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Degree of Approximation of Function in the Hölder Metric (C 1)(e, c) Means of its Fourier Series

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Abstract

We extended a theorem of Das, Ghosh and Ray [4] obtained a result on degree of approximation of function in the Hölder metric by (e, c) mean. In 2022, Rathore, Shrivastava and Mishra [13] has been determined the result on degree of approximation of a function in the Hölder metric by (C, 1) F(a, q) mean of its Fourier series. Further we extend the result on degree of approximation of function in the Hölder metric by (C, 1) (e, c) means of its Fourier series, has been proved.

Keywords: Fourier series; Hölder metric; Banach Spaces; Lebesgue integral; (C, 1) (e, c) mean.
2020 Mathematics Subject Classification: 42B05, 42B08.

1. Introduction

Chandra [3] was first to extend the result of Prössdorf's [9]. In 1983, Mohapatra and Chandra [8] result to find the degree of approximation in the Hölder metric using matrix transform. In this direction we studied on approximation of f belong to many classes also Hölder metric by Cesaro, Nörlund, Euler mean has been discussed by several researchers like respectively Das, Ghosh and Ray [4], Lal and Kushwaha [7], Rathore and Shrivastava [10], Rathore, Shrivastava and Mishra [12] etc. In 2022, Rathore, Shrivastava and Mishra [11] has been determined on approximation of function in the Hölder metric by (C, 1)$[F, d_n]$ product summability of Fourier series. Recently Rathore, Shrivastava and Mishra [13] determined a theorem on the degree of approximation of function in the Hölder Metric by (C, 1) F(a, q) means. Further we extend the result on the degree of approximation of function in the Hölder metric by (C 1)(e, c) means of its Fourier series, has been proved.

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2. Definition and Notation

Let f be a periodic function and integrable in the Lebesgue sense over $[-\pi, \pi]$. Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

Let $C_{2\pi}$ denote the Banach Spaces of all $2\pi-$ periodic continuous function under ''sup" norm for $0 < \alpha \leq 1$ and some positive constant K, the function space H_α is given by the following

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leq K|x-y|^\alpha\} \tag{2}$$

The space H_α is a Banach space (see Prossdorf's [9]) with the norm $|\cdot|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{x,y} \Delta\alpha[f(x,y)] \tag{3}$$

where,

$$\|f\|_c = \sup_{-\pi \leq x \leq \pi} |f(x)| \tag{4}$$

and

$$\Delta^\alpha\{f(x,y)\} = |x-y|^{-\alpha}|f(x) - f(y)|, \ \ (x \neq y) \tag{5}$$

We shall use the convention that $\Delta^0 f(x,y) = 0$. The metric induced (3) on the H_α is called the Hölder metric. If can be seen that $\|f\|_\beta \leq (2\pi)^{\alpha-\beta}\|f\|_\alpha$ for $0 \leq \beta < \alpha \leq 1$. Thus $\{(H_\alpha, \|\cdot\|_\alpha)\}$ is a family of Banach spaces. We write

$$\phi_x(t) = \{f(x+t) + f(x-t) - 2f(x)\} \tag{6}$$

Let $S_k(f;x)$ be the k^{th} partial sum of (1). Then (see Titchmarsh [16]).

$$S_k(f;x) - f(x) = \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}} \sin\left(k + \frac{1}{2}\right)t \ \ dt \tag{7}$$

and

$$\lim_{n\to\infty} e_n^c = \lim_{n\to\infty} \sqrt{\frac{c}{\pi n}} \sum_{r=-\infty}^{\infty} \exp\left(\frac{-cr^2}{k}\right) S_{k+r} \tag{8}$$

where, $S_{k+r} = 0$, for $k + r < 0$ and

$$\|e_n^c - f\| = \sup_{-\pi \leq x \leq \pi} |e_n^c(f:x) - f(x)| \tag{9}$$

where $e_n^c(f:x)$ is n^{th} (e, c) means of f at x, we have

$$e_n^c(f;x) - f(x) = \frac{1}{\pi}\sqrt{\frac{c}{\pi n}} \int_0^\pi \frac{\emptyset_x(\mathrm{t})}{\sin\frac{t}{2}} \left[\sum_{r=-k}^{\infty} \exp\left(-\frac{cr^2}{k}\right) \sin\left(k + r + \frac{1}{2}\right)t\right] dt \tag{10}$$

Then the infinite series $\sum_{n=0}^{\infty} u_n$ with partial sum S_n is said to be summable by (e, c) method to a definite number S (Hardy [5]). A product of (C, 1) and (e, c) mean defines (C, 1) (e, c) means and denoted by $C_n^1 e_n^c$. Thus if

$$C_n^1 e_n^c = \frac{1}{n+1} \sum_{k=0}^{n} e_k^c \to s \quad as \quad n \to \infty \tag{11}$$

then $\sum_{n=0}^{\infty} u_n$ is summable by (C, 1) (e, c) means.

