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Study of Some Inequalities and Applications on 2-norms and Derived Norms

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Abstract

Most of the results on the space of 2-norms generalised to n-normed space. Here, we investigate some results on the nature of sequences of 2-norms and derived norms. But unable to generalise it for n-norms. Therefore we are taking particular case of n-norms. In this paper, we investigated Minkowski type inequalities for 2-norms, and applied these inequalities in the analysis of convergence of sequences of 2-norms, derived norms with respect to sequence of vectors.

Keywords: Minkowski Inequality; Sequences; Cauchy Sequences; Normed space; 2-normed space; Derived norms.

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1. Introduction

Initial investigations on 2-normed spaces was done by Gahler [1] and generalized to n-normed spaces studied by [2–4,6] and others also. In [2,3] Gunawan and others derived formula for obtaining norms (known as derived norms) by 2-norms (In general by n-norms also) with the help of LID (Means linearly independent) vectors. Obviously changing LID vectors, derived norms change. Keeping this fact, we studied the results on sequences of 2-norms and derived norms obtained by sequence of vectors in space.

Definition 1.1. *If* $\mathbf{X}$ *is a real or complex vector space of dimension* $d \geq 2$ *then non-negative real valued function* $\|\cdot,\cdot\|$ *on* $\mathbf{X^2}$, *having four properties:*

(P-1) $\|x^1, x^2\| = 0$ *iff* x^1, x^2 *are linearly dependent;*

(P-2) $\|x^1, x^2\| = \|x^2, x^1\|$;

(P-3) $\|\beta \cdot x^1, x^2\| = |\beta| \cdot \|x^1, x^2\|$ *for all real or complex* β;

(P-4) $\|x^1 + x', x^2\| \leq \|x^1, x^2\| + \|x', x^2\|$;

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$\forall x^1, x^2, x' \in \mathbf{X}$, *is called 2-norm on X and pair* $(\mathbf{X}, \|\cdot,\cdot\|)$ *2-normed space.*

Definition 1.2. *If* $(\mathbf{X}, \|\cdot,\cdot\|)$ *is a 2-normed space. The sequence* $\left(x^l\right)_{l=1}^{\infty} \subset \mathbf{X}$ *is said to be a Cauchy sequence in* $(\mathbf{X}, \|\cdot,\cdot\|)$ *if* $\|x^l - x^{l'}, a\| \to 0$ *as* $l, l' \to \infty$ *for all* $a \in \mathbf{X}$.

Definition 1.3. *The sequence* $\left(x^l\right)_{l=1}^{\infty}$ *is called convergent (with respect to* $\|\cdot,\cdot\|$*) at* $x \in \mathbf{X}$ *if* $\|x^l - x, a\| \to 0$ *as* $l \to \infty$ *for all* $a \in \mathbf{X}$.

Definition 1.4. *Let* $(\mathbf{X}, \|\cdot,\cdot\|)$ *be a* 2-normed space *and* g, h *are two LID vectors, let us define* $\overline{\|\cdot\|}_\infty, \overline{\|\cdot\|}_p : \mathbf{X} \to \mathbb{R}$ *as-*

(D1) $\overline{\|x\|}_\infty = max\{\|g, x\|, \|h, x\|\}$

(D2) $\overline{\|x\|}_p = (\|g, x\|^p + \|h, x\|^p)^{1/p}; 1 \le p < \infty$

In [2,3], Gunawan have proved that both functions ($\overline{\|\cdot\|}_\infty$ *and* $\overline{\|\cdot\|}_p$*) form norms on the vector space* $\mathbf{X}$ *known as derived norms and they are equivalent.*

Definition 1.5. *The sequence* $\left(\|\cdot\|^k\right)_{k=1}^{\infty}$ *of norms on* $\mathbf{X}$ *is called convergent on norm* $\|\cdot\|$ *on* $\mathbf{X}$ *if* $\|x\|^k \to \|x\|; \forall x \in \mathbf{X}$. *Similarly, sequence* $\left(\|\cdot,\cdot\|^k\right)_{k=1}^{\infty}$ *of 2-norms on* $\mathbf{X}$ *converges to 2-norm* $\|\cdot,\cdot\|$ *on* $\mathbf{X}$ *if* $\|x, y\|^k \to \|x, y\|; \forall x, y \in \mathbf{X}$.

