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A Study of Appell Function of Two Variables Using Hurwitz - Lerch Zeta Function of Two Variables

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Abstract

Firstly, the Appell's Hypergeometric Function of two variables $F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]$ is introduced using Hurwitz-Lerch Zeta Function of two variables $\phi_{a_1,a_2,b_1,b_2;c_1c_2}(x_1,x_2,s,p)$. Then several integral representations and differential formula are investigated for this function $F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]$. The function $F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]$ that we have introduced & defined here has also been represented in term of generalized Hypergeometric Function ${}_pF_q$. To strengthen our main results, we have also considered some important special cases.

Keywords: Generalized Hurwitz-Lerch Zeta Function; Gamma Function; Beta Function; Hypergeometric Function; Binomial Series; Eulerian Integral.

1. Introduction

A class of Mathematical Functions that arise in the solution of various classical problems of mathematical physics are termed as Special Functions, for example some Special Functions arise in solving the equation of heat flow or wave propagation in cylindrical co-ordinates, and in many other such physical problems. Special functions have also applications in number theory, for example the Hypergeometric functions are useful in constructing conformal mapping of polygonal regions whose sides are circular areas. In the recent past, some applications are also seen in quantum mechanics and in the angular momentum theory for example Gegenbauer polynomials are used in the developments of four -dimensional spherical harmonics. In 2012, Pathan and Dawan [11] have given further Generalization of Hurwitz-Lerch Zeta Function as:

$$\phi_{a_1,a_2,b_1,b_2;c_1,c_2}\left(x_1,x_2,s,p\right)=\sum_{n_1,n_2=0}^{\infty}\frac{\left(a_1\right)_{n_1}\left(a_2\right)_{n_2}\left(b_1\right)_{n_1}\left(b_2\right)_{n_2}}{n_1!n_2!\left(c_1\right)_{n_1}\left(c_2\right)_{n_2}}\frac{\left(x_1\right)^{n_1}\left(x_2\right)^{n_2}}{\left(n_1+n_2+p\right)^s} \tag{1}$$

$(a_1,a_2,b_1,b_2\in\mathbb{C};s,x_1,x_2\in\mathbb{C})\,p,\ \ c_1,c_1\ \neq\ \{0,-1,-2,\ldots\}$ when $|x_1|<1$ and $|x_2|<1$; and $\mathbb{R}\left(s+c_1+c_2a_1-a_2-b_1-b_2\right)>0$ when $|x_1|=1$ and $|x_2|=1$. Also, in the year 2014 has H.M.

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Srivastava [8] has given a new family of the $\lambda-$ Generalized Hurwitz-Lerch Zeta Function with applications as:

$$\phi_b^a(x,s,p;c) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \exp\left(-pt - \frac{c}{t^a}\right) \cdot \left(1 - xe^{-t}\right)^{-b} dt \tag{2}$$

$(\min\{\mathbb{R}(p), \mathbb{R}(s)\} > 0;\ \mathbb{R}(c) \geq 0; a \geq 0; b \in \mathbb{C})$. Srivastava and Choi [9] have given series associated with the Zeta and Related Functions in the year 2001 by the following equation:

$$\phi(s, a-t) = \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \phi(s+n, a) t^n, \quad (|t| < |a|) \tag{3}$$

Also, Srivastava and Choi [10] have given Zeta and q-Zeta Functions and associated series and Integrals in 2012. Srivastava, Saxena and Pogany [12] have given Integral and computational representations of the extended Hurwitz-Lerch Zeta function in 2011:

$$\phi_{a,b;c}^{\rho,\sigma,k}(x,s,p) = \sum_{n=0}^{\infty} \frac{(a)_{\rho n}(b)_{\sigma n}}{(c)_{kn}} \frac{x^n}{(n+p)^s} \tag{4}$$

$a, b \in c; p, c \in c \mid z_0^-; \rho, \sigma, k \in R^+;$

$$k - \rho - \sigma > -1 \text{ when } s, x \in C;$$
$$k - \rho - \sigma = -1 \text{ and } s \in c \text{ when } |x| < \delta^* = \rho^{-\rho}\sigma^{-\sigma}k^{-k};$$
$$k - \rho - \sigma = -1 \text{ and } R(s + c - a - b) > 1 \text{ when } |x| = \delta^*;$$

