International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **11(4)**(2023), 127–139
Available Online: http://ijmaa.in

Some Integrals Involving Extended Hypergeometric Function

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Abstract

Our purpose in this present paper is to investigate generalized integration formulas containing the extended generalized hypergeometric function and obtained results are expressed in terms of extended hypergeometric function. Certain special cases of the main results presented here are also pointed out for the extended Gauss' hypergeometric and confluent hypergeometric functions.

Keywords: Extended hypergeometric function; Lavoie-Trottier Integral formula; Gauss hypergeometric function.

2020 Mathematics Subject Classification: Primary 33C45; Secondary 33C60, 33B15, 33C05.

1. Introduction

Numerous applications of applied mathematics need scientists and engineers to use various types of special functions. Continuous progress in mathematical physics, probability theory, and other fields has resulted in the discovery of new classes of special functions, as well as their extensions and generalizations. Recent research has focused on the study and creation of special functions, one of which is often referred to as generalized hypergeometric functions. As a part of the theory of confluent hypergeometric functions, these functions are significant special functions, and their closely related variants are frequently employed in physics and engineering. Schwarz and Goursat investigated the unique features of one-variable hypergeometric function. For further information on contemporary research in the fields of dynamical systems theory, stochastic systems, non-equilibrium statistical mechanics, and quantum mechanics, readers may consult the researchers' recent work [6–12] and the sources given therein. Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}^-$ and $\mathbb{C}$ the sets of positive integers, negative integers and complex numbers, respectively, and also let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}$. The generalized hypergeometric function with p numerator and q denominator

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parameters ${}_pF_q(p;q \in \mathbb{N}_0)$ is defined by

$$
{}_pF_q\left[\begin{array}{c} a_1,a_2,a_3,\ldots,a_p \\ b_1,b_2,\ldots,b_q \end{array};z\right] = \sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots.(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n}\frac{z^n}{n!} \tag{1}
$$

$$
= \sum_{n=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(\alpha_j)_n}{\prod\limits_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!} \tag{2}
$$

$$
= {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) \tag{3}
$$

The Pochhammer symbol $(\alpha)_\nu(\alpha,\nu \in \mathbb{C})$ is defined, in terms of Gamma function Γ by

$$
(\alpha)_\nu = \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)} \quad (\alpha+\nu \in \mathbb{C}\setminus\mathbb{Z}_0^-,\nu \in \mathbb{C}\setminus\{0\};\alpha \in \mathbb{C}\setminus\mathbb{Z}_0^-,\nu = 0)
$$

$$
(\alpha)_\nu = \begin{cases} \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1) & (\nu = n \in \mathbb{N},\alpha \in \mathbb{C}) \\ 1 \quad (\nu = 0,\alpha \in \mathbb{C}\setminus\mathbb{Z}_0^-) & \end{cases}
$$

it being understood that $(0)_0 = 1$. Here it is supposed that the variable z, the numerator parameters $\alpha_1,\ldots,\alpha_p$; and the denominator parameters $\beta_1,\ldots,\beta_j$ take on complex values, provided that

$$
(\beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-;j = 1,\ldots,q)
$$

Then, if a numerator parameter is in $\mathbb{Z}_0^-$, the series ${}_pF_q$ is found to terminate and becomes a polynomial in z. With none of the numerator and denominator parameters being zero or a negative integer, the series ${}_pF_q$ in 4.

(i) diverges for all $z \in \mathbb{C}\setminus\{0\}$, if $p > q+1$;

(ii) converges for all $z \in \mathbb{C}$, if $p \leq q$;

(iii) converges for $|z| < 1$ and diverges for $|z| > 1$ if $p = q+1$;

(iv) converges absolutely for $|z| = 1$, if $p = q+1$ and $\mathcal{R}(\varpi) > 0$;

(v) converges conditionally for $|z| = 1(z \neq 1)$, if $p = q+1$ and $-1 < \mathcal{R}(\varpi) \leq 0$;

(vi) diverges for $|z| = 1$, if $p = q+1$ and $\mathcal{R}(\varpi) \leq -1$, where $\varpi = \sum\limits_{j=1}^{q}\beta_j - \sum\limits_{j=1}^{p}\alpha_j$ which is called the parametric excess of the series.

The extended generalized hypergeometric function is defined by Srivastava et al. [16] (See [5,14]:

$$
{}_rF_s\left[\begin{array}{c} (a_1,p),a_2,a_3,\ldots,a_r \\ b_1,b_2,\ldots,b_s \end{array};z\right] = \sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots.(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n}\frac{z^n}{n!} \tag{4}
$$

where, in terms of generalized Pochammer symbol $(\mu; p)_v$ [16]:

$$(\mu; p)_v = \begin{cases} \frac{\Gamma_p(\mu+v)}{\Gamma(v)}, & (\mathcal{R}(p) > 0, \mu, v \in \mathbb{C}) \\ (\mu)_v, & (p = 0, \mu, v \in \mathbb{C}) \end{cases} \tag{5}$$

Here $\Gamma_p(z)$ is the generalized gamma function introduced by Chaudhry and Zubair [2] as follows:

$$\Gamma_p(z) = \begin{cases} \int_0^\infty t^{(z-1)} e^{-p-p/t} dt, & (\mathcal{R}(p) > 0, z \in \mathbb{C}) \\ \Gamma(z), (p = 0, \mathcal{R}(z) > 0) \end{cases} \tag{6}$$

