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Bernoulli Numbers and the Exact Values of Riemann Zeta Function at Positive Even Integers

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Abstract

In this note we present a self-contained discussion of the Bernoulli numbers and apply the Summation Theorem to show an alternative method to verifying the identity for Riemann zeta function at positive even integers.

Keywords: Bernoulli numbers; residue calculus; Summation Theorem; Riemann zeta function.
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1. Introduction

The formula known as the Riemann zeta function at positive even integers is

$$\zeta(2q) = \frac{(-1)^{q-1} B_{2q}\ \pi^{2q}\ 2^{(2q-1)}}{(2q)!}, \text{ for any positive integer q and } \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The complete derivation of the formula is omitted in many undergraduate textbooks and suggested as projects; see Rudin [1], and Kumanduri and Romero [2]. Yet, this exact value is well known and can be derived using different methods [3–6]. For example, Alladi and Defant [3] used Parseval's identity for the Fourier coefficients of x^q, while Tuo et. al. [4] used the hyperbolic cotangent function to find the value. In this note, we discuss the Bernoulli numbers and alternatively apply the Summation Theorem to derive the exact formula. The approach is simple, self-contained, and can be discussed in undergraduate courses on Elementary Analysis. First, we present a discussion of the Bernoulli numbers.

2. Bernoulli Numbers

The Bernoulli numbers were discovered by Jacob Bernoulli (1654-1705), and the numbers have applications in many branches of Mathematics and the sciences. The first published sophisticated

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computer program was about Bernoulli numbers (see, Kim and Toole [7]). The Bernoulli numbers B_n are a sequence of rational numbers where the odd-numbered $B's$ vanish, except for $B_1 = -\frac{1}{2}$ or $B_1 = \frac{1}{2}$ which are explicitly defined by the following respective Generating Functions:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \text{ with } B_1 = -\frac{1}{2} \text{ or } \frac{z}{1 - e^{-z}} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \text{ with } B_1 = \frac{1}{2}$$

Many literatures have discussed the Bernoulli numbers; see [8] and [9]. Apostol [8] credits Leonhard Euler (1707 - 1783) for discovering that the Bernoulli numbers occur as coefficients in the Generating Function given by the following power series expansions:

$$f(z) = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \text{ for } |z| < 2\pi \text{ and } B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} j^n \text{ for } n \geq 0, \tag{1}$$

with limiting value $B_0 = 1, f(0) \doteq \lim_{z \to 0} \frac{z}{e^z - 1} = 1$; and $B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, \ldots$. One way to obtain the formula for B_n is by considering the Taylor series expansion of $ln(x)$ at $x = 1$ [6].

$$\ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(x-1)^k}{k};$$

so that

$$\ln(1-x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\left([1-x]-1\right)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(-x)^k}{k} = -\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

Then observe that $\ln\left[1 - \left(1 - e^t\right)\right] = \ln\left(e^t\right) = t$, so

$$t = ln\left[1 - \left(1 - e^t\right)\right] = -\sum_{k=1}^{\infty} \frac{\left(1 - e^t\right)^k}{k}, \; and \; x = -\sum_{k=1}^{\infty} \frac{(1 - e^x)^k}{k} \quad \text{for } |1 - e^x| < 1.$$

Therefore,

$$f(x) = \frac{x}{e^x - 1} = -\frac{1}{e^x - 1} \sum_{k=1}^{\infty} \frac{(1 - e^x)^k}{k} = \sum_{k=1}^{\infty} \frac{(1 - e^x)^k}{k\,(1 - e^x)} = \sum_{k=1}^{\infty} \frac{(1 - e^x)^{k-1}}{k} = \sum_{k=0}^{\infty} \frac{(1 - e^x)^k}{k+1}.$$

By considering the Taylor series expansion for $f(x) = \frac{x}{e^x - 1}$ at $x = 0$, and matching results with the k^{th} term in the resulting summation above

$$f(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n;$$

where

$$\begin{aligned} B_n &= f^{(n)}(0) = \frac{d^n}{dx^n}\left(\frac{x}{e^x - 1}\right)\bigg|_{x=0} = \frac{d^n}{dx^n}\left(\sum_{k=0}^{\infty} \frac{(1 - e^x)^k}{k+1}\right)\bigg|_{x=0} = \frac{d^n}{dx^n}\left[\sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (-1)^j e^{jx}\right]\Bigg|_{x=0} \\ &= \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (-1)^j \frac{d^n}{dx^n}\left(e^{jx}\right)\bigg|_{x=0} = \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (-1)^j j^n .1 = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (-1)^j j^n; \end{aligned}$$

and $f^{(n)}(0) = 0\ for\ k \geq n+1$. Hence

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j j^n \right] x^n \text{ and } B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n\ for\ n \geq 0.$$

3. Example (Calculating a Few Terms of Bernoulli Numbers Using the Formula for B_n)

$$B_1 = \sum_{k=0}^{1} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^1 = \frac{1}{0+1} \sum_{j=0}^{0} (-1)^j \binom{0}{j} j^1 + \frac{1}{1+1} \sum_{j=0}^{1} (-1)^j \binom{1}{j} j^1$$
$$= 0 + \frac{1}{2}.(0-1) = -\frac{1}{2}$$
$$B_3 = \frac{1}{0+1} \sum_{j=0}^{0} (-1)^j \binom{0}{j} j^3 + \frac{1}{1+1} \sum_{j=0}^{1} (-1)^j \binom{1}{j} j^3 + \frac{1}{2+1} \sum_{j=0}^{2} (-1)^j \binom{2}{j} j^3 + \frac{1}{3+1} \sum_{j=0}^{3} (-1)^j \binom{3}{j} j^3$$
$$= 0$$

4. Theorem (Summation Theorem)

Let $H(z) = \frac{P(z)}{Q(z)}$ be analytic in the complex plane $\mathbb{C}$ except for some finite set of poles $z_1, z_2, \ldots, z_m$ that are not integers and furthermore that $H(z)$ is a rational function with $\text{degQ}(z) - degP(z) \geq 2$. Then, we have the summation formulas:

$$\sum_{n=-\infty}^{\infty} H(n) = -\sum_{j=1}^{m} \text{Res}\left[\pi H(z) \cot(\pi z)\,; z = z_j\right],$$
$$\sum_{n=-\infty}^{\infty} (-1)^n H(n) = \sum_{j=1}^{m} Res[\pi H(z) \csc(\pi z)\,; z = z_j]$$

A statement and discussion of the summation theorem is found in many analysis textbooks; see [10].

