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Common Fixed Points on Multiplicative b-metric Spaces for Weakly Commuting Mappings

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Abstract

In this paper, we introduce the notion of multiplicative b-metric space. We will prove a common fixed point theorem for multiplicative b-metric space. Our results improve and generalize the results of X [3].

Keywords: Weak commutative mappings; multiplicative b-metric space; common fixed points.
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1. Introduction

The study of fixed point and common fixed point has been a subject of great interest since Banach [1] proved the Banach contraction principle in 1922. In the past year many authors generalized the Banach contraction principle in various space as symmetric spaces, partial metric space, cone metric space etc. In 1976, Jungck [4] used the notion of commuting maps to prove the existence of a common fixed point theorems on a metric space (X, d). Many authors have invested various concept of commuting maps, like weakly commuting maps in 2008, Bashirov [2] introduced the notion of multiplicative metric space and studied the concept of multiplication calculus and proved the fundamental theorem of multiplicative calculus. In 2012, Ozavsar et al. [6] investigated the multiplicative metric space by remarking its topological properties and introduced the concept of multiplicative contraction mapping and some fixed-point theorem of multiplicative, contraction mappings on multiplicative metric space. They recently proved a common fixed-point theorem for four self-mappings in multiplicative metric space. Kang [5] introduced the notion of compatible mappings and its various in multiplicative metric space and proved some common fixed-point theorem for these mappings in his paper. We present some definition and result in common fixed-point theorem for commuting and compatible mappings in complete multiplicative b-metric space. For, we have introduced the notion of b-metric in multiplicative metric space.

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2. Preliminaries

Definition 2.1 ([3]). *Let X be a nonempty set. A multiplicative metric is a mapping* $d : X \times X \to R^+$ *satisfying the following conditions:*

(i) $d(x, y) \geq 1$ *for all* $x, y \in X$ *and* $d(x, y) = 1$ *if and only if* $x = y$*;*

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*

(iii) $d(x, y) \leq d(x, z)d(z, y)$ *for all* $x, y \in X$ *(multiplicative triangle inequality).*

We use the following definition for our main result:

Definition 2.2. *Let X be a nonempty set. A multiplicative b-metric is a mapping* $d : X \times X \to R^+$ *satisfying the following conditions:*

(i) $d(x, y) \geq 1$ *for all* $x, y \in X$ *and* $d(x, y) = 1$ *if and only if* $x = y$*;*

(ii) $d(x, y) \leq (y, x)$ *for all* $x, y \in X$*;*

(iii) $d(x, y) \leq b.d(x, z).d(z, y)$ *for all* $x, y, z \in X$ *(multiplicative triangle inequality), where* $b \geq 1$.

Definition 2.3 ([3]). *Let* (X, d) *be a multiplicative metric space,* $\{x_n\}$ *be a sequence in X and* $x \in X$*. If for every multiplicative open ball* $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$, $\epsilon > 1$*, there exists a natural number N such that* $n \geq N$ *, then* $x_n \in B(x)$*. The sequence* $\{x_n\}$*is said to be multiplicative converging to x, denoted by* $x_n \to x$ $(n \to \infty)$*.*

Definition 2.4 ([3]). *Let* (X, d) *be a multiplicative metric space and* $\{x_n\}$ *be a sequence in X. The sequence is called a multiplicative Cauchy sequence if it holds that for all* $\epsilon > 1$ *, there exists* $N \in \mathbf{N}$ *such that* $d(x_n, x_m) < \epsilon$ *for all* $m, n > N$*.*

Definition 2.5 ([3]). *We call a multiplicative metric space complete if every multiplicative Cauchy sequence in it is multiplicative convergence to* $x \in X$*.*

Definition 2.6 ([3]). *Suppose that* S, T *are two self-mappings of a multiplicative metric space* (X, d)*;* S, T *are called commutative mappings if it holds that for all* $x \in X$, $STx = TSx$*.*

