International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **12(2)**(2024), 87–93
Available Online: http://ijmaa.in

Bessel Polynomials Through Linear Algebra

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Abstract

In the present note, Bessel polynomials are obtained through linear algebra methods. A matrix corresponding to the Bessel differential operator is found and its eigenvalues are obtained. The elements of the eigenvector obtained correspond to the Bessel polynomials.

Keywords: Special functions; Bessel polynomials; Bessel; eigenvalues; eigenvector.
2020 Mathematics Subject Classification: 42C05, 33C10, 33C99, 15A18.

1. Introduction

In many areas of applied mathematics, various types of Special functions become essential tools for scientists and engineers. The Continuous development of mathematical physics, probability theory and other areas has led to new classes of Special functions and their extensions and generalizations have been done by many researchers in the recent past. Very recently in the year 2017, Aboites [1, 2] obtained Laguerre polynomials and Hermite polynomials through the Linear Algebra method. Motivated by the recent work, the authors in the present paper have applied Linear Algebra method for obtaining Bessel polynomials. Bessel polynomials are the solutions of the following Hypergeometric equation:

$$s\left(x\right)F^{''}\left(x\right)+t\left(x\right)F'\left(x\right)+\lambda F\left(x\right)=0 \tag{1}$$

In which $F\left(x\right)$ is a real function of a real variable $F:U\rightarrow\mathbb{R}$, where $U\subset\mathbb{R}$ is an open subset of the real line, and $\lambda\,\epsilon\mathbb{R}$ a corresponding eigenvalue, and the function $s\left(x\right)$ and $t\left(x\right)$ are real polynomials of at most second order and first order, respectively. The general self-adjoint form of an eigenvalue equation in Sturm-Liouville Theory is taken as:

$$\mathcal{L}u\left(x\right)=\frac{d}{dx}\left[px\frac{du\left(x\right)}{dx}\right]+\ q\left(x\right)u\left(x\right),$$

where λ is an eigenvalue, and $w\left(x\right)$ is a density function also known as a weight function.

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In the recent past, this Sturm-Liouville Theory has also been studied with the Linear Algebra concept and it is seen that the diagonalization of a real symmetric matrix corresponds to the solution of ODE defined as a self-adjoint operator $\mathcal{L}$, in terms of its eigenfunctions, which are the continuous analog of the eigenvectors. For a detailed study in this regard, one can consult the recent works of V. Aboites [1].

1.1 Special cases of equation (1)

(i). when $s(x)$ =constant equation (1) may be expressed as:

$$F''(x) - 2\alpha x F'(x) + \lambda F(x) = 0 \tag{2}$$

If we take $\alpha = 1$, in the above equation (2), we get Hermite polynomials.

(ii). when $s(x)$ =a polynomial of first degree, equation (1) takes the form:

$$xF''(x) + (-\alpha x + \beta + 1) F'(x) + \lambda F(x) = 0 \tag{3}$$

If we take $\alpha = 1$ and $\beta = 0$, in equation (3), we arrive at Laguerre polynomials.

(iii). When $s(x)$ =a polynomial of the second degree, equation (1), takes the form:

$$\left(1 - x^2\right) F''(x) + [\beta - \alpha - (\alpha + \beta + 2) x] F'(x) + \lambda F(x) = 0 \tag{4}$$

Equation (4) gives Jacobi equations, and correspondingly the following polynomials are generated for different values of α and β in equation (4).

(a). When $\alpha = \beta$, we get Gegenbauer polynomials.

(b). When $\alpha = \beta = 0$, we get Legendre polynomials.

(c). When $\alpha = -1, \beta = 0$, we get Bessel polynomials.

(d). When $\alpha = \beta = \pm\frac{1}{2}$, we get Chebyshev first and second kind polynomials.

2. Bessel Polynomials through Matrix Algebra

The algebraic polynomial of degree n.

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots a_n x^n \tag{5}$$

with $a_0, a_1, a_2, \ldots, a_n \in \Re$, is represented by the Column vector:

$$A_n = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{bmatrix} \tag{6}$$

Taking first derivative of the above polynomial (5) we obtain the polynomial:

$$\frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots a_n x^n\right) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots n a_n x^{n-1}, \tag{7}$$

which may be written as:

$$\frac{d}{dx} A_n = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ . \\ . \\ . \\ n a_n \\ 0 \end{bmatrix} \tag{8}$$

Taking second derivative of the above polynomial (5) we obtain the polynomial:

$$\frac{d^2}{dx^2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots a_n x^n\right) = 2a_2 + 6a_3 x + 12a_4 x^2 + \ldots n\,(n-1)\,a_n x^{n-2}, \tag{9}$$

or,

$$\frac{d^2}{dx^2} A_n = \begin{bmatrix} 2a_2 \\ 6a_3 \\ 12a_4 \\ . \\ . \\ . \\ n(n-1)\,a_n \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

Equation (7) may be written as:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & n \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \dots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \dots \\ na_n \\ 0 \end{bmatrix} \tag{11}$$

Therefore the first derivative operator of A_n may be written as:

$$\frac{d}{dx} \to \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & n \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{12}$$