The corresponding extensions of Gauss's hypergeometric and confluent Hypergeometric functions are as follows:

$${}_2F_1 \left\{ \begin{matrix} (a_1, p), \alpha \\ \beta \end{matrix} ; z \right\} = \sum_{n=0}^{\infty} \frac{(a_1; p)_n . (\alpha)_n}{(\beta)_n} \frac{z^n}{n!} \tag{7}$$

and

$${}_1F_1 \left\{ \begin{matrix} (a_1, p) \\ \beta \end{matrix} ; z \right\} = \sum_{n=0}^{\infty} \frac{(a_1; p)_n}{(\beta)_n} \frac{z^n}{n!} \tag{8}$$

where, $\alpha, \beta \in \mathbb{C}$. In this paper, we derives some new integral formulas involving the generalized hypergeometric function 4. Further we give corollaries as special cases for the extended Gauss' hypergeometric and confluent hypergeometric functions. For the present investigation, we need the following result of Oberhettinger [12].

$$\int_0^\infty (\xi - z)^\alpha z^{\beta-1} dz = \xi^{\alpha+\beta} B(\alpha + 1, \beta) \tag{9}$$

$\xi > z$, $(\sim) > -1$, $(\odot) > 0$. For various other investigations involving certain special functions, interested reader may be referred to several recent papers on the subject (see, for example, [1,3,4,17–19] and the references cited in each of these papers.

2. Main Result

In this section, the generalized integral formulas involving the extended generalized hypergeometric function defined in 4 are established here by inserting with the suitable argument in the integrand of 9 and we express the obtained result in terms of an extended Wright-type hypergeometric function.

Theorem 2.1. *Let* $a_1, a_2, \ldots, a_r, \alpha, \beta, p \in \mathbb{C}$; *and* $b_1, b_2, \ldots, b_s \in \mathbb{C} \setminus \mathbb{Z}_0^-$; *with* $\mathcal{R}(p) > 0; \mathcal{R}(\alpha) > \mathcal{R}(\beta) >$

0; $p \geq 0$ *and* $z > 0$. *Then the following formula holds true:*

$$\int_0^\infty (\xi - z)^\alpha z^{\beta-1} {}_rF_s \begin{Bmatrix} (a_1, p), a_2, a_3, \ldots, a_r & ; z \\ b_1, b_2, \ldots, b_s & \end{Bmatrix} dz$$

$$= \xi^{\alpha+\beta} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} {}_{r+1}F_{s+1} \begin{Bmatrix} \beta, (a_1, p), a_2, a_3, \ldots, a_r & ; \xi \\ (\alpha+\beta+1), b_1, b_2, \ldots, b_s & \end{Bmatrix} \tag{10}$$

Proof. From the definition (4), we have

$${}_rF_s \begin{Bmatrix} (a_1, p), a_2, a_3, \ldots, a_r & ; z \\ b_1, b_2, \ldots, b_s & \end{Bmatrix} = \sum_{n=0}^{\infty} \frac{(a_1; p)_n.(a_2)_n.(a_3)_n \ldots (a_r)_n}{(b_1)_n (b_2)_n \ldots .(b_s)_n} \frac{z^n}{n!}$$

Let us consider the Left side of (10),

$$\int_0^\infty (\xi - z)^\alpha z^{\beta-1} {}_rF_s \begin{Bmatrix} (a_1, p), a_2, a_3, \ldots, a_r & ; z \\ b_1, b_2, \ldots, b_s) & \end{Bmatrix} dz$$

$$= \int_0^\infty (\xi - z)^\alpha z^{\beta-1} \sum_{n=0}^{\infty} \frac{(a_1; p)_n.(a_2)_n.(a_3)_n \ldots (a_r)_n}{(b_1)_n (b_2)_n \ldots (b_s)_n} \frac{z^n}{n!} dz$$

$$= \sum_{n=0}^{\infty} \frac{(a_1; p)_n.(a_2)_n.(a_3)_n \ldots (a_r)_n}{(b_1)_n (b_2)_n \ldots (b_s)_n n!} \int_0^\infty (\xi - z)^\alpha z^{\beta+n-1} dz$$

By applying (9) we have

$$\xi^{\alpha+\beta} \sum_{n=0}^{\infty} \frac{(a_1; p)_n.(a_2)_n.(a_3)_n \ldots (a_r)_n)}{(b_1)_n (b_2)_n \ldots (b_s)_n n!} B(\alpha+1, \beta+n) \xi^n \tag{11}$$

$$= \xi^{\alpha+\beta} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \sum_{n=0}^{\infty} \frac{(\beta)_n (a_1; p)_n.(a_2)_n.(a_3)_n \ldots (a_r)_n}{(\alpha+\beta+1)_n (b_1)_n (b_2)_n \ldots (b_s)_n n!} \xi^n$$

$$= \xi^{\alpha+\beta} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} {}_{r+1}F_{s+1} \begin{Bmatrix} \beta, (a_1, p), a_2, a_3, \ldots, a_r & ; \xi \\ (\alpha+\beta+1), b_1, b_2, \ldots, b_s & \end{Bmatrix}$$

