International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **11(4)**(2023), 175–180
Available Online: http://ijmaa.in

Some Results of Generalized Reverse Biderivations on Prime Rings

V. S. V. Krishna Murty[1,*], C. Jaya Subba Reddy[1]

[1]*Department of Mathematics, SV University, Tirupathi, Andhra Pradesh, India*

Abstract

Let R be a ring with center $Z(R)$. In the present paper, we obtain some results in connection with symmetric generalized reverse biderivations on prime ring R.

Keywords: Prime ring; Generalized biderivation; Generalized Reverse Biderivation; Bimultiplier; Reverse Bimultiplier.

2020 Mathematics Subject Classification: 16N60, 13N15.

1. Introduction

The concept of bi-derivation was introduced by Maksa in [8]. It is shown in [9] that bi-derivations are related to general solution of some functional equations. Some results concerning symmetric bi-derivations in prime rings can be found in [10,11,13]. Some theorems on symmetric generalized biderivations of prime ring are proved in [1,7]. Daif, Haetinger and Tammam El-Sayiad showed that any reverse biderivation on R prime ring is also a biderivation in [12]. Lateron, Many results have been found concerning reverse biderivations [2,6]. C.Jaya Subba Reddy et. al. [3–5,14], have studied some results concerning reverse biderivations on prime and semi prime rings. In the present paper, we obtain some results in connection with the symmetric generalized reverse biderivations on prime ring R.

2. Preliminaries

Let R be a ring and $Z(R)$ be its center. Remember that R is prime if $x_1 R x_2 = \{0\}$ implies $x_1 = 0$ or $x_2 = 0$ for any $x_1, x_2 \in R$. Also, R is semi-prime if $x_1 R x_1 = \{0\}$ implies $x_1 = 0$ for any $x_1 \in R$. For $x_1, x_2 \in R$, the notation $[x_1, x_2]$ denotes the commutator $x_1 x_2 - x_2 x_1$. Recall that a map D from $R \times R$ into R is termed symmetric if $D(x_1, x_2) = D(x_2, x_1)$ for all $x_1, x_2 \in R$. A symmetric biadditive map

*Corresponding author (krishnamurty.vadrevu@gmail.com (Research Scholar))

(symmetric in both arguments) D from $R \times R$ into R is termed a biderivation if

$$D(x_1x_2, x_3) = D(x_1, x_3)x_2 + x_1D(x_2, x_3) \quad \text{and} \quad D(x_1, x_2x_3) = D(x_1, x_2)x_3 + x_2D(x_1, x_3)$$

for all $x_1, x_2, x_3 \in R$. Several authors have studied biderivations and investigated properties of biderivations. During the course of time, researchers have introduced the definitions of reverse biderivation, left (similarly right) biderivation, and Jordan left (similarly right) biderivation. A biadditive map D from $R \times R$ into R is termed a symmetric reverse biderivation if

$$D(x_1x_2, x_3) = D(x_2, x_3)x_1 + x_2D(x_1, x_3) \quad \text{and} \quad D(x_1, x_2x_3) = D(x_1, x_3)x_2 + x_3D(x_1, x_2)$$

for all $x_1, x_2, x_3 \in R$. A biadditive map $\delta : R \times R$ into R is termed a symmetric generalized biderivation if there is a biderivation $D : R \times R$ into R such that

$$\delta(x_1x_2, x_3) = \delta(x_1, x_3)x_2 + x_1D(x_2, x_3) \quad \text{and} \quad \delta(x_1, x_2x_3) = \delta(x_1, x_2)x_3 + x_2D(x_1, x_3)$$

for all $x_1, x_2, x_3 \in R$. A biadditive map $\delta : R \times R$ into R is termed a symmetric generalized reverse biderivation if there is a reverse biderivation $D : R \times R$ into R such that