3. Inequalities

We require the following inequalities.

$$\sum_{r=k+1}^{\infty} r \exp\left(-\frac{cr^2}{k}\right) \leq \frac{k}{2c} \exp(-ck) \tag{12}$$

$$\left| \sum_{r=k+1}^{\infty} \exp\left(-\frac{cr^2}{k}\right) \sin\left(k+r+\frac{1}{2}\right) t \right| \leq \frac{kt}{2c} \exp(-ck) \tag{13}$$

$$\sum_{r=k+1}^{\infty} \exp\left(-\frac{cr^2}{k}\right) \cos(rt) = O\left\{\frac{\exp(-ck)}{t}\right\} \tag{14}$$

$$1 + 2\sum_{r=1}^{\infty} \exp\left(-\frac{cr^2}{k}\right) \cos rt = \sqrt{\frac{\pi k}{c}} \left\{\exp\left(\frac{-kt^2)}{4c}\right) + O\left(\exp\left(\frac{-k\pi}{4c}\right)\right)\right\} \tag{15}$$

The inequality (13) follow from (12), (14) may be obtained and using Able's Lemma and (15) may be obtained by classical formula for theta function (see siddiqui [15]) and (12) is due to Shrivastava & Verma [14].

4. Some Theorems

In 1928, Alexits [1] proved the following theorem.

Theorem 4.1. *If* $f \in C_{2\pi} \cap$ *Lip* α, $(0 < \alpha \leq 1)$ *then*

$$\left|\sigma_n{}^{\delta} - f\right| = O\left(n^{-a} \log n\right) \tag{16}$$

where $0 < \alpha \leq \delta \leq 1$ *and* $\sigma_n^{\delta}(f;x)$ *is* (C,δ) *mean of* $\{S_n(f;x)\}$.

The case $\alpha = \delta = 1$ was proved by Bernstein [2]. Theorem 4.1, was extended by several workers such as Holland, Sahney and Tzimbalario [6]. Replacing (C,δ) mean by $(E,q)(q > 0)$ Chandra [3] obtained the following result:

Theorem 4.2. *Let* $0 \leq \beta < \alpha \leq 1$ *and let* $f \in H_{\alpha}$. *Then*

$$\|E_n{}^q(f) - f\|_{\beta} = \mathrm{O}\left\{n^{\beta-\alpha} \log n\right\} \tag{17}$$

where, $E_n{}^q(f,x)$ *denotes* (E,q) *transform of* $S_n(f;x)$.

Das, Ghosh and Ray [4] extended the result of Hölder metric by (e, c) means. Their as follows theorem.

Theorem 4.3. *Let* $0 \le \beta < \alpha \le 1$ *and* $f \in H_\alpha$. *Then*

$$\|e_n(f) - f\|_\beta = O\left(n^{\beta-\alpha}\log n\right). \tag{18}$$

Rathore and Shrivastava [10] obtained the degree of approximation of function of belonging to weighted class by $(C,1)(e,c)$ means. We have proved.

Theorem 4.4. *If* $f : R \to R$ *is* 2π*-periodic Lebesgue integrable on* $[-\pi,\pi]$ *and belonging to the Lipschitz class then approximation of* f *by the* $(C,1)(e,c)$ *means of its Fourier series satisfies for* $n = 0,1,2,\ldots,$

$$\|(C,e)_n^c(x) - f(x)\|_\infty = O(n+1)^{-\alpha} \quad \textit{for} \quad 0 < \alpha < 1 \tag{19}$$

We extend the above results

5. Main Results

Lemma 5.1. *Let* $\Phi_x(t)$ *be defined in* (6) *and for* $f \in H_\alpha$, *then*

$$\left|\Phi_x(t) - \Phi_y(t)\right| \le 4k|x-y|^a \tag{20}$$

$$\left|\Phi_x(t) - \Phi_y(t)\right| \le 4k|t|^a \tag{21}$$

$$|F(t)| = \left|\phi_x(t) - \phi_y(t)\right| \tag{22}$$

Lemma 5.2. *Let* $M_n(t) = \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$ *then* $M_n(t) = O(n+1)$, *for* $0 \le t \le \frac{\pi}{n+1}$.