This complete prove of Theorem 2.1. □

Theorem 2.2. *Let* $a_1, a_2, \ldots, a_r, \alpha, \beta, p \in \mathbb{C}$; *and* $b_1, b_2, \ldots, b_s \in \mathbb{C} \setminus \mathbb{Z}_0^-$; *with* $\mathcal{R}(p) > 0; \mathcal{R}(\alpha) > \mathcal{R}(\beta) > 0$; $p \geq 0$ *and* $z > 0$. *Then the following formula holds true:*

$$\int_0^\infty (\xi - z)^\alpha z^{\beta-1} {}_rF_s \begin{Bmatrix} (a_1, p), a_2, a_3, \ldots, a_r & ; (\xi - z) \\ b_1, b_2, \ldots, b_s & \end{Bmatrix} dz$$

$$= \xi^{\alpha+\beta} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} {}_{r+1}F_{s+1} \begin{Bmatrix} (\alpha+1), (a_1, p), a_2, a_3, \ldots, a_r & ; \xi \\ (\alpha+\beta+1), b_1, b_2, \ldots, b_s & \end{Bmatrix} \tag{12}$$

Proof. From the definition (4), we have

$$ {}_rF_s\left\{\begin{matrix}(a_1,p),a_2,a_3,\dots,a_r \\ b_1,b_2,\dots,b_s\end{matrix};z\right\} = \sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\dots(a_r)_n}{(b_1)_n(b_2)_n\dots.(b_s)_n}\frac{z^n}{n!} $$

Let us consider the Left side of (12),

$$ \int_0^\infty(\xi-z)^\alpha z^{\beta-1}F_s^r\left\{\begin{matrix}(a_1,p),a_2,a_3,\dots,a_r \\ b_1,b_2,\dots,b_s)\end{matrix};(\xi-z)\right\}dz $$

$$ =\int_0^\infty(\xi-z)^\alpha z^{\beta-1}\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\dots(a_r)_n}{(b_1)_n(b_2)_n\dots(b_s)_n}\frac{(\xi-z)^n}{n!}dz $$

$$ =\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\dots(a_r)_n}{(b_1)_n(b_2)_n\dots(b_s)_n n!}\int_0^\infty(\xi-z)^{\alpha+n}z^{\beta-1}dz $$

By applying (9) we have,

$$ =\xi^{\alpha+\beta}\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\dots(a_r)_n)}{(b_1)_n(b_2)_n\dots(b_s)_n n!}B(\alpha+n+1,\beta)\xi^n \tag{13} $$

$$ =\xi^{\alpha+\beta}\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\sum_{n=0}^{\infty}\frac{(\alpha+1)_n(a_1;p)_n.(a_2)_n.(a_3)_n\dots(a_r)_n}{(\alpha+\beta+1)_n(b_1)_n(b_2)_n\dots(b_s)_n n!}\xi^n $$

$$ =\xi^{\alpha+\beta}\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}{}_{r+1}F_{s+1}\left\{\begin{matrix}(\alpha+1),(a_1,p),a_2,a_3,\dots,a_r \\ (\alpha+\beta+1),b_1,b_2,\dots,b_s\end{matrix};\xi\right\} $$

This complete prove of Theorem 2.2. □

Theorem 2.3. *Let* $a_1,a_2,\dots,a_r,\alpha,\beta,p\in\mathbb{C}$; *and* $b_1,b_2,\dots,b_s\in\mathbb{C}\setminus/\mathbb{Z}_0^-$; *with* $\mathcal{R}(p)>0;\mathcal{R}(\alpha)>\mathcal{R}(\beta)>0$; $p\geq 0$ *and* $z>0$. *Then the following formula holds true:*

$$ \int_0^\infty(\xi-z)^\alpha z^{\beta-1}{}_rF_s\left\{\begin{matrix}(a_1,p),a_2,a_3,\dots,a_r \\ b_1,b_2,\dots,b_s\end{matrix};z(\xi-z)\right\}dz $$

$$ =\xi^{\alpha+\beta}\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}{}_{r+1}F_{s+1}\left\{\begin{matrix}(\alpha+1),(a_1,p),a_2,a_3,\dots,a_r \\ \left(\frac{\alpha+\beta+1}{2}\right),\left(\frac{\alpha+\beta+2}{2}\right)),b_1,b_2,\dots,b_s\end{matrix};\left(\frac{\xi}{4}\right)\right\} \tag{14} $$

Proof. From the definition (4), we have

$$ {}_rF_s\left\{\begin{matrix}(a_1,p),a_2,a_3,\dots,a_r \\ b_1,b_2,\dots,b_s\end{matrix};z\right\} = \sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\dots(a_r)_n}{(b_1)_n(b_2)_n\dots.(b_s)_n}\frac{z^n}{n!} $$

Let us consider the Left side of (14),

$$\int_0^\infty (\xi - z)^\alpha z^{\beta-1} {}_rF_s \left\{ \begin{matrix} (a_1, p), a_2, a_3, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; z(\xi - z) \right\} dz$$

$$= \int_0^\infty (\xi - z)^\alpha z^{\beta-1} \sum_{n=0}^\infty \frac{(a_1; p)_n.(a_2)_n.(a_3)_n \ldots (a_r)_n}{(b_1)_n (b_2)_n \ldots (b_s)_n} \frac{z^n (\xi - z)^n}{n!} dz$$

$$= \sum_{n=0}^\infty \frac{(a_1; p)_n.(a_2)_n.(a_3)_n \ldots (a_r)_n}{(b_1)_n (b_2)_n \ldots .(b_s)_n n!} \int_0^\infty (\xi - z)^{\alpha+n} z^{\beta+n-1} dz$$