$$\delta(x_1x_2, x_3) = \delta(x_2, x_3)x_1 + x_2D(x_1, x_3) \quad \text{and} \quad \delta(x_1, x_2x_3) = \delta(x_1, x_3)x_2 + x_3D(x_1, x_2)$$

for all $x_1, x_2, x_3 \in R$. A mapping $f : R \to R$ defined by $f(x) = D(x, x)$ for all $x \in R$, where $D : R \times R \to R$ is a symmetric and biadditive (i.e., additive in both arguments) mapping, is called the trace of D. An additive mapping $h : R \to R$ is called a left (resp. right) multiplier of R if $h(x_1x_2) = h(x_1)x_2$ (resp. $h(x_1x_2) = x_1h(x_2)$) for all $x_1, x_2 \in R$. A biadditive mapping $D : R \times R \to R$ is said to be a left (resp. right) bi-multiplier of R if $D(x_1, x_2x_3) = D(x_1, x_2)x_3$ (resp. $D(x_1x_2, x_3) = x_1D(x_2, x_3)$) for all $x_1, x_2, x_3 \in R$. A biadditive mapping $D : R \times R \to R$ is said to be a reverse left (resp. right) bi-multiplier of R if $D(x_1, x_2x_3) = D(x_1, x_3)x_2$ (resp. $D(x_1x_2, x_3) = x_2D(x_1, x_3)$) holds for all $x_1, x_2, x_3 \in \mathbb{R}$.

3. Main Results

Theorem 3.1. *Let R be a prime ring of characteristic not two. U be an ideal of R and $n \geq 1$, a fixed integer. Consider $\delta_1, \delta_2, \delta_3, \ldots, \delta_n : R \times R \to R$ are symmetric generalized reverse biderivations with associated reverse biderivations $D_1, D_2, D_3, \ldots, D_n : R \times R \to R$ respectively such that*

$$\delta_1(x_1, x_1)\delta_2(y_1, y_1)\delta_3(z_1, z_1)\ldots\delta_n(u_1, u_1) = 0, \quad \forall x_1, y_1, z_1, \ldots, u_1 \in U.$$

Then one of the following holds:

1. $\delta_1(x_1, x_1) = 0, \forall\ x_1 \in U.$

2. *All* $\delta_{n+1}(u_1, u_1)$ *act as a reverse left bimultiplier on R, for all* $n \geq 1$.

Proof. We shall use Mathematical Induction to prove this theorem. Let $n = 1$, then we get $\delta_1(x_1, x_1) = 0$ and hence proved. Let $n = 1, 2$, then we get

$$\delta_1(x_1, x_1)\delta_2(y_1, y_1) = 0 \ \forall \ x_1, y_1 \in U \tag{1}$$

By linearizing in y_1, we get, $\delta_1(x_1, x_1)(\delta_2(y_1, y_1) + \delta_2(y_1, z_1) + \delta_2(z_1, y_1) + \delta_2(z_1, z_1)) = 0$. Using (1) and the fact that Char $R \neq 2$, we get

$$\delta_1(x_1, x_1)\delta_2(y_1, z_1) = 0 \ \forall \ x_1, y_1, z_1 \in U. \tag{2}$$

Replace z_1 by rz_1 in the above equation and using (2), we get, $\delta_1(x_1, x_1)z_1D_2(y_1, r) = 0 \ \forall \ x_1, y_1, z_1 \in U$ and $r \in R$. By the primeness of R, we get either $\delta_1(x_1, x_1) = 0$ or $D_2(y_1, r) = 0$. If $\delta_1(x_1, x_1) = 0$, then we are done. Suppose

$$D_2(y_1, r) = 0 \ \forall \ y_1 \in U \text{ and } r \in R \tag{3}$$

Replacing r by rx_1 in $\delta_2(y_1, r)$ and using (3), we get $\delta_2(y_1, rx_1) = \delta_2(y_1, x_1)r$. Therefore δ_2 acts as a left reverse bimultiplier. Suppose $n = 1, 2, 3$

$$\delta_1(x_1, x_1)\delta_2(y_1, y_1)\delta_3(z_1, z_1) = 0 \ \forall \ x_1, y_1, z_1 \in U \tag{4}$$