Proof. $\sin nt \le n \sin t$ for $0 \le t \le \frac{\pi}{n+1}$

$$\begin{aligned} M_n(t) &= \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \frac{(2k+1)\sin\frac{t}{2}}{\sin\frac{t}{2}} \\ &= \frac{1}{\pi(n+1)} \sum_{k=0}^{n} (2k+1) \\ &= \mathrm{O}(n+1)^n \end{aligned} \tag{23}$$

□

Lemma 5.3. *Let* $M_n(t) = \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$ *then* $M_n(t) = O\left(\frac{1}{t}\right)$ *for* $\frac{\pi}{n+1} \le t \le \pi$.

Proof. $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$ and $\sin kt \le 1$ for $\frac{\pi}{n+1} \le t \le \pi$

$$M_n(t) = \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \frac{1}{t/\pi}$$

$$= O\left(\frac{1}{t}\right) \tag{24}$$

□

Theorem 5.4. *Let* $0 \leq \beta < \alpha \leq 1$ *and* $f \in H_\alpha$ *then*

$$|(C,e)_n^c - f(x)|_\beta = O\left[(n+1)^{\beta-\alpha}\log(n+1)\right], \tag{25}$$

where, $(C,e)_n^c$ *is the product summability* $(C,1)(e,c)$ *mean of* $S_n(f,x)$*.*

Proof. Titchmarsh [16] and using Riemann - Lebesgue theorem,

$$S_n(f;x) - f(x) = \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}}\sin\left(n+\frac{1}{2}\right)tdt \tag{26}$$

Using (1) the (e,c) mean (e_n^c) of $S_n(f;x)$ is

$$e_n^c - f(x) = \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}}\sqrt{\frac{c}{\pi n}}\sum_{r=-k}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\sin\left(k+\frac{1}{2}\right)tdt \tag{27}$$

We have

$$\begin{aligned}
e_n^c(t) &= \sqrt{\frac{c}{\pi n}}\sum_{r=-k}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\sin\left(k+\frac{1}{2}\right)t \qquad (28)\\
&= \sqrt{\frac{c}{\pi n}}\left[\begin{array}{c}\left\{1+2\sum_{r=1}^{k}\exp\left(\frac{-cr^2}{k}\right)\cos rt\right\}\sin\left(k+\frac{1}{2}\right)t+\\ \sum_{r=k+1}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t\end{array}\right]\\
&= \sqrt{\frac{c}{\pi n}}\left\{1+2\sum_{r=1}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\cos rt\right\}\sin\left(k+\frac{1}{2}\right)t\\
&- \sqrt{\frac{c}{\pi n}}2\sum_{r=k+1}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\cos rt\sin\left(k+\frac{1}{2}\right)t\\
&+ \sqrt{\frac{c}{\pi n}}\sum_{r=k+1}^{\infty}\exp\left(\frac{-cr^2}{k}\right)\sin\left(k+r+\frac{1}{2}\right)t\\
&= J_n(t) + K_n(t) + L_n(t)
\end{aligned}$$

Product (C, 1) (e, c) mean of $S_n(f;x)$ as $C_n^1 e_n^c$. We write

$$C_n^1 e_n^c - f(x) = \frac{1}{\pi(n+1)}\sum_{k=0}^{n}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}}\{J_n(t)+K_n(t)+L_n(t)\}\,dt \tag{29}$$

Writing

$$I_n(x) = C_n^1 e_n^c - f(x) = \frac{1}{\pi(n+1)}\sum_{k=0}^{n}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}}\{J_n(t)+K_n(t)+L_n(t)\}\,dt \tag{30}$$

We have

$$|I_n(x)| = \left|C_n^1 e_n^c - f(x)\right| = \left|\frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \int_0^{\pi} \frac{\phi_x(t)}{\sin\frac{t}{2}} \{J_n(t) + K_n(t) + L_n(t)\}\, dt\right|$$