Properties & particular cases of Hurwitz-Lerch Zeta function are found in [9,13,14] various type of generalizations, extensions of Hurwitz-Lerch Zeta function can be also found in [8,10,12,15,16,17,18]. More details about generalizations, extensions, properties and cases of hypergeometric functions are found in [2-6]. Zeta function is one of the special functions that is widely used in number theory and is defined as [8]:

$$\phi(s,p) = \sum_{n=0}^{\infty} \frac{1}{(n+p)^s}; \qquad R(s) > 1 \tag{5}$$

For $p = 0$ the zeta function reduces to Riemann Zeta Function [8]:

$$\phi(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}; \qquad R(s) > 1 \tag{6}$$

The Hurwitz - Lerch Zeta Function is defined as [9,10]:

$$\phi(z,s,p) = \sum_{n=0}^{\infty} \frac{x^n}{(n+p)^s} \tag{7}$$

$(p \in c \backslash z_0; s \in C)$ when $|x| < 1$ and $R(s) > 1$ when $|x| = 1$. The Generalization of Hurwitz - Lerch Zeta Function is defined [8]:

$$\phi_{(b,c)}^{(\rho,\sigma)}(x,s,p) = \sum_{n=0}^{\infty} \frac{(b)_{\rho n}}{(c)_{\sigma n}} \frac{x^n}{(n+p)^s} \tag{8}$$

$(p \in C \backslash Z_0; s \in C), (b \in C)$ and $(\rho, \sigma \in \mathbb{R}^+)\, \rho < \sigma$ when $s, x \in C; \rho = \sigma$ and $s \in C$ when $|x| < 1; \rho = \sigma$ and $R(s + c - b) > 0$ when $|x| < 1$. Motivated by the above recent works, in the field of applications of special functions and their extensions and generalizations, the authors in the present paper have defined the Appel Hypergeometric Function of two variables in terms of Hurwitz-Lerch Zeta Function of two variables by the following equation:

$$F_2\left[a, b_1, b_2; c_1, c_2; x_1, x_2\right] = \sum_{n_1, n_2 = 0}^{\infty} \frac{(a)_{n_1+n_2} \left(n_1 + n_2 + p\right)^s}{\left(a_1\right)_{n_1} \left(a_2\right)_{n_2}} \phi_{a_1, a_2, b_1, b_2; c_1, c_2}\left(x_1, x_2, s, p\right) \tag{9}$$

$(a_1, a_2, b_1, b_2 \in C; s, x_1, x_2 \in C)p,\ c_1, c_1 \neq \{0, -1, -2, \ldots\} \mathbb{R}(c) > 0,\ \mathbb{R}(b) > 0$ when $|x_1| < 1, |x_2| < 1$ and $|x_1| + |x_2| < 1$ and $\mathbb{R}(s + c_1 + c_2 - a_1 - a_2 - b_1 - b_2) > 0$ when $|x_1| = 1, |x_2| = 1$, where $\phi_{a_1, a_2, b_1, b_2; c_1 c_2}\left(x_1, x_2, s, p\right)$ is the generalized Hurwitz-Lerch Zeta Function of two variables defined by Pathan and Dawan [11] by (1):

2. Preliminaries

Definition 2.1. *The Hypergeometric Function ${}_2F_1(a, b; c; x)$ is defined as [1]:*

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} = 1 + \frac{ab}{c} \frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{x^2}{2!} + \cdots \tag{10}$$

$|x| < 1$ *and* $\mathbb{R}(c) > 0, \mathbb{R}(b) > 0.$

Definition 2.2. *The Generalized Hypergeometric Function is defined as [1]:*

$${}_pF_q\left(a_1, a_2, \ldots a_p; b_1, b_2, \ldots b_q; x\right) = \sum_{n=0}^{\infty} \frac{\left(a_1\right)_n \cdots \left(a_p\right)_n}{\left(b_1\right)_n \cdots \left(b_q\right)_n} \frac{x^n}{n!}, \tag{11}$$

where Pochhammer Symbol are defined for $n \in C$:

1. $(a)_n = \frac{\Gamma(a+n)}{\Gamma a} = \begin{cases} 1 & (n = 0; a \neq 0) \\ a(a+1)(a+2) - \{a + (n-1)\} & (n \in N; a \in C) \end{cases}$, $N = \{1, 2, 3 \ldots\}$ *and* $Z_0^- = \{0, -1, -2, \ldots\}$.