Definition 2.7 ([3]). *Suppose that* S, T *are two self-mappings of a multiplicative metric space* (X, d)*;* S, T *are called weak commutative mappings if it holds that for all* $x \in X$, $d(STx, TSx) \leq d(Sx, Tx)$*.*

Definition 2.8 ([3]). *Let* (X, d) *be a multiplicative metric space. A mapping* $f : X \to X$ *is called a multiplicative contraction if there exists a real constant* $\lambda \in [0, 1)$ *such that* $d(f(x_1), f(x_2)) \leq d(x_1, x_2)^\lambda$ *for all* $x, y \in X$*.*

3. Main Results

Theorem 3.1. *Let* S, T, A *and B be self-mappings of a complete multiplicative metric space X; which satisfy the following conditions:*

(i) $SX \subset BX, TX \subset AX$;

(ii) *A and S are weak commutative, B and T also are weak commutative;*

(iii) *One of S, T , A and B is continuous;*

(iv) $d(Sx, Ty) \leq [b\{\max\{d(Ax, By), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ax, Ty), d^*(Ax, By), d^*(Sx, Ty)\}\}]^{\lambda}$, $\lambda \in (0, \frac{1}{2})$ $\forall$ $x, y \in X$, *where* $d^*(Ax, By) = \min\{1, d(Ax, By)\}$, $d^*(Sx, Ty) = \min\{1, d(Sx, Ty)\}$.

Then S, T, A *and* B *have a unique common fixed point.where* $b \geq 1$. *Provided* $b^k \to 1$ *as* $k \to \infty$.

Proof. Since $SX \subset BX$, and $T(X) \subset AX$, consider a point $x_0 \in X$, then $\exists x_1 \in X$, such that $Sx_0 = Bx_1 = y_0$ (say) and $Tx_1 = Ax_2 = y_1$; continuing this inductive, we have, $\exists x_2 \in X$ such that $Tx_1 = Ax_2 = y_1, \ldots$; $\exists x_{2n+1} \in X$ such that $Bx_{2n+1} = y_{2n}$, $\exists x_{2n+2} \in X$ such that $Tx_{2n+1} = Ax_{2n+2} = y_{2n+1}, \ldots$; $\forall n = 0, 1, 2, \ldots, \infty$. Now we can define a sequence $\{y_n\} \in X$, we obtain by putting $x = x_{2n}, y = x_{2n+1}$ in condition (iv) we obtain,

$$
\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\leq \{b \max\{d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n,}Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Sx_{2n}, Bx_{2n+1}), \\
&d(Ax_{2n}, Tx_{2n+1}), d^*(Ax_{2n}, Bx_{2n+1}), d^*(Sx_{2n}, Tx_{2n+1})\}\}^{\lambda} \\
&\leq \{b \max\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n+1}), \\
&d^*(y_{2n-1}, y_{2n}), d^*(y_{2n}, y_{2n+1})\}\}^{\lambda} \\
&\leq \{b \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), 1, d(y_{2n-1}, y_{2n}, d(y_{2n}, y_{2n+1}), d^*(y_{2n-1}, y_{2n}), \\
&d^*(y_{2n}, y_{2n+1}), d^*(y_{2n-1}, y_{2n}), d^*(y_{2n}, y_{2n+1}\}\}^{\lambda} \\
&= b^{\lambda} d^{\lambda}(y_{2n-1}, y_{2n}) d^{\lambda}(y_{2n}, y_{2n+1}).
\end{aligned}
$$