By applying (9) we have,

$$= \xi^{\alpha+\beta} \sum_{n=0}^\infty \frac{(a_1; p)_n.(a_2)_n.(a_3)_n \ldots (a_r)_n)}{(b_1)_n (b_2)_n \ldots .(b_s)_n n!} B(\alpha + n + 1, \beta + n) \xi^n \tag{15}$$

$$= \xi^{\alpha+\beta} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \sum_{n=0}^\infty \frac{(\alpha+1)_n, (\beta)_n, (a_1; p)_n.(a_2)_n.(a_3)_n \ldots (a_r)_n}{\left(\frac{\alpha+\beta+1}{2}\right)_n, \left(\frac{\alpha+\beta+2}{2}\right)_n, (b_1)_n (b_2)_n \ldots .(b_s)_n n!} \left(\frac{\xi}{4}\right)^n$$

$$= \xi^{\alpha+\beta} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} {}_{r+1}F_{s+1} \left\{ \begin{matrix} (\alpha+1), (a_1, p), a_2, a_3, \ldots, a_r \\ \left(\frac{\alpha+\beta+1}{2}\right), \left(\frac{\alpha+\beta+2}{2}\right)), b_1, b_2, \ldots, b_s \end{matrix} ; \left(\frac{\xi}{4}\right) \right\}$$

This complete prove of Theorem 2.3. □

3. Extension

In this section we prove some extension formulae of Theorem 2.1, Theorem 2.2 and Theorem 2.3 by using the property of Beta function,

$$B(\alpha, \beta) = B(\alpha + 1, \beta) + B(\alpha, \beta + 1) \tag{16}$$

Theorem 3.1. *Let* $a_1, a_2, \ldots, a_r, \alpha, \beta, p \in \mathbb{C}$; *and* $b_1, b_2, \ldots, b_s \in \mathbb{C} \setminus \mathbb{Z}_0^-$; *with* $\mathcal{R}(p) > 0; \mathcal{R}(\alpha) > \mathcal{R}(\beta) > 0; p \geq 0$ *and* $z > 0$. *Then the following formula holds true:*

$$\int_0^\infty (\xi - z)^\alpha z^{\beta-1} F_s^r \left\{ \begin{matrix} (a_1, p), a_2, a_3, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; z \right\} dz$$

$$= \xi^{\alpha+\beta} B(\alpha + 2, \beta)_{r+1} F_{s+1} \left\{ \begin{matrix} (\beta), (a_1, p), a_2, a_3, \ldots, a_r \\ (\alpha+\beta+2), b_1, b_2, \ldots, b_s \end{matrix} ; \xi \right\}$$

$$+ \xi^{\alpha+\beta} B(\alpha + 1, \beta + 1)_{r+1} F_{s+1} \left\{ \begin{matrix} (\beta+1), (a_1, p), a_2, a_3, \ldots, a_r \\ (\alpha+\beta+2), b_1, b_2, \ldots, b_s \end{matrix} ; \xi \right\} \tag{17}$$

Proof. From the Definition (4), we have

$$ {}_rF_s\left\{\begin{matrix}(a_1,p),a_2,a_3,\ldots,a_r \\ b_1,b_2,\ldots,b_s\end{matrix};z\right\} = \sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots.(b_s)_n}\frac{z^n}{n!} $$

Let us consider the Left side of (17),

$$ \begin{aligned} &= \int_0^{\infty}(\xi-z)^{\alpha}z^{\beta-1}F_s^r\left\{\begin{matrix}(a_1,p),a_2,a_3,\ldots,a_r \\ b_1,b_2,\ldots,b_s)\end{matrix};z\right\}dz \\ &= \int_0^{\infty}(\xi-z)^{\alpha}z^{\beta-1}\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n}\frac{z^n}{n!}dz \\ &= \sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n n!}\int_0^{\infty}(\xi-z)^{\alpha}z^{\beta+n-1}dz \end{aligned} $$

By using (9) and (16) we have

$$ = \xi^{\alpha+\beta}\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n n!}B(\alpha+1,\beta+n)\xi^n \tag{18} $$

$$ \begin{aligned} &= \xi^{\alpha+\beta}\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n n!}[B(\alpha+2,\beta+n)+B(\alpha+1,\beta+n+1)]\xi^n \\ &= \xi^{\alpha+\beta}B(\alpha+2,\beta)\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n n!}\frac{(\beta)_n}{(\alpha+\beta+2)_n}\xi^n \\ &+ \xi^{\alpha+\beta}B(\alpha+1,\beta+1)\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n n!}\frac{(\beta+1)_n}{(\alpha+\beta+2)_n}\xi^n \\ &= \xi^{\alpha+\beta}\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\sum_{n=0}^{\infty}\frac{(\beta)_n(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n}{(\alpha+\beta+1)_n(b_1)_n(b_2)_n\ldots(b_s)_n n!}\xi^n \\ &= \xi^{\alpha+\beta}B(\alpha+2,\beta)_{r+1}F_{s+1}\left\{\begin{matrix}(\beta),(a_1,p),a_2,a_3,\ldots,a_r \\ (\alpha+\beta+2),b_1,b_2,\ldots,b_s\end{matrix};\xi\right\} \\ &+ \xi^{\alpha+\beta}B(\alpha+1,\beta+1)_{r+1}F_{s+1}\left\{\begin{matrix}(\beta+1),(a_1,p),a_2,a_3,\ldots,a_r \\ (\alpha+\beta+2),b_1,b_2,\ldots,b_s\end{matrix};\xi\right\} \end{aligned} $$