2. $(a)_{n+m} = (a)_n (a+n)_m$, $n \in N$; $a \in C$; $N = \{1, 2, 3 \ldots\}$ *and* $Z_0^- = \{0, -1, -2, \ldots\}$.

Definition 2.3. *The Appell's Hypergeometric Functions F_2 of two variables that was introduced by Poul Appell (1880) [3-8]:*

$$F_2\left(a, b_1, b_2; c_1, c_2; x_1, x_2\right) = \sum_{n_1, n_2 = 0}^{\infty} \frac{(a)_{n_1+n_2} \left(b_1\right)_{n_1} \left(b_2\right)_{n_2}}{\left(c_1\right)_{n_1} \left(c_2\right)_{n_2}} \frac{\left(x_1\right)^{n_1}}{n_1!} \frac{\left(x_2\right)^{n_2}}{n_2!} \tag{12}$$

$|x_1| + |x_2| < 1$ *and* $R(c) > 0, R(b) > 0.$

Definition 2.4. *The integral representation of the Pochhammer symbol $\left(a_1\right)_{n_1}$ & $\left(a_2\right)_{n_2}$ is defined as:*

$$\left(a_1\right)_{n_1} = \frac{1}{\Gamma a_1} \int_0^{\infty} y^{a_1 + n_1 - 1} e^{-y} dy \tag{13}$$

$$(a_2)_{n_2} = \frac{1}{\Gamma a_2} \int_0^{\infty} z^{a_2+n_2-1} e^{-z} dz \tag{14}$$

Definition 2.5. *The following identity*

$$(a)_{n+1} = a(a+1)_n \tag{15}$$

Several Integral Representations, and Differential Formula are obtained for our function introduced in (9). In this next section we mention some of the known formulae and results which we need in the proofs of our main results.

3. Main Results

3.1 Integral Representations

Theorem 3.1. *The following integral representation for* $F_2[a, b_1, b_2; c_1, c_2; x_1, x_2]$ *holds true:*

$$\mathbf{F}_2[a, b_1, b_2; c_1, c_2; x_1, x_2] = \int_0^{\infty} \int_0^{\infty} \frac{1}{\Gamma a_1 \Gamma a_2} y^{a_1-1} z^{a_2-1} e^{-(y+z)} \times \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2}(n_1+n_2+p)^s}{\left[(a_1)_{n_1}(a_2)_{n_2}\right]^2} \phi_{a_1,a_2,b_1,b_2;c_1c_2}(x_1 y, x_2 z, s, p)\, dydz \tag{16}$$

where $R(c) > 0$, $R(b) > 0$ *provided* $|x_1| < 1$ *and* $|x_2| < 1$; $R(s) > 0$, $R(a) > 0$ *provided* $|x_1| \leq 1$ *and* $|x_2| \leq 1$; $R(s) > 1$, *provided* $x_1 = 1$ *and* $x_2 = 1$.

Proof. Using (12) and multiplying and dividing by $\left\{(a_1)_{n_1}(a_2)_{n_2}\right\}$ on right hand side we get:

$$F_2[a, b_1, b_2; c_1, c_2; x_1, x_2] = \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1} + n_2 (b_1)_{n_1}(b_2)_{n_2}}{(c_1)\, n_{n_1}(c_2)_{n_2}} \frac{(x_1)^{n_1}}{n_1!} \frac{(x_2)^{n_2}}{n_2!} \frac{(a_1)_{n_1}(a_2)_{n_2}}{(a_1)_{n_1}(a_2)_{n_2}} \tag{17}$$

Using (13) and (14) for $(a_1)_{n_1}$ and $(a_2)_{n_2}$ on right -hand side of (17) we get:

$$F_2[a, b_1, b_2; c_1, c_2; x_1, x_2] = \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2}(b_1)_{n_1}(b_2)_{n_2}}{(a_1)_{n_1}(a_2)_{n_2}(c_1)_{n_1}(c_2)_{n_2}} \frac{(x_1)^{n_1}}{n_1!} \frac{(x_2)^{n_2}}{n_2!} \times \frac{1}{\Gamma a_1} \int_0^{\infty} y^{a_1+n_1-1} e^{-y} dy \frac{1}{\Gamma a_2} \int_0^{\infty} z^{a_2+n_2-1} e^{-z} dz \tag{18}$$