5. Application of Summation Theorem to Evaluating $\zeta(2q)$ for $q \in Z^+$

We will use residue calculus to show that,

$$\zeta(2q) = \frac{(-1)^{q-1} B_{2q}\, \pi^{2q}\, 2^{(2q-1)}}{(2q)!} \quad \text{where } q \in \mathbb{Z}^+, \quad \text{and } \frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}$$

Let $H(z) = \frac{1}{z^{2q}}$. We observe that $H(z)$ has a pole of order $2q$ at $z = 0$ and $\cot(z)$ also has simple pole at $z = 0$. Hence we generate the Laurent series for $\pi\, H(z) \cot(\pi z)$ to help determine the required residue. We will first show that,

$$\pi z \cot(\pi z) = \sum_{k=0}^{\infty} B_{2k} \frac{(-4)^k \pi^{2k} z^{2k}}{(2k)!} + \frac{2\pi z}{2}.$$

Consider

$$\frac{z}{e^z-1}+\frac{z}{2}=z\left(\frac{1}{e^z-1}+\frac{1}{2}\right)=z\left(\frac{2+e^z-1}{2\left(e^z-1\right)}\right)=\frac{z}{2}\left(\frac{e^z+1}{e^z-1}\right)=\frac{z}{2}\left(\frac{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}\right)=\frac{z}{2}\coth\left(\frac{z}{2}\right).$$

We can substitute $z=2t$ into $\frac{z}{2}\coth\left(\frac{z}{2}\right)=\frac{z}{e^z-1}+\frac{z}{2}$ and observe that $t\coth(t)$ is even function so every odd-number coefficient in the power series expansion of $t\coth(t)$ must be 0; then apply (1) above to obtain:

$$t\coth(t)=\frac{2t}{e^{2t}-1}+\frac{2t}{2}=\sum_{k=0}^{\infty}B_{2k}\frac{(2t)^{2k}}{(2k)!}+\frac{2t}{2},\quad \text{since } B_3=B_5=B_7=\ldots=0.$$

Also

$$z\cot(z)=iz\coth(iz)=\left(\sum_{k=0}^{\infty}B_{2k}\frac{(2iz)^{2k}}{(2k)!}+\frac{2iz}{2}\right)=\sum_{k=0}^{\infty}B_{2k}\frac{\left(-4z^2\right)^k}{(2k)!}+\frac{2iz}{2}=\sum_{k=0}^{\infty}B_{2k}\frac{(-4)^kz^{2k}}{(2k)!}+\frac{2iz}{2}$$

Hence,

$$\pi z\cot(\pi z)=\sum_{k=0}^{\infty}B_{2k}\frac{(-4)^k\pi^{2k}z^{2k}}{(2k)!}+\frac{2i\pi z}{2},$$

$$\pi\cot(\pi z)=\frac{1}{z}\left(\sum_{k=0}^{\infty}B_{2k}\frac{(-4)^k\pi^{2k}z^{2k}}{(2k)!}+\frac{2i\pi z}{2}\right),$$

so that,

$$\begin{aligned}\pi\ H(z)\cot(\pi z)&=\frac{\pi\cot(\pi z)}{z^{2q}}\\&=\frac{1}{z^{2q}.z}\left(\sum_{k=0}^{\infty}B_{2k}\frac{(-4)^k\pi^{2k}z^{2k}}{(2k)!}+\frac{2i\pi z}{2}\right)\\&=\frac{1}{z}\left(\sum_{k=0}^{\infty}B_{2k}\frac{(-4)^k\pi^{2k}z^{2k-2q}}{(2k)!}+\frac{2i\pi z^{1-2q}}{2}\right)\end{aligned}$$

In the Laurent series expansion above, the coefficient of $\frac{1}{z}$ is attained when $2k-2q=0$, in which case the

$$\text{Res}\left(\pi\frac{1}{z^{2q}}\cot(\pi z);z=0\right)=B_{2q}\frac{(-4)^q\pi^{2q}}{(2q)!}=\frac{(-1)^qB_{2q}\ \pi^{2q}\ 2^{(2q)}}{(2q)!}.$$

So

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^{2q}}=-Res\left(\pi\frac{1}{z^{2q}}\cot(\pi z);z=0\right)$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^{2q}}=-\frac{1}{2}\text{Res}\left(\pi\frac{1}{z^{2q}}\cot(\pi z);z=0\right),$$

and hence,

$$\zeta(2q)=\sum_{n=1}^{\infty}\frac{1}{n^{2q}}=-\frac{1}{2}\frac{(-1)^qB_{2q}\ \pi^{2q}\ 2^{(2q)}}{(2q)!}=\frac{(-1)^{q-1}B_{2q}\ \pi^{2q}\ 2^{(2q-1)}}{(2q)!}$$

and

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The special case for $k = 1$ in which $\zeta(2) = \sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, was first derived by Euler in 1734 [5].

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