This complete prove of Theorem 3.1. □

Theorem 3.2. *Let* $a_1,a_2,\ldots,a_r,\alpha,\beta,p\in\mathbb{C}$; *and* $b_1,b_2,\ldots,b_s\in\mathbb{C}\setminus\mathbb{Z}_0^-$; *with* $\mathcal{R}(p)>0;\mathcal{R}(\alpha)>\mathcal{R}(\beta)>0; p\geq 0$ *and* $z>0$. *Then the following formula holds true:*

$$ \int_0^{\infty}(\xi-z)^{\alpha}z^{\beta-1}{}_sF_r\left\{\begin{matrix}(a_1,p),a_2,a_3,\ldots,a_r \\ b_1,b_2,\ldots,b_s\end{matrix};(\xi-z)\right\}dz $$

$$= \xi^{\alpha+\beta} B(\alpha+2,\beta)_{r+1}F_{s+1}\left\{\begin{matrix}(\alpha+2),(a_1,p),a_2,a_3,\ldots,a_r \\ (\alpha+\beta+2),b_1,b_2,\ldots,b_s\end{matrix};\xi\right\}$$

$$+ \xi^{\alpha+\beta} B(\alpha+1,\beta+1)_{r+1}F_{s+1}\left\{\begin{matrix}(\alpha+1),(a_1,p),a_2,a_3,\ldots,a_r \\ (\alpha+\beta+2),b_1,b_2,\ldots,b_s\end{matrix};\xi\right\} \qquad (19)$$

Proof. From the Definition (4), we have

$$_rF_s\left\{\begin{matrix}(a_1,p),a_2,a_3,\ldots,a_r \\ b_1,b_2,\ldots,b_s\end{matrix};z\right\} = \sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots.(b_s)_n}\frac{z^n}{n!}$$

Let us consider the Left side,

$$= \int_0^\infty (\xi-z)^\alpha z^{\beta-1}{}_rF_s\left\{\begin{matrix}(a_1,p),a_2,a_3,\ldots,a_r \\ b_1,b_2,\ldots,b_s)\end{matrix};(\xi-z)\right\}dz$$

$$= \int_0^\infty (\xi-z)^\alpha z^{\beta-1}\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n}\frac{(\xi-z)^n}{n!}dz$$

$$= \sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n n!}\int_0^\infty (\xi-z)^{\alpha+n}z^{\beta-1}dz$$

By using (9) and 16) we have

$$= \xi^{\alpha+\beta}\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n)}{(b_1)_n(b_2)_n\ldots(b_s)_n n!}B(\alpha+n+1,\beta)\xi^n \qquad (20)$$

$$= \xi^{\alpha+\beta}\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n)}{(b_1)_n(b_2)_n\ldots(b_s)_n n!}[B(\alpha+n+2,\beta)+B(\alpha+n+1,\beta+1)]\xi^n$$

$$= \xi^{\alpha+\beta}\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n)}{(b_1)_n(b_2)_n\ldots(b_s)_n n!}B(\alpha+n+2,\beta)\xi^n$$

$$+ \xi^{\alpha+\beta}\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n)}{(b_1)_n(b_2)_n\ldots(b_s)_n n!}B(\alpha+n+1,\beta+1)\xi^n$$

$$= \xi^{\alpha+\beta}B(\alpha+2,\beta)\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n)}{(b_1)_n(b_2)_n\ldots(b_s)_n n!}\frac{(\alpha+2)_n}{(\alpha+\beta+2)_n}\xi^n$$

$$+ \xi^{\alpha+\beta}B(\alpha+1,\beta+1)\sum_{n=0}^{\infty}\frac{(a_1;p)_n.(a_2)_n.(a_3)_n\ldots(a_r)_n)}{(b_1)_n(b_2)_n\ldots(b_s)_n n!}\frac{(\alpha+1)_n}{(\alpha+\beta+2)_n}\xi^n$$

$$= \xi^{\alpha+\beta}B(\alpha+2,\beta)_{r+1}F_{s+2}\left\{\begin{matrix}(\alpha+2),(a_1,p),a_2,a_3,\ldots,a_r \\ (\alpha+\beta+2),b_1,b_2,\ldots,b_s\end{matrix};\xi\right\}$$

$$+ \xi^{\alpha+\beta}B(\alpha+1,\beta+1)_{r+1}F_{s+2}\left\{\begin{matrix}(\alpha+1),(a_1,p),a_2,a_3,\ldots,a_r \\ (\alpha+\beta+2),b_1,b_2,\ldots,b_s\end{matrix};\xi\right\}$$

This complete prove of Theorem 3.2. □

Theorem 3.3. *Let $a_1, a_2, \ldots, a_r, \alpha, \beta, p \in \mathbb{C}$; and $b_1, b_2, \ldots, b_s \in \mathbb{C} \setminus \mathbb{Z}_0^-$; with $\mathcal{R}(p) > 0; \mathcal{R}(\alpha) > \mathcal{R}(\beta) > 0$; $p \geq 0$ and $z > 0$. Then the following formula holds true:*

$$\int_0^\infty (\xi - z)^\alpha z^{\beta-1} {}_rF_s \left\{ \begin{matrix} (a_1, p), a_2, a_3, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; z(\xi - z) \right\} dz$$

$$= \xi^{\alpha+\beta} B(\alpha+2, \beta) {}_{r+1}F_{s+2} \left\{ \begin{matrix} (\alpha+2), (\beta), (a_1, p), a_2, a_3, \ldots, a_r \\ \left(\frac{\alpha+\beta+1}{2}\right), \frac{\alpha+\beta+2}{2}, b_1, b_2, \ldots, b_s \end{matrix} ; \left(\frac{\xi}{4}\right) \right\}$$

$$+ \xi^{\alpha+\beta} B(\alpha+1, \beta+1) {}_{r+1}F_{s+2} \left\{ \begin{matrix} (\alpha+1), (\beta+1), (a_1, p), a_2, a_3, \ldots, a_r \\ \left(\frac{\alpha+\beta+1}{2}\right), \frac{\alpha+\beta+2}{2}, b_1, b_2, \ldots, b_s \end{matrix} ; \left(\frac{\xi}{4}\right) \right\} \quad (21)$$