This implies that $d(y_{2n}, y_{2n+1}) \leq b^h d^{\frac{\lambda}{1-\lambda}}(y_{2n-1}, y_{2n})$. Let $\frac{\lambda}{1-\lambda} = h$, *where* $\lambda \in (0, \frac{1}{2})$, then

$$d(y_{2n}, y_{2n+1}) \leq b^h d^h(y_{2n-1}, y_{2n}), \tag{1}$$

similarly, by putting $x = x_{2n+2}$, $y = x_{2n+1}$ on (iv), we may obtain

$$d(y_{2n+1}, y_{2n+2}) \leq b^h d^h(y_{2n}, y_{2n+1}). \tag{2}$$

From (1) and (2),

$$d(y_n, y_{n+1}) \leq b^h d^h(y_{n-1}, y_n) \leq b^{h^2} d^{h^2}(y_{n-1}, y_n) \leq \cdots \leq b^{h^n} d^{h^n}(y_1, y_0), \forall n \geq 2.$$

Let $m, n \in \mathbb{N}$ such that $m \geq n$, then we get

$$d(y_m, y_n) \leq d(y_m, y_{m-1}).d(y_{m-1}, y_{m-2})...d(y_{n+1}, y_n)$$

$$\leq d^{h^{(m-1)}}(y_1, y_0), d^{h^{(m-2)}}(y_1, y_0)...d^{h^n}(y_1, y_0)$$
$$\leq Bd^{\frac{h^n}{1-h}}(y_1, y_0), \quad (where\ B\ is\ constant)$$
$$\leq Bd(y_1, yo), \ \ as \ \ 0 \leq \frac{h}{1-h} \leq 1,$$

where, $B = b^{h^{(m-1)}} b^{h^{(m-2)}} b^{h^{(m-3)}} ... b^{h^{(m-n)}}$, $as\ b^k \to 1$. This implies that $d(y_m, y_n) \to 1\ as\ m, n \to \infty$. Hence $\{y_n\}$ is a multiplicative Cauchy sequence in X. By the completeness of X, there exists $z \in X$ such that $y_n \to z\ as\ n \to \infty$. Moreover, since $\{Sx_{2n}\} = \{Bx_{2n+1}\} = \{y_{2n}\}$ and $\{Tx_{2n+1}\} = \{Ax_{2n+2}\} = \{y_{2n+1}\}$, are subsequence of $\{y_n\}$, so we obtain, $\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Ax_{2n+2} = z$. Taking condition (ii) and (iii) we obtain following cases;

Case 1: Suppose that A is continuous then, $\lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} A^2x_{2n} = Az$. Since A and S are weakly commuting, then $d(ASx_{2n}, SAx_{2n}) \leq d(Sx_{2n}, Ax_{2n})$. Let $n \to \infty$, we get $\lim_{n\to\infty} d(SAx_{2n}, Az) \leq d(z, z) = 1$, i.e., let $\lim_{n\to\infty} SAx_{2n} = Az$,

$$d(SAx_{2n}, Tx_{2n+1}) \leq [b\{max\{d(A^2x_{2n}, Bx_{2n+1}), d(A^2x_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}),$$
$$d(SAx_{2n}, Bx_{2n+1}), d(A^2x_{2n}, Tx_{2n+1})d^*(A^2x_{2n}, Bx_{2n+1}), d^*(SAx_{2n}, Tx_{2n+1}\}\}]^\lambda.$$

Let $n \to \infty$, we can obtain

$$d(Az, z) \leq [b\{max\{d(Az, z), d(Az, Az), d(z, z), d(Az, z), d(Az, z), d^*(Az, z)d^*(Az, z)\}\}]^\lambda$$
$$= [b\{max\{d(Az, z), 1\}\}]^\lambda$$
$$= b^\lambda d^\lambda(Az, z).$$

This implies that $d(Az, z) = 1$, i.e., $Az = z$,

$$d(Sz, Tx_{2n+1}) \leq [b\{max\{d(Az, , Bx_{2n+1}), d(Az, Sz)d(Bx_{2n+1}, Tx_{2n+1}), d(Sz, Bx_{2n+1}), d(Az, Tx_{2n+1}),$$
$$d^*(A^2z, Bx_{2n+1}), d^*(SAx_{2n}, Tx_{2n+1})\}\}]^\lambda.$$