Interchanging the order of integration and summation on the right - hand of (18) we get:

$$F_2[a, b_1, b_2; c_1, c_2; x_1, x_2] = \int_0^{\infty} \int_0^{\infty} \frac{1}{\Gamma a_1} \frac{1}{\Gamma a_2} \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2}(b_1)_{n_1}(b_2)_{n_2}}{\left(a_{n_1}(a_2)_{n_2}(c_1)_{n_1}(c_2)_{n_2}\right.} \frac{1}{n_1! n_2!} \times y^{a_1-1}(x_1 y)^{n_1} z^{a_2-1}(x_2 z)^{n_2} e^{-(y+z)} dydz \tag{19}$$

Again, multiplying and dividing by $\left\{(a_1)_{n_1}(a_2)_{n_2}(n_1+n_2+p)^s\right\}$ on right hand side of (19) we get:

$$F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]=\int_0^\infty\int_0^\infty\frac{1}{\Gamma a_1\Gamma a_2}y^{a_1-1}z^{a_2-1}e^{-(y+z)}\sum_{n_1,n_2=0}^{\infty}\frac{\left(aa_{n_1}+n_2(b_1)_{n_1}(b_2)_{n_2}\right.}{\left(a_{n_1}(a_2)_{n_2}(c_1)_{n_1}(c_2)_{n_2}\right.}$$
$$\times\frac{1}{n_1!n_2!}(x_1y)^{n_1}(x_2z)^{n_2}\frac{\left[(a_1)_{n_1}(a_2)_{n_2}(n_1+n_2+p)^s\right]}{\left[(a_1)_{n_1}(a_2)_{n_2}(n_1+n_2+p)^s\right]}dydz \tag{20}$$

On comparison with (1)

$$F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]=\int_0^\infty\int_0^\infty\frac{1}{\Gamma a_1\,\lceil a_2}y^{a_1-1}z^{a_2-1}e^{-(y+z)}\sum_{n_1,n_2=0}^{\infty}\frac{(a)_{n_1+n_2}(n_1+n_2+p)^s}{\left[(a_1)_{n_1}(a_2)_{n_2}\right]^2}$$
$$\times\,\phi_{a_1,a_2,b_1,b_2;c_1c_2}(x_1y,x_2z,s,p)\,dydz \tag{21}$$

Hence we get the desired result, that is (16), which is wanted to prove. □

3.2 Differential Formula

Theorem 3.2. *The following differential Formula for $F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]$ holds true*

$$\frac{\delta^2}{\delta x_1\delta x_2}\{F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]\}=\sum_{n_1,n_2=0}^{\infty}\frac{(a)_{n_1+n_2+2}a_1a_2b_1b_2}{(a_1)_{n_1+1}(a_2)_{n_2+2}c_1c_2}(n_1+n_2+p+2)^s$$
$$\times\,\phi_{a_1+1,a_2+1,b_1+1,b_2+1;c_1+1,c_2+1}(x_1,x_2,s,p+2) \tag{22}$$

where $R(c)>0$, $R(b)>0$ provided $|x_1|<1$ and $|x_2|<1$; $R(s)>0, R(a)>0$ provided $|x_1|\leq 1$ and $|x_2|\leq 1$; $R(s)>1$, provided $x_1=1$ and $x_2=1$.

Proof. Partial derivative of (12) with respect to x_2 yields:

$$\frac{\delta}{\delta x2}\{F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]\}=\sum_{n_1,n_2=0}^{\infty}\frac{(a)_{n_1+n_2}(b_1)_{n_1}(b_2)_{n_2}}{(c_1)_{n_1}(c_2)_{n_2}}\frac{(x_1)^{n_1}}{n_1!}\frac{n_2(x_2)^{n_2-1}}{n_2!} \tag{23}$$

And again, partial derivative of above (23) with respect to x_1 yields:

$$\frac{\delta^2}{\delta x_1\delta x_2}\{F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]\}=\sum_{n_1,n_2=0}^{\infty}\frac{(a)_{n_1+n_2}(b_1)_{n_1}(b_2)_{n_2}}{(c_1)_{n_1}(c_2)_{n_2}}\frac{n_1n_2(x_1)^{n_1-1}}{n_1!}\frac{(x_2)^{n_2-1}}{n_2!} \tag{24}$$