Now

$$\begin{aligned} |I_n(x) - I_n(y)| &= \left|\frac{1}{\pi(n+1)} \sum_{k=0}^{n} \int_0^{\pi} \frac{\phi_x(t) - \phi_y(t)}{\sin\frac{t}{2}} \{J_n(t) + K_n(t) + L_n(t)\}\, dt\right| \\ &= \left|\frac{1}{\pi(n+1)} \sum_{k=0}^{n} \int_0^{\pi} \frac{F(t)}{\sin^t /2} \{J_n(t) + K_n(t) + L_n(t)\}\, dt\right| \\ &= |I_1 + I_2 + I_3| \end{aligned} \tag{31}$$

Now

$$|I_1| = \left|\frac{1}{\pi(n+1)} \sum_{k=0}^{n} \int_0^{\pi} \frac{F(t)}{\sin\frac{t}{2}} \{J_n(t)\}\, dt\right|$$

$$|I_1| \leq \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \int_0^{\pi} \frac{|F(t)|}{\sin\frac{t}{2}} \sqrt{\frac{c}{\pi k}} \left\{1 + 2\sum_{r=1}^{\infty} \exp\left(\frac{-cr^2}{k}\right) \cos rt\right\} \sin\left(k+\frac{1}{2}\right) tdt \tag{32}$$

$$= \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \int_0^{\pi} \frac{|F(t)|}{\sin\frac{t}{2}} \sqrt{\frac{c}{\pi k}} \sqrt{\frac{\pi k}{c}} \left\{\exp\left(\frac{-kt^2}{4c}\right) + O\left(\exp\left(\frac{-k\pi}{4c}\right)\right)\right\} \sin\left(k+\frac{1}{2}\right) t$$

$$= \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \int_0^{\pi} \frac{|F(t)|}{\sin\frac{t}{2}} \left\{\exp\left(\frac{-kt^2}{4c}\right) + O\left(\exp\left(\frac{-k\pi}{4c}\right)\right)\right\} \sin\left(k+\frac{1}{2}\right) t \tag{33}$$

$$= I_{1.1} + I_{1.2} \tag{34}$$

Now

$$\begin{aligned} I_{1.1} &= \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \int_0^{\pi} \frac{|F(t)|}{\sin\frac{t}{2}} \exp\left(\frac{-kt^2}{4c}\right) \sin\left(k+\frac{1}{2}\right) t \\ &= O\left(\exp\left(\frac{-nt^2}{4c}\right)\right) \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \int_0^{\pi} \frac{|F(t)|}{\sin\frac{t}{2}} \sin\left(k+\frac{1}{2}\right) t \\ &= O(1) \left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}\right) |F(t)| M_n(t) dt \\ &= I_{1.11} + I_{1.12} \end{aligned} \tag{35}$$

Now

$$\begin{aligned} I_{1.11} &= \int_0^{\pi/n+1} |F(t)| M_n(t) dt \\ &= O(n+1) \int_0^{\pi/n+1} |t|^{\alpha} dt \\ &= O(n+1)^{-\alpha} \end{aligned} \tag{36}$$

Now

$$\begin{aligned} I_{1.12} &= \int_{\pi/n+1}^{\pi} |F(t)| M_n(t) dt \\ &= \int_{\pi/n+1}^{\pi} |t|^{\alpha} O\left(\frac{1}{t}\right) dt \\ &= \int_{\pi/n+1}^{\pi} |t|^{\alpha-1} dt \\ &= O(n+1)^{-\alpha} \end{aligned} \tag{37}$$

Now

$$\begin{aligned} I_{1.2} &= \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \int_0^{\pi} \frac{|F(t)|}{\sin\frac{t}{2}} \left\{ O\left(\exp\left(\frac{-k\pi}{4c}\right)\right) \right\} \sin\left(k+\frac{1}{2}\right) t dt \\ &= O\left(\exp\left(\frac{-n\pi}{4c}\right)\right) \int_0^{\pi} |F(t)| M_n(t) dt \\ &= O(1) \left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi} \right) |F(t)| M_n(t) dt \\ &= O(n+1)^{-\alpha} \end{aligned} \tag{38}$$