Let $n \to \infty$, we can obtain

$$d(Sz, z) \leq [b\{max\{d(Az, z), d(z, Sz), d(z, z), d(Sz, z), d(z, z), d^*(Az, z)d^*(Az, z)\}\}]^\lambda$$
$$= [b\{max\{d(Sz, z, 1)\}\}]^\lambda$$
$$= b^\lambda d^\lambda(Sz, z),$$

which implies that $d(Sz, z) = 1$, i.e., $Sz = z$, $z = Sz \in SX \subseteq BX$, so, $\exists z^* \in X$ such that $z = Bz^*$

$$d(z, Tz^*) = d(Sz, Tz^*)$$
$$\leq [b\{max\{d(Az, Bz^*), d(Az, Sz), d(Bz^*, Tz^*), d(Sz, Bz^*), d(Az, Tz^*),$$

$$d^*(Az, Bz^*), d(Az, Sz), d^*(Bz^*, Tz^*)\}\}^\lambda$$
$$= [b\{max\{d(z, Tz^*), 1)\}\}]^\lambda$$
$$= b^\lambda d^\lambda (z, Tz^*),$$

which implies $d(Sz, z) = 1$ i.e., $Tz^* = z$. Since B and T are weakly commuting mappings then

$$d(Bz, Tz) = d(BTz^*, TBz^*) \leq d(Bz^*, Tz^*) = d(z, z) = 1,$$

so, $Bz = Tz$, z is a fixed point of T. Using condition (iv), we have

$$\begin{aligned} d(z, Tz) &= d(Sz, Tz) \\ &\leq [b\{max\{d(Az, Bz), d(Az, Sz), d(Bz, Sz), d(Sz, Bz), d(Az, Tz), d^*(Az, Bz), d^*(Sz, Tz)\}\}]^\lambda \\ &= [b\{max\{d(z, Tz), 1)\}\}]^\lambda \\ &= b^\lambda d^\lambda (z, Tz), \end{aligned}$$

which implies $d(Tz, z) = 1$ i.e., $Tz = z$.

Case 2: Suppose that B is continuous, we can obtain the same result by the way of Case 1.

Case 3: Suppose that S is continuous then $\lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} S^2x_{2n} = Sz$. Since A and S are weak commutative, then $d(ASx_{2n}, SAx_{2n}) \leq d(Sx_{2n}, Ax_{2n})$. Let $n \to \infty$ then $\lim_{n\to\infty} (ASx_{2n}, Sz) \leq d(z, z) = 1$, i.e., $\lim_{n\to\infty} ASx_{2n} = Sz$,

$$\begin{aligned} d(S^2x_{2n}, Tx_{2n+1}) \leq [b\{max\{&d(ASx_{2n}, Bx_{2n+1}), d(ASx_{2n}, S^2x_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \\ &d(S^2x_{2n}, Bx_{2n+1}), d(ASx_{2n}, Tx_{2n+1}), d^*(ASx_{2n}, Bx_{2n+1}), d^*(S^2x_{2n}, Tx_{2n+1})\}\}]^\lambda. \end{aligned}$$

Let $n \to \infty$ we can obtain

$$\begin{aligned} d(Sz, z) &\leq [b\{max\{d(Sz, z), d(Sz, Sz), d(z, z), d(Sz, z), d(Sz, z), d^*(Sz, z), d^*(Sz, Sz)\}\}]^\lambda \\ &= [b\{max\{d(Sz, z), 1\}\}]^\lambda \\ &= b^\lambda d^\lambda (Sz, z), \end{aligned}$$

which implies $d(Sz, z)$ = 1 i.e., $Sz = z$. $z = Sz \in SX \subseteq BX$, so $\exists z^* \in X$ such that $z = Bz^*$

$$\begin{aligned} d(S^2x, Tz^*) \leq [b\{max\{&d(ASx_{2n}, Bz^*), d(ASx_{2n}, S^2x_{2n}), d(Bz^*, Tz^*), \\ &d(S^2x_{2n}, Bz^*), d(ASx_{2n}, Tz^*), d^*(ASx_{2n}, Bz^*), d^*(S^2x_{2n}, Tz^*)\}\}]^\lambda, \end{aligned}$$