Or

$$\frac{\delta^2}{\delta x_1\delta x_2}\{F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]\}=\sum_{n_1,n_2=0}^{\infty}\frac{(a)_{n_1+n_2}(b_1)_{n_1}(b_2)_{n_2}}{(c_1)_{n_1}(c_2)_{n_2}}\frac{(x_1)^{n_1-1}}{(n_1-1)!}\frac{(x_2)^{n_2-1}}{(n_2-1)!} \tag{25}$$

Replacing n_1 by (n_1+1) and n_2 by (n_2+1) on right - hand side of above (25) we get:

$$\frac{\delta^2}{\delta x_1\delta x_2}\{F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]\}=\sum_{n_1,n_2=0}^{\infty}\frac{(a)_{n_1+n_2+2}(b_1)_{n_1+1}(b_2)_{n_2+1}}{(c_1)_{n_1+1}(c_2)_{n_2+1}}\frac{(x_1)^{n_1}}{(n_1)!}\frac{(x_2)^{n_2}}{(n_2)!} \tag{26}$$

Multiplying and dividing by $\left\{(a_1)_{n_1+1}(a_2)_{n_2+1}(n_1+n_2+p+2)^s\right\}$ on right hand side of (26) we get:

$$\frac{\delta^2}{\delta x_1 \delta x_2}\{F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]\} = \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2+2}(b_1)_{n_1+1}(b_2)_{n_2+1}}{(c_1)_{n_1+1}(c_2)_{n_2+1}} \frac{(x_1)^{n_1}}{(n_1)!}\frac{(x_2)^{n_2}}{(n_2)!} \times \frac{(a_1)_{n_1+1}(a_2)_{n_2+1}}{(a_1)_{n_1+1}(a_2)_{n_2+1}}\frac{(n_1+n_2+p+2)^s}{(n_1+n_2+p+2)^5} \tag{27}$$

Or,

$$= \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2+2}}{(a_1)_{n_1+1}(a_2)_{n_2+1}}\frac{(x_1)^{n_1}}{(n_1)!}\frac{(x_2)^{n_2}}{(n_2)!}\frac{(a_1)_{n_1+1}(a_2)_{n_2+1}(b_1)_{n_1+1}(b_2)_{n_2+1}}{(c_1)_{n_1+1}(c_2)_{n_2+1}}\frac{(n_1+n_2+p+2)^5}{(n_1+n_2+p+2)^5} \tag{28}$$

Or,

$$= \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2+2}}{(a_1)_{n_1+1}(a_2)_{n_2+1}}\frac{(a_1)_{n_1+1}(a_2)_{n_2+1}(b_1)_{n_1+1}(b_2)_{n_2+1}}{(c_1)_{n_1+1}(c_2)_{n_2+1}}\frac{(x_1)^{n_1}(x_2)^{n_2}(n_1+n_2+p+2)^s}{(n_1)!(n_2)!(n_1+n_2+p+2)^s} \tag{29}$$

Using the identity (15) on above (29) we get:

$$\frac{\delta^2}{\delta x_1 \delta x_2}\{F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]\} = \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2+2}}{(a_1)_{n_1+1}(a_2)_{n_2+1}}\frac{a_1a_2b_1b_2}{c_1c_2} \times \frac{(a_1+1)_{n_1}(a_2+1)_{n_2}(b_1+1)_{n_1}(b_2+1)_{n_2}}{(c_1+1)_{n_1}(c_2+1)_{n_2}}\frac{(x_1)^{n_1}(x_2)^{n_2}(n_1+n_2+p+2)^s}{(n_1)!(n_2)!(n_1+n_2+p+2)^s} \tag{30}$$

On comparison with (1)

$$\frac{\delta^2}{\delta x_1 \delta x_2}\{F_2[a,b_1,b_2;c_1,c_2;x_1,x_2]\} = \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2+2}a_1a_2b_1b_2}{(a_1)_{n_1+1}(a_2)_{n_2+2}c_1c_2}(n_1+n_2+p+2)^s \times \phi_{a_1+1,a_2+1,b_1+1,b_2+1;c_1+1,c_2+1}(x_1,x_2,s,p+2) \tag{31}$$