Linearizing in z_1 and using (4), we get

$$\begin{aligned} 2\delta_1(x_1, x_1)\delta_2(y_1, y_1)\delta_3(z_1, u_1) &= 0 \\ \delta_1(x_1, x_1)\delta_2(y_1, y_1)\delta_3(z_1, u_1) &= 0 \quad (\text{Since Char } R \neq 2) \end{aligned} \tag{5}$$

Replacing u_1 by su_1 in (5) and using (5), we get $\delta_1(x_1, x_1)\delta_2(y_1, y_1)u_1D_3(z_1, s) = 0$, $\forall \ x_1, y_1, z_1, u_1 \in U$ and $s \in R$. Using the primeness of R, we have $\delta_1(x_1, x_1)\delta_2(y_1, y_1) = 0$ or $D_3(z_1, s) = 0$. If $\delta_1(x_1, x_1)\delta_2(y_1, y_1) = 0, \forall \ x_1, y_1 \in U$, then the result is evident. Suppose that

$$D_3(z_1, s) = 0, \forall z_1 \in U, s \in R \tag{6}$$

Replacing s by st in $\delta_3(z_1, s)$ using (6), we get, $\delta_3(z_1, st) = \delta_3(z_1, t)s$. Therefore δ_3 acts as a reverse left bimultiplier. Let the statement be true for $n = k$. That means if $\delta_1(x_1, x_1)\delta_2(y_1, y_1)\ldots\delta_k(u_1, u_1) = 0$, then

1. $\delta_1(x_1, x_1) = 0, \quad \forall x_1 \in U$

2. $\delta_{k+1}(u_1, u_1)$ act as a reverse left bimultiplier on R, for all $n \geq 1$

Let us prove for $n = k+1$. Suppose

$$\delta_1(x_1,x_1)\delta_2(y_1,y_1)\delta_3(z_1,z_1)\ldots\delta_k(u_1,u_1)\delta_{k+1}(v_1,v_1) = 0 \quad \forall \;\; x_1,y_1,z_1,v_1,u_1 \in U \tag{7}$$

Linearizing (7) in v_1 and using the fact that $Char \;\; R \neq 2$, we get

$$\delta_1(x_1,x_1)\delta_2(y_1,y_1)\delta_3(z_1,z_1)\ldots\delta_k(u_1,u_1)\delta_{k+1}(v_1,w_1) = 0, \quad \forall x_1,y_1,z_1,\ldots,u_1,v_1,w_1 \in U \tag{8}$$

Replacing w_1 by tw_1 and using (8), we get, $\delta_1(x_1,x_1)\delta_2(y_1,y_1)\delta_3(z_1,z_1)\ldots\delta_k(u_1,u_1)w_1D_{k+1}(v_1,t) = 0$, $\forall \;\; x_1,y_1,z_1,\ldots,u_1,v_1,w_1 \in U$ and $t \in R$. By primeness of R, we have, $\delta_1(x_1,x_1)\delta_2(y_1,y_1)\delta_3(z_1,z_1)$ $\ldots\delta_k(u_1,u_1) = 0$ or $D_{k+1}(v_1,t) = 0$. Suppose that $\delta_1(x_1,x_1)\delta_2(y_1,y_1)\delta_3(z_1,z_1)\ldots\delta_k(u_1,u_1) = 0$, then the result is evident. Suppose that

$$D_{k+1}(v_1,t) = 0, \quad \forall \;\; v_1 \in U \text{ and } t \in R \tag{9}$$

Replacing t by tp in $\delta_{k+1}(v_1,t)$ and using (9), then we get, $\delta_{k+1}(v_1,tp) = \delta_{k+1}(v_1,p)t$. Therefore δ_{k+1} acts as a reverse left bimultiplier. This completes the proof of the theorem. □

Theorem 3.2. *Let R be a prime ring of characteristic not two and $k \geq 1$, a fixed positive integer. Consider symmetric reverse biderivations $D_n : R \times R \to R$ for $n = 1,2,3,\ldots,k$ such that*

$$D_1(x_1,x_1)D_2(y_1,y_1), D_3(z_1,z_1)\ldots D_n(u_1,u_1) = 0$$

for all $x_1,y_1,z_1\ldots,u_1 \in R$. Then it follows that $D_n = 0$ for some n.