$$\begin{aligned} d(z, Tz^*) &= d(Sz, Tz^*) \\ &\leq [b\{max\{d(Sz, z), d(Sz, Sz), d(z, Tz^*), d(Sz, z), d(sz, Tz^*), d^*(Sz, z), d^*(Sz, Tz^*))\}\}]^\lambda \end{aligned}$$

$$= [b\{max\{d(z, Tz^*), 1\}\}]^\lambda$$
$$= b^\lambda d^\lambda(z, Tz^*),$$

which implies that $d(z, Tz^*) = 1,$ *i.e.,* $Tz^* = z$. Since T and B are weak commutative, then $d(Tz, Bz) = d(TBz^*, BTz^*) \leq d(Tz^*, Bz^*) = d(z, z) = 1$, so $Bz = Tz$,

$$d(Sx_{2n}, Tz) \leq [b\{max\{d(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), d(Sx_{2n}, Bz), d(Ax_{2n}, Tz), d^*(Ax_{2n}, Bz), d^*(Sx_{2n}, Tz)\}\}]^\lambda.$$

Let $n \to \infty$ we can obtain

$$d(z, Tz) \leq [b\{max\{d(z, Tz), d(z, z), d(Tz, Tz), d(z, Tz), d(z, Tz), d^*(z, Tz), d^*(z, Tz)\}\}]\lambda$$
$$= [b\{\max\{d(z, Tz), 1\}\}]^\lambda$$
$$= b^\lambda d^\lambda(z, Tz).$$

which implies $d(z, Tz) = 1$ i.e., $Tz = z$. $z = Tz \in TX \subseteq AX$, so $\exists z^{**} \in X$, such that $z = Az^{**}$

$$d(Sz^{**}, z) = d(Sz^{**}, Tz)$$
$$\leq [b\{max\{d(Az^{**}, Bz), d(Az^{**}, Sz^{**}), d(Bz, Tz) d(Sz^{**}, Bz),$$
$$d(Az^{**}, Tz), d^*(Az^{**}, Bz), d^*(Sz^{**}, Tz)\}\}]^\lambda = [b\{max\{d(z, z), d(z, Sz^{**}), d(Bz, Bz), d(Sz^{**}, z),$$
$$d(z, z), d^*(z, z), d^*(Sz^{**}, Tz)\}\}]^\lambda$$
$$= [b\{\max\{d(Sz^{**}, z), 1\}\}]^\lambda$$
$$= b^\lambda d^\lambda(Sz^{**}, z).$$

This implies that $d(Sz^{**}, z) = 1$ i.e., $Sz^{**} = z$. Since S and A are weak commutative, then $d(Az, Sz) = d(ASz^{**}, SAz^{**}) \leq d(Az^{**}, Sz^{**}) = d(z, z) = 1$, so $Az = Sz$. We obtain $Sz = Tz = Az = Bz = z$, so z is common fixed point of S, T, A and B.

Case 4: Suppose that T is continuous, we can obtain the same result by the way of Case 3. In addition we prove that S, T, A and B have a unique common fixed point. suppose that $w \in X$ is also a common fixed point of S, T, A and B, then we obtain

$$d(z, w) = d(Sz, Tw)$$
$$\leq [b\{max\{d(Az, Bw), d(Az, Sz), d(Bw, Tw), d(Sz, Bw), d(Az, Tw), d^*(Az, Bw), d^*(Sz, Tw)\}\}]^\lambda$$
$$= [b\{max\{d(z, w), 1\}]^\lambda$$
$$= [b\{max\{d(z, w), 1\}]^\lambda$$
$$= b^\lambda d^\lambda(z, w)$$

This is a contradiction as $d(z,w) > 1$, when $z = w$. Thus z is a unique common fixed point of $A, B, S, T \subset X$. □

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