□

4. Special Cases

Case 1: If we put $a = a_1 = a_2 = 1$ in (9) we obtain:

$$F[1,b_1,b_2;c_1,c_2;x_1,x_2] = \sum_{n_1,n_2=0}^{\infty} \frac{(n_1+n_2)!(n_1+n_2+p)^s}{n_1!n_2!}\phi_{1,1,b_1,b_2;c_1,c_2}(x_1,x_2,s,p) \tag{32}$$

$(b_1,b_2 \in C; s,x_1,x_2 \in c)$ and $p,c_1,c_1 \neq \{0,-1,-2,\ldots\}$ when $|x_1| < 1$ and $|x_2| < 1$ and $R(s+c_1+c_2-b_1-b_2) > 0$ when $|x_1| = 1$ and $|x_2| = 1$.

Case 2: If we put $b_1 = c_1, b_2 = c_2$ in (9) we obtain:

$$F[a;x_1,x_2] = \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2}(n_1+n_2+p)^s}{(a_1)_{n_1}(a_2)_{n_2}}\phi_{a_1,a_2}(x_1,x_2,s,p) \tag{33}$$

$(a_1, a_2 \in c; s, x_1, x_2 \in c)$ and $p \neq \{0, -1, -2, \ldots\}$ when $|x_1| < 1$ and $|x_2| < 1$ and $R(s - a_1 - a_2) > 0$ when $|x_1| = 1$ and $|x_2| = 1$, where $\phi_{a_1,a_2}(x_1, x_2, s, p)$ is generalized Hurwitz - Lerch Zeta function in [11].

Case 3: If $a = a_1 = a_2 = 1$ and $b_1 = c_1, b_2 = c_2$ in (9) we obtain:

$$F[1; x_1, x_2] = \sum_{n_1,n_2=0}^{\infty} \frac{(n_1+n_2)!\,(n_1+n_2+p)^s}{n_1!n_2!}\phi_{1,1}(x_1, x_2, s, p) \tag{34}$$

$(s, x_1, x_2 \in c)$ and $p \neq \{0, -1, -2, \ldots\}$ when $|x_1| < 1$ and $|x_2| < 1$ and $R(s) > 0$ when $|x_1| = 1$ and $|x_2| = 1$, where $\phi_{a_{1,a_2}}(x_1, x_2, s, p)$ is Hurwitz - Lerch Zeta function of two variables of $\phi(x, s, p)$ is defined in [9, 10].

Case 4: If $b_1 \to \infty$ in (9) we obtain:

$$F[a, b_1, b_2; c_1, c_2; x_1, x_2] = \lim_{b_1 \to \infty} F\left[a, b_1, b_2; c_1, c_2; \frac{x_1}{b_1}, x_2\right] \tag{35}$$

$$= \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2}(b_2)_{n_2}(x_1)^{n_1}(x_2)^{n_2}}{(c_1)_{n_1}(c_2)_{n_2}(n_1)!\,(n_2)!} \tag{36}$$

$$= \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2}(b_2)_{n_2}(x_1)^{n_1}(x_2)^{n_2}}{(c_1)_{n_1}(c_2)_{n_2}(n_1)!\,(n_2)!} \frac{\left[(a_1)_{n_1}(a_2)_{n_2}(n_1+n_2+p)^s\right]}{\left[(a_1)_{n_1}(a_2)_{n_2}(n_1+n_2+p)^5\right]} \tag{37}$$

$$F[a, b_1, b_2; c_1, c_2; x_1, x_2] = \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2}(n_1+n_2+p)^s}{(a_1)_{n_1}(a_2)_{n_2}}\phi_{a_1,a_2,b_2;c_1c_2}(x_1, x_2, s, p) \tag{38}$$

$(a_1, a_2, b_2 \in C;\ s, x_1, x_2 \in C)$ and $p, c_1, c_1 \neq \{0, -1, -2, \ldots\}$ when $|x_1| < 1$ and $|x_2| < 1$ and $R(s + c_1 + c_2 - a_1 - a_2 - b_2) > 0$ when $|x_1| = 1$ and $|x_2| = 1$.