2. Main Results

Before going to our main results, we establish a Minkowski type inequalities for 2-norms in the form of Lemma.

Lemma 2.1. *In 2-normed space* $(\mathbf{X}, \|\cdot,\cdot\|)$ *following inequality holds* $\forall x^1, x^2, y^1, y^2, z \in \mathbf{X}$ *and for all* $1 \le p < \infty$*:*

$$\left| \left(\|x^1, z\|^p + \|y^1, z\|^p\right)^{1/p} - \left(\|x^2, z\|^p + \|y^2, z\|^p\right)^{1/p} \right| \le \left(\|x^1 - x^2, z\|^p + \|y^1 - y^2, z\|^p\right)^{1/p}$$

Proof. By axioms of 2-norm $\forall u, v, w \in \mathbf{X}$, we have

$$\|u, w\| = \|(u - v) + v, w\| \le \|u - v, w\| + \|v, w\|$$

gives

$$\|u, w\| - \|v, w\| \le \|u - v, w\|$$

Interchange of u, v gives

$$\|v, w\| - \|u, w\| \le \|v - u, w\| = \|u - v, w\|$$

Therefore

$$|\|u, w\| - \|v, w\|| \le \|u - v, w\|$$

Consequentially

$$\big|\|u,w\| - \|v,w\|\big|^p \le \|u-v,w\|^p \tag{1}$$

Again for every normed space $(\mathbf{W}, \|\cdot\|)$, we have

$$\big|\|a\| - \|a'\|\big| \le \|a-a'\|; \forall a, a' \in \mathbf{W} \tag{2}$$

Taking $a_1 = \|x^1, z\|, a_2 = \|y^1, z\|, a'_1 = \|x^2, z\|, a'_2 = \|y^2, z\|$ then obviously $a = (a_1, a_2), a' = (a'_1, a'_2)$ are vectors of normed space $(\mathbb{R}^2, \|\cdot\|_p)$. Therefore, in view of (2), we have $\big|\|a\|_p - \|a'\|_p\big| \le \|a-a'\|_p$. That is

$$\begin{aligned}\left|\left(\|x^1,z\|^p + \|y^1,z\|^p\right)^{1/p} - \left(\|x^2,z\|^p + \|y^2,z\|^p\right)^{1/p}\right| &\le \left(\Big|\|x^1,z\| - \|x^2,z\|\Big|^p + \Big|\|y^1,z\| - \|y^2,z\|\Big|^p\right)^{1/p} \\ &\le \left(\|x^1-x^2,z\|^p + \|y^1-y^2,z\|^p\right)^{1/p} \quad \text{By (1)}\end{aligned}$$

□

Lemma 2.2. *If $\|\cdot,\cdot\|_1$ and $\|\cdot,\cdot\|_2$ are two different norms on $\mathbf{X}$ then $\forall u^1, u^2, v^1, v^2, w^1, w^2, z^1, z^2 \in \mathbf{X}$, we have*

$$\begin{aligned}&\left|\left(\|u^1,v^1\|_1^p + \|w^1,z^1\|_1^p\right)^{1/p} - \left(\|u^2,v^2\|_2^p + \|w^2,z^2\|_2^p\right)^{1/p}\right| \\ &\qquad \le \left(\Big|\|u^1,v^1\|_1 - \|u^2,v^2\|_2\Big|^p + \Big|\|w^1,z^1\|_1 - \|w^2,z^2\|_2\Big|^p\right)^{1/p}\end{aligned}$$