Then

$$I_1 = O(n+1)^{-\alpha} \tag{39}$$

Now

$$|I_2| = \left| \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \int_0^{\pi} \frac{F(t)}{\sin\frac{t}{2}} K_n(t) dt \right|$$

$$\begin{aligned} |I_2| &\leq -\frac{2}{\pi(n+1)} \sum_{k=0}^{n} \int_0^{\pi} \frac{|F(t)|}{\sin\frac{t}{2}} \sqrt{\frac{c}{\pi k}} \sum_{r=k+1}^{\infty} \exp\left(\frac{-cr^2}{k}\right) \cos rt \sin\left(k+\frac{1}{2}\right) t dt \\ &= -\frac{2}{\pi(n+1)} \sum_{k=0}^{n} \sqrt{\frac{c}{\pi k}} \int_0^{\pi} \frac{|F(t)|}{\sin\frac{t}{2}} \sin\left(k+\frac{1}{2}\right) t . O\left(\frac{\exp(-ck)}{t}\right) dt \quad \text{using inequality (14)} \qquad (40) \\ &= O\left(n^{-1/2} \exp(-cn)\right) \int_0^{\pi} \frac{|F(t)|}{t} M_n(t) dt \\ &= O(1) \left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi} \right) \frac{|F(t)|}{t} M_n(t) dt \\ &= O(n+1)^{-\alpha} \end{aligned} \tag{41}$$

Now

$$\begin{aligned} |I_3| &\leq \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \int_0^{\pi} \frac{|F(t)|}{\sin\frac{t}{2}} L_n(t) dt \\ &= \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \sqrt{\frac{c}{\pi k}} \int_0^{\pi} \frac{|F(t)|}{\sin\frac{t}{2}} \sum_{r=k+1}^{\infty} \exp\left(\frac{-cr^2}{k}\right) \sin\left(k+r+\frac{1}{2}\right) t dt \\ &= \frac{1}{\pi(n+1)} \sum_{k=0}^{n} \sqrt{\frac{c}{\pi k}} \int_0^{\pi} \frac{|F(t)|}{\sin\frac{t}{2}} \cdot \frac{kt}{2c} \exp(-ck) dt \quad \text{using inequality (13)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{O\left(n^{-1/2}\exp(-cn)\right)}{\pi(n+1)} \sum_{k=0}^{n} \sqrt{\frac{c}{\pi k}} \int_0^{\pi} \frac{|F(t)|}{\sin\frac{t}{2}} \cdot \frac{t}{2c} \exp(-ck) dt \\
&= O\left(\frac{\exp(-cn)}{\sqrt{n}}\right) \int_0^{\pi} |F(t)| dt \quad \text{since} \quad |\sin\frac{t}{2}| \le \frac{t}{2} \\
&= O(1) \left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}\right) |F(t)| dt \\
&= O(n+1)^{-\alpha} \qquad (42)
\end{aligned}
$$

Now using

$$
\begin{aligned}
|F(t)| &= \left|\Phi_x(t) - \Phi_y(t)\right| \\
&= O\left(|x-y|^{\alpha}\right) \qquad (43) \\
I_{1.11} &= \int_0^{\pi/n+1} |F(t)| M_n(t) dt \\
&= \int_0^{\pi/n+1} O\left(|x-y|^{\alpha}\right) M_n(t) dt \\
&= O\left(|x-y|^{\alpha}\right) O(n+1) \\
&= O\left(|x-y|^{\alpha}\right) \qquad (44) \\
I_{1.12} &= \int_{\pi/n+1}^{\pi} \cdot O\left(|x-y|^{\alpha}\right) M_n(t) dt \\
&= O\left(|x-y|^{\alpha}\right) \int_{\pi/n+1}^{\pi} \cdot O\left(\frac{1}{t}\right) dt \\
&= O\left(|x-y|^{\alpha}\right) \log(n+1) \qquad (45) \\
I_{1.1} &= O\left(|x-y|^{\alpha}\right) + O\left(|x-y|^{\alpha}\right) \log(n+1) \\
&= O\left(|x-y|^{\alpha}\right) \log(n+1) \qquad (46)
\end{aligned}
$$