Case 5: If $b_1 \to \infty, b_2 \to \infty$ in (9) we obtain:

$$F[a, b_1, b_2; c_1c_2; x_1, x_2] = \lim_{b_1,b_2 \to \infty} F\left[a, b_1, b_2; c_1c_2; \frac{x_1}{b_1}, \frac{x_2}{b_2}\right] \tag{39}$$

$$= \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2}(x_1)^{n_1}(\mathrm{X}_2)^{n_2}}{(c_1)_{n_1}(c_2)\,\mathrm{n}_2\,(n_1)!\,(n_2)!}$$

$$= \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1+n_2}(x_1)^{n_1}(x_2)^{n_2}}{(c_1)_{n_1}(c_2)_{n_2}(n_1)!\,(n_2)!} \frac{\left[(a_1)_{n_1}(a_2)_{n_2}(n_1+n_2+p)^s\right]}{\left[(a_1)_{n_1}(a_2)_{n_2}(n_1+n_2+p)^s\right]}$$

$$F[a, b_1, b_2; c_1c_2; x_1, x_2] = \sum_{n_1,n_2=0}^{\infty} \frac{(a)_{n_1} + n_2\,(n_1+n_2+p)^s}{(a_1)_{n_1}(a_2)_{n_2}}\phi_{a_1,a_2;c_1c_2}(x_1, x_2, s, p) \tag{40}$$

$(a_1, a_2 \in C; s, x_1, x_2 \in C)$ and $p, c_1, c_1 \neq \{0, -1, -2, \ldots\}$ when $|x_1| < 1$ and $|x_2| < 1$ and $R(s + c_1 + c_2 - a_1 - a_2) > 0$ when $|x_1| = 1$ and $|x_2| = 1$.

5. Conclusion

We have introduced the Appell's Hypergeometric Function of two variables in terms of Hurwitz - Lerch Zeta Function of two variables and thereafter we have obtained Integral representation and

differential Formula for this function $F\left[a, b_1, b_2; c_1, c_2; x_1, x_2\right]$. And then some of the special cases of our main results are also considered which give rise to some other new interesting results.
Integral representation:

$$F_2\left[a, b_1, b_2; c_1, c_2; x_1, x_2\right] = \int_0^\infty \int_0^\infty \frac{1}{\Gamma a_1 \lceil a_2} y^{a_1-1} z^{a_2-1} e^{-(y+2)} \\ \times \sum_{n_1, m_2=0}^{\infty} \frac{(a)_{n_1+n_2}\left(n_1+n_2+p\right)^2}{\left[\left(a_1\right)_{n_1}\left(a_2\right)_{n_2}\right]^2} \phi_{a_1, a_2 b_2, b_2, c_1 c_2}\left(x_1 y, x_2 z, s, p\right) dy dz$$

(i) $R(c) > 0, R(b) > 0$ provided $|x_1| < 1$ and $|x_2| < 1$;

(ii) $R(s) > 0, R(a) > 0$ provided $|x_1| \leq 1$ and $|x_2| \leq 1$;

(iii) $R(s) > 1$, provided $x_1 = 1$ and $x_2 = 1$.

In the course of our above study, we have also obtained Differential Formula for $F\left[a, b_1, b_2; C_1, C_2; x_1, x_2\right]$.
Differential Formula:

$$\frac{\delta^2}{\sigma x_2 \delta x_2}\left\{F\left[a, b_1, b_2; c_1, c_2; x_1, x_2\right]\right\} = \sum_{n_2, n_2-0}^{\infty} \frac{(a)_{n_1+n_2+2} a_1 a_2 b_2 b_2}{\left(a_1\right)_{n_1+1}\left(a_2\right)_{n_2+2} c_1 c_2}\left(n_1+n_2+p+2\right)^s \\ \times \Phi_{a_1+1, a_2+1, b_2+1, b_2+1 x_1+1, c_2+1}\left(x_1, x_2, s, p+2\right)$$

(i) $R(c) > 0, R(b) > 0$ provided $|x_1| < 1$ and $|x_2| < 1$;

(ii) $R(s) > 0, R(a) > 0$ provided $|x_1| \leq 1$ and $|x_2| \leq 1$;

(iii) $R(s) > 1$, provided $x_1 = 1$ and $x_2 = 1$.

Lastly to strengthen our main results we have given and discussed some special cases.

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