Proof. Taking $a_1 = \|u^1, v^1\|_1$, $a_2 = \|w^1, z^1\|_1$, $b_1 = \|u^2, v^2\|_2$, $b_2 = \|w^2, z^2\|_2$ then obviously $a = (a_1, a_2), b = (b_1, b_2)$ are vectors of normed space $(\mathbb{R}^2, \|\cdot\|_p)$. Therefore, in view of (2), we have $\big|\|a\|_p - \|b\|_p\big| \le \|a-b\|_p$. That is

$$\begin{aligned}&\left|\left(\|u^1,v^1\|_1^p + \|w^1,z^1\|_1^p\right)^{1/p} - \left(\|u^2,v^2\|_2^p + \|w^2,z^2\|_2^p\right)^{1/p}\right| \\ &\qquad \le \left(\Big|\|u^1,v^1\|_1 - \|u^2,v^2\|_2\Big|^p + \Big|\|w^1,z^1\|_1 - \|w^2,z^2\|_2\Big|^p\right)^{1/p}\end{aligned}$$

□

Theorem 2.3. *If $(x^k)_{k=1}^{\infty}$ is Cauchy sequence but not convergent in the 2-normed space $(\mathbf{X}, \|\cdot,\cdot\|)$, such that $\{x^k, x^{k+1}\}$,set of consecutive 2 terms of sequence, is LID $\forall 1 \le k < \infty$, then the sequence $\left(\overline{\|\cdot\|}_p^k\right)_{k=1}^{\infty}$ of derived norms defined by $\overline{\|x\|}_p^k = \left(\|x^k, x\|^p + \|x^{k+1}, x\|^p\right)^{1/p}; 1 \le p < \infty$ converges to a semi-norm on $\mathbf{X}$.*

Proof. Since, sequence is Cauchy therefore $\|x^l - x^{l'}, x\| \to 0$ as $l, l' \to \infty; \forall x \in \mathbf{X}$. By Lemma 2.1

$$\begin{aligned}\left|\overline{\|x\|}_p^k - \overline{\|x\|}_p^{k'}\right| &= \left|\left(\|x^k,x\|^p + \|x^{k+1},x\|^p\right)^{1/p} - \left(\|x^{k'},x\|^p + \|x^{k'+1},x\|^p\right)^{1/p}\right| \\ &\le \left(\|x^k - x^{k'},x\|^p + \|x^{k+1} - x^{k'+1},x\|^p\right)^{1/p}.\end{aligned}$$

Therefore $\left|\overline{\|x\|}_p^k - \overline{\|x\|}_p^{k'}\right| \to 0$ as $k, k' \to \infty; \forall x \in \mathbf{X}$. Shows $\left(\overline{\|x\|}_p^k\right)_{k=1}^{\infty}$ is a Cauchy sequence of non-negative terms in $\mathbb{R}; \forall x \in \mathbf{X}$. Consequently converges uniquely, say, to non-negative $l_x \in \mathbb{R}$. Define $\|\cdot\| : \mathbf{X} \to \mathbb{R}$ as

$$\|x\| = \lim \overline{\|x\|}_p^k. \tag{3}$$

Then $\|\cdot\|$ is well-defined and non-negative function, which satisfies the followings:

(N1) Since $\overline{\|\alpha \cdot x\|}_p^k = |\alpha| \cdot \overline{\|x\|}_p^k; \forall k$ therefore $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|; \forall \alpha \in \mathbb{K}$.

(N2) $\overline{\|x+y\|}_p^k \leq \overline{\|x\|}_p^k + \overline{\|y\|}_p^k \Rightarrow \|x+y\| \leq \|x\| + \|y\|; \forall x, y \in \mathbf{X}$.

(N3) Obviously for $x = 0, \overline{\|x\|}_p^k = 0; \forall k$ therefore $\|x\| = 0$.