Proof. We prove the theorem by Mathematical Induction. For $n = 1$, we have

$$D_1(x_1,x_1) = 0, \;\; \forall x_1 \in \mathbb{R} \;\text{ and hence }\; D_1 = 0 \tag{10}$$

For $n = 2$, we have

$$D_1(x_1,x_1)D_2(y_1,y_1) = 0 \;\; \forall \;\; x_1,y_1 \in \mathbb{R} \tag{11}$$

By linearizing y_1 and using the fact that Char($\mathbb{R}$) $\neq 2$, we get

$$D_1(x_1,x_1)D_2(y_1,z_1) = 0 \;\; \forall \;\; x_1,y_1,z_1 \in \mathbb{R} \tag{12}$$

Put $z_1 = uz_1$ in (12) and using (12)

$$D_1(x_1,x_1)z_1D_2(y_1,u) = 0 \;\; \forall \;\; x_1,y_1,z_1,u \in R$$

In particular, $D_1(x_1,x_1)z_1D_2(x_1,x_1) = 0$ which implies $D_1 = 0$ or $D_2 = 0$. For $n = 3$

$$D_1(x_1,x_1)D_2(y_1,y_1)D_3(z_1,z_1) = 0 \ \forall \ x_1,y_1,z_1 \in \mathbb{R} \tag{13}$$

By linearizing (13) in z_1 and using the fact that Char($\mathbb{R}$) $\neq 2$, we get

$$D_1(x_1,x_1)D_2(y_1,y_1)D_3(z_1,w) = 0 \ \forall \ x_1,y_1,z_1,w \in \mathbb{R} \tag{14}$$

Replacing z_1 by uz_1 and using (14)

$$D_1(x_1,x_1)D_2(y_1,y_1)z_1D_3(u,w) = 0 \ \forall \ x_1,y_1,z_1,u,w \in \mathbb{R}$$

By the Primeness of $\mathbb{R}$, we get either $D_1(x_1,x_1)D_2(y_1,y_1) = 0$ or $D_3(u,w) = 0$. Suppose $D_1(x_1,x_1)D_2(y_1,y_1) = 0$, then by the previous case, we have either $D_1 = 0$ or $D_2 = 0$. Suppose $D_3(u,w) = 0$, which means $D_3 = 0$. Hence the theorem is proved for $n = 3$. Let the theorem be true for $n = k$. If we consider $D_1(x_1,x_1)D_2(y_1,y_1)D_3(z_1,z_1)\ldots D_k(u_1,u_1) = 0$, then $D_1 = 0$ or $D_2 = 0$ or $D_3 = 0 \ldots D_k = 0$, $\forall \ x_1,y_1,z_1,\ldots,u_1 \in \mathbb{R}$. For $n = k+1$, by hypothesis, we have

$$D_1(x_1,x_1)D_2(y_1,y_1)D_3(z_1,z_1)\ldots D_k(u_1,u_1)D_{k+1}(v_1,v_1) = 0 \ \forall \ x_1,y_1,z_1,\ldots,u_1,v_1 \in \mathbb{R} \tag{15}$$

By linearization in v_1 and using the fact that $Char(\mathbb{R}) \neq 2$

$$D_1(x_1,x_1)D_2(y_1,y_1)D_3(z_1,z_1)\ldots D_k(u_1,u_1)D_{k+1}(v_1,w_1) = 0 \ \forall \ x_1,y_1,z_1,\ldots,u_1,v_1,w_1 \in \mathbb{R} \tag{16}$$