Proof. From the Definition (4), we have

$${}_rF_s \left\{ \begin{matrix} (a_1, p), a_2, a_3, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; z \right\} = \sum_{n=0}^{\infty} \frac{(a_1; p)_n . (a_2)_n . (a_3)_n \ldots (a_r)_n}{(b_1)_n (b_2)_n \ldots . (b_s)_n} \frac{z^n}{n!}$$

Let us consider the Left side of (21),

$$= \int_0^\infty (\xi - z)^\alpha z^{\beta-1} {}_rF_s \left\{ \begin{matrix} (a_1, p), a_2, a_3, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; z(\xi - z) \right\} dz$$

$$= \int_0^\infty (\xi - z)^\alpha z^{\beta-1} \sum_{n=0}^{\infty} \frac{(a_1; p)_n . (a_2)_n . (a_3)_n \ldots (a_r)_n}{(b_1)_n (b_2)_n \ldots (b_s)_n} \frac{z^n (\xi - z)^n}{n!} dz$$

$$= \sum_{n=0}^{\infty} \frac{(a_1; p)_n . (a_2)_n . (a_3)_n \ldots (a_r)_n}{(b_1)_n (b_2)_n \ldots . (b_s)_n n!} \int_0^\infty (\xi - z)^{\alpha+n} z^{\beta+n-1} dz$$

By using (9) and (16) we have

$$= \xi^{\alpha+\beta} \sum_{n=0}^{\infty} \frac{(a_1; p)_n . (a_2)_n . (a_3)_n \ldots (a_r)_n)}{(b_1)_n (b_2)_n \ldots . (b_s)_n n!} B(\alpha+n+1, \beta+n) \xi^n \quad (22)$$

$$= \xi^{\alpha+\beta} \sum_{n=0}^{\infty} \frac{(a_1; p)_n . (a_2)_n . (a_3)_n \ldots (a_r)_n)}{(b_1)_n (b_2)_n \ldots . (b_s)_n n!} [B(\alpha+n+2, \beta+n) + B(\alpha+n+1, \beta+n+1)] \xi^n$$

$$= \xi^{\alpha+\beta} \sum_{n=0}^{\infty} \frac{(a_1; p)_n . (a_2)_n . (a_3)_n \ldots (a_r)_n)}{(b_1)_n (b_2)_n \ldots . (b_s)_n n!} B(\alpha+n+2, \beta+n) \xi^n$$

$$+ \xi^{\alpha+\beta} \sum_{n=0}^{\infty} \frac{(a_1; p)_n . (a_2)_n . (a_3)_n \ldots (a_r)_n)}{(b_1)_n (b_2)_n \ldots . (b_s)_n n!} B(\alpha+n+1, \beta+n+1) \xi^n$$

$$= \xi^{\alpha+\beta} \sum_{n=0}^{\infty} \frac{(a_1; p)_n . (a_2)_n . (a_3)_n \ldots (a_r)_n)}{(b_1)_n (b_2)_n \ldots . (b_s)_n n!} B(\alpha+2, \beta) \frac{(\alpha+2)_n (\beta)_n}{2^{2n} \left(\frac{\alpha+\beta+2}{2}\right)_n} \xi^n$$

$$+ \xi^{\alpha+\beta} \sum_{n=0}^{\infty} \frac{(a_1; p)_n . (a_2)_n . (a_3)_n \ldots (a_r)_n)}{(b_1)_n (b_2)_n \ldots . (b_s)_n n!} B(\alpha+1, \beta+1) \frac{(\alpha+1)_n (\beta+1)_n}{2^{2n} \left(\frac{\alpha+\beta+2}{2}\right)_n} \xi^n$$

$$= \xi^{\alpha+\beta} B(\alpha+2,\beta) \sum_{n=0}^{\infty} \frac{(a_1;p)_n.(a_2)_n.(a_3)_n \dots (a_r)_n)}{(b_1)_n (b_2)_n \dots .(b_s)_n n!} \frac{(\alpha+2)_n (\beta)_n}{2^{2n} \left(\frac{\alpha+\beta+1}{2}\right)_n \left(\frac{\alpha+\beta+2}{2}\right)_n} \xi^n$$

$$+ \xi^{\alpha+\beta} B(\alpha+1,\beta+1) \sum_{n=0}^{\infty} \frac{(a_1;p)_n.(a_2)_n.(a_3)_n \dots (a_r)_n)}{(b_1)_n (b_2)_n \dots .(b_s)_n n!} \frac{(\alpha+1)_n (\beta+1)_n}{2^{2n} \left(\frac{\alpha+\beta+1}{2}\right)_n \left(\frac{\alpha+\beta+2}{2}\right)_n} \xi^n$$

$$= \xi^{\alpha+\beta} B(\alpha+2,\beta)_{r+1}F_{s+2} \left\{ \begin{matrix} (\alpha+2),(\beta),(a_1,p),a_2,a_3,\dots,a_r \\ \left(\frac{\alpha+\beta+1}{2}\right),\left(\frac{\alpha+\beta+2}{2}\right),b_1,b_2,\dots,b_s \end{matrix} ; \left(\frac{\xi}{4}\right) \right\}$$

$$+ \xi^{\alpha+\beta} B(\alpha+1,\beta+1)_{r+1}F_{s+2} \left\{ \begin{matrix} (\alpha+1),(\beta+1),(a_1,p),a_2,a_3,\dots,a_r \\ \left(\frac{\alpha+\beta+1}{2}\right),\left(\frac{\alpha+\beta+2}{2}\right),b_1,b_2,\dots,b_s \end{matrix} ; \left(\frac{\xi}{4}\right) \right\}$$