Thus $\|\cdot\|$ is a semi-norm on $\mathbf{X}$ and $\overline{\|\cdot\|}_p^k \to \|\cdot\|$. □

Theorem 2.4. *If $\left(\{x^k, y^k\}\right)_{k=1}^{\infty}$ is a sequence of LID sets in $\mathbf{X}$ such that $x^k \to x$ and $y^k \to y$ in 2-normed space $(\mathbf{X}, \|\cdot,\cdot\|)$, where $\{x, y\}$ is linearly independent then $\overline{\|\cdot\|}_p^k \to \overline{\|\cdot\|}_p$ as $k \to \infty$, where*

$$\overline{\|z\|}_p^k = \left(\|x^k, z\|^p + \|y^k, z\|^p\right)^{1/p}; \qquad 1 \leq p < \infty, \tag{4}$$

$$\overline{\|z\|}_p = \left(\|x, z\|^p + \|y, z\|^p\right)^{1/p}; \qquad 1 \leq p < \infty \tag{5}$$

Proof. By lemma 2.1,

$$\begin{aligned}\left|\overline{\|z\|}_p^k - \overline{\|z\|}_p\right| &= \left|\left(\|x^k, z\|^p + \|y^k, z\|^p\right)^{1/p} - \left(\|x, z\|^p + \|y, z\|^p\right)^{1/p}\right| \\ &\leq \left(\|x^k - x, z\|^p + \|y^k - y, z\|^p\right)^{1/p}\end{aligned}$$

and it is assumed that $\|x^k - x, z\| \to 0, \quad \|y^k - y, z\| \to 0 \ \forall z \in \mathbf{X}$ as $k \to \infty$. Therefore $\overline{\|z\|}_p^k \to \overline{\|z\|}_p$ as $k \to \infty; \forall z \in \mathbf{X}$. Hence, $\overline{\|\cdot\|}_p^k \to \overline{\|\cdot\|}_p$ as $k \to \infty$. □

Theorem 2.5. *If $\left(\|\cdot,\cdot\|^k\right)_{k=1}^{\infty}$ is a sequence of 2-norms defined on $\mathbf{X}$ and converges to 2-norm $\|\cdot,\cdot\|$ on $\mathbf{X}$ then for every linearly independent set $\{a, b\}$ in $\mathbf{X}$ sequence of derived norms $\{\overline{\|\cdot\|}_p^k\}_{k=1}^{\infty}$ converges to derived norm $\overline{\|\cdot\|}_p$, where,*

$$\overline{\|z\|}_p^k = \left((\|a, z\|^k)^p + (\|b, z\|^k)^p\right)^{1/p}; \qquad 1 \leq p < \infty, \tag{6}$$

$$\overline{\|z\|}_p = \left(\|a, z\|^p + \|a, z\|^p\right)^{1/p}; \qquad 1 \leq p < \infty. \tag{7}$$

Proof. Taking $\|\cdot,\cdot\|_1 = \|\cdot,\cdot\|^k$, $\|\cdot,\cdot\|_2 = \|\cdot,\cdot\|$ and $u^1 = u^2 = a, w^1 = w^2 = b, v^1 = v^2 = z^1 = z^2 = z$ in Lemma 2.2, we have

$$\left|\overline{\|z\|}_p^k - \overline{\|z\|}_p\right| = \left|\left((\|a, z\|^k)^p + (\|b, z\|^k)^p\right)^{1/p} - \left(\|a, z\|^p + \|b, z\|^p\right)^{1/p}\right|$$

$$\leq \left(\left| \|a,z\|^k - \|a,z\| \right|^p + \left| \|b,z\|^k - \|b,z\| \right|^p \right)^{1/p}.$$

And it is given that $\left| \|x,y\|^k - \|x,y\| \right| \to 0$ as $k \to \infty; \forall x, y \in \mathbf{X}$. Therefore, $\overline{\|z\|}_p^k \to \overline{\|z\|}_p$; $\forall z \in \mathbf{X}$. Hence, $\overline{\|\cdot\|}_p^k \to \overline{\|\cdot\|}_p$. □

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