In a similar manner, equation (9) may be written as:

$$\begin{bmatrix} 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 6 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & n\,(n-1) \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \dots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} 2a_1 \\ 6a_3 \\ 12a_4 \\ \dots \\ n\,(n-1) \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

Therefore the second derivative operator of A_n may be written as:

$$\frac{d^2}{dx^2} \to \begin{bmatrix} 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 6 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & n\,(n-1) \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{14}$$

The Bessel differential operator is given by [3]:

$$x^2 \frac{d^2}{dx^2} + 2\,(x+1)\,\frac{d}{dx} \tag{15}$$

which on using equation (7) and (9) may be written as:

$$
\begin{aligned}
x^2[2a_2 + 6a_3x + 12a_4x^2 + \cdots + n\,(n-1)\,a_nx^{n-2}] + 2\,(x+1)\left[a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1}\right] \\
= [2a_2x^2 + 6a_3x^3 + 12a_4x^4 + \cdots + n\,(n-1)\,a_nx^n] \\
+ [2a_1x + 4a_2x^2 + 6a_3x^3 + 8a_4x^4 + \cdots + 2n\,(n-1)\,a_nx^n] \\
+ [2a_1 + 4a_2x + 6a_3x^2 + 8a_4x^3 \cdots + 2na_nx^{n-1}],
\end{aligned} \tag{16}
$$

which may be written as:

$$
\begin{bmatrix}
0 & 2 & 0 & 0 & \dots & 0 \\
0 & 2 & 4 & 0 & \dots & 0 \\
0 & 0 & 6 & 6 & \dots & 0 \\
0 & 0 & 0 & 12 & \dots & 0 \\
0 & 0 & 0 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & \dots & n\,(n-1)+2n
\end{bmatrix}
\begin{bmatrix}
a_0 \\ a_1 \\ a_2 \\ a_3 \\ \dots \\ a_{n-1} \\ a_n
\end{bmatrix}
=
\begin{bmatrix}
2a_1 \\ 2a_1 + 4a_2 \\ 6a_2 + 6a_3 \\ 12a_3 + 8a_4 \\ \dots \\ \dots \\ \dots \\ n\,(n-1)+2n
\end{bmatrix} \tag{17}
$$

Therefore:

$$
x^2\frac{d^2}{dx^2} + 2\,(x+1)\,\frac{d}{dx} \to
\begin{bmatrix}
0 & 2 & 0 & 0 \\
0 & 2 & 4 & 0 \\
0 & 0 & 6 & 6 \\
0 & 0 & 0 & 12
\end{bmatrix} \tag{18}
$$

And therefore, the Bessel differential operator is represented by the following 4×4 matrix:

$$
\begin{bmatrix}
0 & 2 & 0 & 0 \\
0 & 2 & 4 & 0 \\
0 & 0 & 6 & 6 \\
0 & 0 & 0 & 12
\end{bmatrix} \tag{19}
$$

The eigenvalues of a matrix M are the values that satisfy the equation $(M - \lambda I) = 0$. However since Matrix (19) is a upper triangular matrix, the eigenvalues λ_i of this matrix are the elements of the diagonal, namely: $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 6, \lambda_4 = 12$. The corresponding eigenvectors are the solutions of the equation $(M - \lambda_i I) \cdot \nu = 0$ where the eigenvector $\nu = [a_0, a_1, a_2, a_3]^T$.

$$\begin{bmatrix} 0-\lambda_i & 2 & 0 & 0 \\ 0 & 2-\lambda_i & 4 & 0 \\ 0 & 0 & 6-\lambda_i & 6 \\ 0 & 0 & 0 & 12-\lambda_i \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

Substituting $\lambda_1 = 0$, in equation (20), we obtain the first eigenvector ν_1 as:

$$\nu_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

The elements of this eigenvector correspond to the Bessel polynomials:

$$B_0(x) = 1.$$

Substituting in equation (20) the second eigenvalue $\lambda_2 = 2$, we obtain the second eigenvector ν_2:

$$\nu_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

The elements of this eigenvector correspond to the second Bessel polynomials:

$$B_1(x) = 1 + x.$$

Substituting in equation (20), the third eigenvalue $\lambda_3 = 6$ we obtain the third eigenvector ν_3:

$$\nu_3 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 0 \end{bmatrix} \tag{23}$$

The elements of this eigenvector correspond to the third Bessel polynomials:

$$B_2(x) = 1 + 3x + 3x^2.$$

Substituting in equation (20) the fourth eigenvalue $\lambda_4 = 12$, one obtains the eigenvector ν_4:

$$\nu_4 = \begin{bmatrix} 1 \\ 6 \\ 15 \\ 15 \end{bmatrix} \tag{24}$$

The elements of this eigenvector correspond to the fourth Bessel polynomials:

$$B_3(x) = 1 + 6x + 15x^2 + 15x^3$$

3. Conclusion

The basic linear algebra concepts of obtaining eigenvalues and corresponding eigenvectors of Matrix is applied to obtain the different forms like $B_0(x)$, $B_1(x)$, $B_2(x)$ and $B_3(x)$ of Bessel polynomials. This is done by obtaining 4×4 corresponding Matrix from a second-order differential operator for the Bessel polynomial and it is seen that the elements of eigenvectors correspond to the different Bessel polynomials.

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