This complete prove of Theorem 3.3. □

4. Corollaries

On taking suitable values of $r = 1$ and $s = 1$ in Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 3.1, Theorem 3.2 and Theorem 3.3 we get following corollaries;

Corollary 4.1. *Let $a_1, a_2, \alpha, \beta, p \in \mathbb{C}$; and $b_1, b_2 \in \mathbb{C} \setminus \mathbb{Z}_0^-$; with $\mathcal{R}(p) > 0; \mathcal{R}(\alpha) > \mathcal{R}(\beta) > 0; p \geq 0$ and $z > 0$. Then the following formula holds true:*

$$\int_0^\infty (\xi - z)^\alpha z^{\beta-1} {}_2F_1 \left\{ \begin{matrix} (a_1,p),a_2 \\ b \end{matrix} ; z \right\} dz = \xi^{\alpha+\beta} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} {}_3F_2 \left\{ \begin{matrix} \beta,(a_1,p),a_2 \\ (\alpha+\beta+1),b \end{matrix} ; \xi \right\}$$

Corollary 4.2. *Let $a_1, a_2, \alpha, \beta, p \in \mathbb{C}$; and $b_1, b_2 \in \mathbb{C} \setminus \mathbb{Z}_0^-$; with $\mathcal{R}(p) > 0; \mathcal{R}(\alpha) > \mathcal{R}(\beta) > 0; p \geq 0$ and $z > 0$. Then the following formula holds true:*

$$\int_0^\infty (\xi - z)^\alpha z^{\beta-1} {}_2F_1 \left\{ \begin{matrix} (a_1,p),a_2 \\ b \end{matrix} ; (\xi - z) \right\} dz = \xi^{\alpha+\beta} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} {}_2F_3 \left\{ \begin{matrix} (\alpha+1),(a_1,p),a_2 \\ (\alpha+\beta+1),b \end{matrix} ; \xi \right\}$$

Corollary 4.3. *Let $a_1, a_2, \alpha, \beta, p \in \mathbb{C}$; and $b_1, b_2 \in \mathbb{C} \setminus \mathbb{Z}_0^-$; with $\mathcal{R}(p) > 0; \mathcal{R}(\alpha) > \mathcal{R}(\beta) > 0; p \geq 0$ and $z > 0$. Then the following formula holds true:*

$$\int_0^\infty (\xi - z)^\alpha z^{\beta-1} {}_2F_1 \left\{ \begin{matrix} (a_1,p),a_2 \\ b \end{matrix} ; z(\xi - z) \right\} dz$$

$$= \xi^{\alpha+\beta} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} {}_3F_3 \left\{ \begin{matrix} (\alpha+1),(a_1,p),a_2 \\ \left(\frac{\alpha+\beta+1}{2}\right),\left(\frac{\alpha+\beta+2}{2}\right)),b \end{matrix} ; \left(\frac{\xi}{4}\right) \right\}$$

Corollary 4.4. *Let $a_1, a_2, \alpha, \beta, p \in \mathbb{C}$; and $b_1 \in \mathbb{C} \setminus \mathbb{Z}_0^-$; with $\mathcal{R}(p) > 0; \mathcal{R}(\alpha) > \mathcal{R}(\beta) > 0; p \geq 0$ and $z > 0$.*

Then the following formula holds true:

$$\int_0^\infty (\xi - z)^\alpha z^{\beta-1} {}_2F_1 \begin{Bmatrix} (a_1, p), a_2 & ; z \\ b_1 & \end{Bmatrix} dz = \xi^{\alpha+\beta} B(\alpha+2, \beta) {}_3F_2 \begin{Bmatrix} (\beta), (a_1, p), a_2 & ; \xi \\ (\alpha+\beta+2), b_1 & \end{Bmatrix}$$
$$+ \xi^{\alpha+\beta} B(\alpha+1, \beta+1) {}_3F_2 \begin{Bmatrix} (\beta+1), (a_1, p), a_2 & ; \xi \\ (\alpha+\beta+2), b_1 & \end{Bmatrix} \quad (23)$$

Corollary 4.5. *Let $a_1, a_2, \alpha, \beta, p \in \mathbb{C}$; and $b_1 \in \mathbb{C} \setminus \mathbb{Z}_0^-$; with $\mathcal{R}(p) > 0; \mathcal{R}(\alpha) > \mathcal{R}(\beta) > 0$; $p \geq 0$ and $z > 0$. Then the following formula holds true:*

$$\int_0^\infty (\xi - z)^\alpha z^{\beta-1} {}_2F_1 \begin{Bmatrix} (a_1, p), a_2 & ; (\xi - z) \\ b_1 & \end{Bmatrix} dz = \xi^{\alpha+\beta} B(\alpha+2, \beta) {}_2F_3 \begin{Bmatrix} (\alpha+2), (a_1, p), a_2 & ; \xi \\ (\alpha+\beta+2), b_1 & \end{Bmatrix}$$
$$+ \xi^{\alpha+\beta} B(\alpha+1, \beta+1) {}_3F_2 \begin{Bmatrix} (\alpha+1), (a_1, p), a_2 & ; \xi \\ (\alpha+\beta+2), b_1 & \end{Bmatrix} \quad (24)$$

