star(x)}{1-\gamma(x)}\Big)',$$

where

$$|\nabla m| \in L^\infty(\Omega), \quad |\nabla\gamma| \in L^\infty(\Omega).$$

Conclusions

In this paper, we have studied the existence, uniqueness, and regularity of nonnegative weak solutions to the singular elliptic equations involving a nonlinear singular term with variable exponents and showed that the singular term has regularizing effects on the solutions of the problem by applied method of approximations and the Schauder fixed point theorem; nevertheless, some methods for elliptic equations operators can not directly be applied to our problem by cause of the nonlinearity of the operator A which depends on $p(\cdot)$ and the nonlinear singular term which depends on $\gamma(\cdot)$ that is attached on $m(\cdot)$. Furthermore, we have proved that the existence, uniqueness, and regularity of weak solutions depend on the summability of f, $m(\cdot)$ and the value of $\gamma(\cdot)$.

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