Similarly

$$
I_{1,2} = O\left(|x-y|^{\alpha}\right) \log(n+1) \qquad (47)
$$

Then

$$
\begin{aligned}
I_1 &= O\left(|x-y|^{\alpha}\right) \log(n+1) \qquad (48) \\
I_2 &= O(1) \left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}\right) \frac{|F(t)|}{t} M_n(t) dt \\
&= \int_0^{\pi/n+1} O\left(|x-y|^{\alpha}\right) O(n+1) \frac{1}{t} + \int_{\pi/n+1}^{\pi} .O\left(|x-y|^{\alpha}\right) \frac{1}{t} O\left(\frac{1}{t}\right) \\
&= \left(\int_0^{\pi/n+1} O\left(|x-y|^{\alpha}\right) O(n+1) \frac{1}{t} + \int_{\pi/n+1}^{\pi} \cdot O\left(|x-y|^{\alpha}\right) \frac{1}{t} O\left(\frac{1}{t}\right)\right) dt \\
&= O\left(|x-y|^{\alpha}\right) \log(n+1) + O\left(|x-y|^{\alpha}\right) \int_{\pi/n+1}^{\pi} \cdot t^{-2} dt \\
&= O\left(|x-y|^{\alpha}\right) \log(n+1) + O\left(|x-y|^{\alpha}\right) \left[\frac{t^{-1}}{-1}\right]_{\pi/(n+1)}^{\pi}
\end{aligned}
$$

$$= O\left(|x-y|^{\alpha}\right)\log(n+1) \tag{49}$$

Now

$$\begin{aligned} I_3 &= \left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}\right)|F(t)|dt \\ &= \int_0^{\pi/(n+1)} O\left(|x-y|^{\alpha}\right)dt + \int_{\pi/(n+1)}^{\pi} O\left(|x-y|^{\alpha}\right)dt \\ &= O\left(|x-y|^{\alpha}\right) \end{aligned} \tag{50}$$

Now for $k = 1, 2, 3$ and for $0 \leq \beta < \alpha \leq 1$. We observe that

$$|I_k| = |I_k|^{1-\beta/\alpha}\,|I_k|^{\beta/\alpha} \tag{51}$$

By using (39) and (48) in the first and second factor on the right of the above identify (51) for $k = 1$

$$|I_1| = O\left\{|x-y|^{\beta}(n+1)^{\beta-\alpha}\right\} \tag{52}$$

Again (41) and (49) in the first and second factor on the right of the identify (51) for $k = 2$ we have

$$|I_2| = O\left\{|x-y|^{\beta}(n+1)^{\beta-\alpha}\log(n+1)\right\} \tag{53}$$

By using (42) and (50) in the first and second factor on the right of the identify (51) for $k = 3$ we have

$$|I_3| = O\left\{x-y|^{\beta}\,(n+1)^{\beta-\alpha}\log(n+1)\right\} \tag{54}$$

Thus from (52), (53) and (54) we get

$$\begin{aligned} \sup_{\substack{x,y\\ x\neq y}}\left|\Delta^{\beta} I_n(x,y)\right| &= \sup_{\substack{x,y\\ x\neq y}} \frac{|I_n(x)-l_n(y)|}{(x-y)^{\beta}} \\ &= O\left\{(n+1)^{\beta-\alpha}\log(n+1)\right\} \end{aligned} \tag{55}$$

Now $f \in H_{\alpha} \Rightarrow \emptyset_x(t) = O\left(t^{\alpha}\right)$. Proceeding as above we obtain

$$\begin{aligned} \|I_n\|_c &= \sup_{-\pi\leq x\leq\pi} \|C_n^1 e_n^c - f(x)\| \\ &= O\left\{(n+1)^{-\alpha}\log(n+1)\right\} \end{aligned} \tag{56}$$

Combining (55) and (56) and using (51), we get

$$\|C_n^1 e_n^c - f(x)\|_{\beta} = 0\left\{(n+1)^{\beta-\alpha}\log(n+1)\right\}$$

□

Corollary 5.5. *If $f \in Lip\ \alpha$, when $0 < \alpha \leq 1$. Then for $n > 1$ So.*

$$\|C_n^1 e_n^c - f(x)\|_\beta = O\left\{(n)^{-\alpha}\log n\right\}$$

We put $\beta = 0$ then Theorem 4.2 is particular case of main theorem.

6. Conclusion

The summability method (e, c) includes method of summability like Borel, (E, 1), (E, q), F(a, q) and $[F, d_n]$ then by using the result of main theorem we can derive more generalizing result and also the result of [13] can be derived directly.

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