Corollary 4.6. *Let $a_1, a_2 \alpha, \beta, p \in \mathbb{C}$; and $b_1 \in \mathbb{C} \setminus \mathbb{Z}_0^-$; with $\mathcal{R}(p) > 0$; $\mathcal{R}(\alpha) > \mathcal{R}(\beta) > 0$; $p \geq 0$ and $z > 0$. Then the following formula holds true:*

$$\int_0^\infty (\xi - z)^\alpha z^{\beta-1} {}_2F_1 \begin{Bmatrix} (a_1, p), a_2 & ; z(\xi - z) \\ b_1 & \end{Bmatrix} dz$$
$$= \xi^{\alpha+\beta} B(\alpha+2, \beta) {}_3F_3 \begin{Bmatrix} (\alpha+2), (\beta), (a_1, p), a_2 & ; \left(\frac{\xi}{4}\right) \\ \left(\frac{\alpha+\beta+1}{2}\right), \frac{\alpha+\beta+2}{2}, b_1 & \end{Bmatrix}$$
$$+ \xi^{\alpha+\beta} B(\alpha+1, \beta+1) {}_3F_3 \begin{Bmatrix} (\alpha+1), (\beta+1), (a_1, p), a_2 & ; \left(\frac{\xi}{4}\right) \\ \left(\frac{\alpha+\beta+1}{2}\right), \frac{\alpha+\beta+2}{2}, b_1 & \end{Bmatrix} \quad (25)$$

Acknowledgment

The authors would like to express their gratitude to the reviewers for their insightful comments which ultimately improve the quality of the paper.

References

[1] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Sci., 3(2012), 10-16.

[2] M. A. Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, CRC Press Company, London, (2001).

[3] J. Choi and P. Agarwal, *Certain unified integrals associated with Bessel functions*, Boundary Value Problems, 2013(2013), 95.

[4] J. Choi, P. Agarwal, S. Mathur and S. D. Purohit, *Certain new integral formulas involving the generalized Bessel functions*, Bull. Korean Math. Soc., 51(2014), 995-1003.

[5] Junesang Choi, *Certain applications of generalized Kummer's summation formulas for 2F1*, Symmetry, 13(8)(2021).

[6] R. Gorenflo and F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer Series on CSM Courses and Lectures, vol. 378, Springer-Verlag, Wien, (1997), 223-276.

[7] R. Gorenflo, F. Mainardi and H. M. Srivastava, *Special functions in fractional relaxationoscillation and fractional diffusion-wave phenomena*, in: D. Bainov (Ed.), Proceedings of the Eighth International Colloquium on Differential Equations (Plovdiv, Bulgaria; August 1823, 1997), VSP Publishers, Utrecht and Tokyo, (1998), 195-202.

[8] K. S. Nisar, S. D. Purohit and S. R. Mondal, *Generalized fractional kinetic equations involving generalized Struve function of the first kind*, J. King Saud Univ. Sci., 28(2016), 167-171.

[9] K. S. Nisar and S. R. Mondal, *Certain unified integral formulas involving the generalized modified k-Bessel function of first kind*, arXiv: 1601.06487 [math.CA].

[10] K. S. Nisar, P. Agarwal and S. Jain, *Some unified integrals associated with Bessel-Struve kernel function*, arXiv:1602.01496v1 [math.CA].

[11] I. Podlubny, *Fractional Differential Equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Academic Press, San Diego, CA, (1999).

[12] T. R. Prabhakar, *A singular integral equation with a generalized MittagLeffler function in the kernel*, Yokohama Math. J., 19(1971), 715.

[13] F. Oberhettinger, *Tables of Mellin transforms*, Springer, New York, (1974).

[14] Junesang Choi, Mohd Idris Qureshi, Aarif Hussain Bhat and Javid Majid, *Reduction formulas for generalized hypergeometric series associated with new sequences and applications*, Fractal Fract., 5(4)(2021), Article ID 150.

[15] S. C. Sharma and Menu Devi, *Certain Properties of Extended Wright Generalized Hypergeometric Function*, Annals of Pure and Applied Mathematics, 9(1)(2015), 45-51.

[16] H. M. Srivastava, A. Cetinkaya and Keyamaz I Onur, *A certain generalized Pochhammer symbol and its applications to hypergeometric functions*, Appl. Math. Comput., 226(2014), 484-491.

[17] H. M. Srivastava, *A note on the integral representation for the product of two generalized Rice polynomials*, Collect. Math., 24(1973), 117-121.

[18] H. M. Srivastava and C. M. Joshi, *Integral Representation For The Product Of A Class Of Generalized Hypergeometric Polynomials*, Acad. Roy. Belg. Bull. Cl. Sci. (Ser. 5), 60(1974), 919-926.

[19] H. M. Srivastava and Z. Tomovski, *Fractional calculus with an integral operator containing a generalized MittagLeffler function in the kernel*, Appl. Math. Comput., 211(2009), 189-210.

[20] L. Gross, *Abstract Wiener spaces*, Proc. 5th Berkeley Symp. Math. Stat. and Probab., 2(1)(1965), 31-42.

[21] K. Itô, *Stochastic integral*, Proc. Imp. Acad. Tokyo, 20(1944), 519-524.

[22] H. P. Mc Kean, Stochastic Integrals, Academic Press, New York, (1969).