Replacing v_1 by v_1t and using (16), we get

$$D_1(x_1,x_1)D_2(y_1,y_1)D_3(z_1,z_1)\ldots D_k(u_1,u_1)tD_{k+1}(v_1,w_1) = 0 \ \forall \ x_1,y_1,z_1,\ldots,u_1,v_1,w_1,t \in \mathbb{R}$$

By the primeness of $\mathbb{R}$, we have either $D_1(x_1,x_1)D_2(y_1,y_1)D_3(z_1,z_1)\ldots D_k(u_1,u_1) = 0$ or $D_{k+1}(v_1,w_1) = 0$. Suppose $D_1(x_1,x_1)D_2(y_1,y_1)D_3(z_1,z_1)\ldots D_k(u_1,u_1) = 0$, then by the previous discussion, we have $D_1 = 0$ or $D_2 = 0$ or $\ldots$ or $D_k = 0$. Suppose $D_{k+1}(v_1,w_1) = 0$, and in particular $D_{k+1}(v_1,v_1) = 0$ then $D_{k+1} = 0$. Hence, the theorem is proved. □

References

[1] A. Ali, V. D. Filippis and F. Shujat, *Results concerning symmetric generalized biderivations Of Prime and semiprime rings*, Matematiqki Vesnik, 66(4)(2014), 410-417.

[2] B. Albayrak and N. AydinK, *Reverse and Jordan (α,β)-biderivation on Prime and Semiprime Rings*, Adıyaman University Journal of Science, 9(1)(2019), 149–164.

[3] C. J. S. Reddy, A. S. K. Kumar and B. R. Reddy, *Ideals and symmetric reverse bi-derivations of prime and semiprime rings*, Malaya Journal of Matematik, 6(1)(2018), 291–293.

[4] C. J. S. Reddy, A. S. Kumar and B. R. Reddy, *Results of symmetric reverse bi-derivations on prime rings*, Annals of Pure and Applied Mathematics, 16(1)(2018), 1–6.

[5] C. J. S. Reddy and R. Naik, *Symmetric reverse bi-derivations on prime rings*, Research Journal of Pharmacy and Technology, 9(9)(2016), 1496–1500.

[6] E. K. Sogutcu and O. Golbaşı, *some results on lie ideals with symmetric reverse bi-derivations in semiprime rings*, Facta Universitatis, Series: Mathematics and Informatics, (2021), 309–319.

[7] F. Shujat and A. Fallatah, *On the iterations of generalized bi-derivation on prime ring*, Boletim da Sociedade Paranaense de Matemática, 40(2022), 1–5.

[8] G. Maksa, *A remark on symmetric biadditive functions having nonnegative diagonalization*, Glasnik Math, 15(35)(1980), 279–282.

[9] G. Maksa, *On the trace of symmetric bi-derivations*, CR Math. Rep. Acad. Sci. Canada, 9(9)1987), 303–307.

[10] J. Vukman, *Two results concerning symmetric bi-derivations on prime rings*, Aeq. Math., 40(1)(1990), 181–189.

[11] J. Vukman, *Symmetric bi-derivations on prime and semi-prime rings*, Aeq. Math., 38(2–3)(1989), 245–254.

[12] M. N. Daif, M. S. T. El-Sayiad and C. Haetinger, *Reverse, Jordan and left biderivations*, Oriental Journal Of Mathematics, 2(2)(2010), 65–81.

[13] N. Parveen, *Product of Traces of Symmetric Bi-Derivations in Rings*, Palestine Journal of Mathematics, 11(1)2022.

[14] Sk. Haseena and C. J. S. Reddy, *symmetric generalized reverse* $(\alpha, 1)$ *biderivations in rings*, JP Journal of Algebra, Number Theory and Applications, 58(2022), 37–43.