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Few Integrals Involving H-function Of One Variable

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Abstract

This paper contains eight integrals involving H-function of one variable with proper conditions of validity. The special cases for G-functions have also been obtained.

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1. Introduction and Preliminaries

It can be taken from our previous paper [1].

$$H^{l,\,m+1}_{p+1,\,q+1}\left[z\left|\begin{matrix}(\mu,\lambda) & ((a_p,A_p))\\ ((b_q,B_q)) & (\gamma,\delta)\end{matrix}\right.\right]$$

will be abbreviated as

$$H^{l,\,m+1}_{p+1,\,q+1}\left[z\left|\begin{matrix}(\mu,\lambda), & ((\quad))\\ ((\quad)), & (\gamma,\delta)\end{matrix}\right.\right]$$

Here $(a_p,A_p)\equiv(a_j,A_j)_{1,p}\equiv(a_1,A_1),(a_2,A_2),...,(a_p,\ A_p)$.

1.1 Some Known Results

The following known results will be required in the proof of the integrals (involving H-function of one variable) to be evaluated.

(i) $\int_0^\infty e^{-\alpha t}\left[\sin\ h(\beta t)\right]^\gamma dt=\beta^{-1}2^{-\gamma-1}B\left(\frac{1}{2}\frac{\alpha}{\beta}-\frac{1}{2}\gamma,1+\gamma\right)=\beta^{-1}2^{-\gamma-1}\frac{\Gamma\left(\frac{1}{2}\frac{\alpha}{\beta}-\frac{1}{2}\gamma\right)\Gamma(1+\gamma)}{\Gamma\left(\frac{1}{2}\frac{\alpha}{\beta}+\frac{1}{2}\gamma+1\right)}$ [2]

$$\text{provided } Re(\gamma)>-1, Re(\beta)>0, Re\left(\frac{\alpha}{\beta}\right)>Re(\gamma) \tag{1}$$

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(ii) $\int_0^1 x^{\sigma}\left(1-x^2\right)^{\frac{-\mu}{2}} P_v^{\mu}(x)dx = \dfrac{2^{\mu-1}\Gamma\left(\frac{1}{2}+\frac{1}{2}\sigma\right)\Gamma\left(1+\frac{1}{2}\sigma\right)}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}v-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\sigma+\frac{1}{2}v-\frac{1}{2}\mu+\frac{3}{2}\right)}$ [2]

$$\text{provided } Re(\mu) < 1, Re(\sigma) > -1 \tag{2}$$

(iii) $\int_1^{\infty} x^{-\rho}\left(x^2-1\right)^{\frac{-\mu}{2}} P_v^{\mu}(x)dx = \dfrac{2^{\rho+\mu-2}\Gamma\left(\frac{\rho+\mu+v}{2}\right)\Gamma\left(\frac{\rho+\mu-v-1}{2}\right)}{\pi^{\frac{1}{2}}\Gamma(\rho)}$ [4]

$$\text{provided } Re(\mu) < 1, Re(\rho+\mu+v) > 0, Re(\rho+\mu-v) > 1 \tag{3}$$

(iv) Integral Containing Tchebichef Polynomial.

$$\int_{-1}^{1}(1-x)^{\frac{1}{2}}(1+x)^{\alpha}\,U_n(x)dx = \frac{\pi^{\frac{1}{2}}2^{\alpha+2n+\frac{3}{2}}\left\{(n+1)!\right\}^2\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\alpha+1\right)}{(2n+2)!\Gamma\left(\alpha+n+\frac{5}{2}\right)\Gamma\left(\alpha-n+\frac{1}{2}\right)}$$

[4]

$$\text{provided } Re(\alpha) > -1; n = 0, 1, 2, \ldots \tag{4}$$

(v) $\int_0^1 x^{\lambda} P_{2r}(x)dx = \dfrac{(-1)^r\left(\frac{-\lambda}{2}\right)_r}{2\left(\frac{1}{2}+\frac{1}{2}\lambda\right)_{r+1}} = \dfrac{(-1)^r\Gamma\left(r-\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)}{2\Gamma\left(r+\frac{\lambda}{2}+\frac{3}{2}\right)\Gamma\left(-\frac{\lambda}{2}\right)}$ [4]

$$\text{provided } Re(\lambda) > -1;\ \text{ r is non-negative integer} \tag{5}$$

(vi) $\int_0^1 x^{\lambda} P_{2v+1}(x)dx = \dfrac{(-1)^v\left(\frac{1}{2}-\frac{1}{2}\lambda\right)_v}{2\left(1+\frac{1}{2}\lambda\right)_{v+1}} = \dfrac{(-1)^v\Gamma\left(\frac{1}{2}-\frac{1}{2}\lambda+v\right)\Gamma\left(1+\frac{1}{2}\lambda\right)}{2\Gamma\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\Gamma\left(2+v+\frac{1}{2}\lambda\right)}$ [4]

$$\text{provided } Re(\lambda) > -2 \text{ and } v \text{ is non negative integer} \tag{6}$$

(vii) Integral associated with generalized Laguerre polynomial

$$\int_0^{\infty} x^{\beta-1}e^{-x}L_n^{(\alpha)}(x)dx = \frac{\Gamma\left(\alpha-\beta+n+1\right)\Gamma\left(\beta\right)}{\Gamma\left(\alpha-\beta+1\right)\ \ n!}$$

[4]

$$\text{provided } Re(\beta) > 0;\ \text{ n - non negative integer} \tag{7}$$

2. Single Integral Involving H-function of One Variable

The integrals to be evaluated here are expressed in the form of the following theorems:

Theorem 2.1. *If the following conditions are satisfied:*

(i) If $Re\left(\gamma\right) > -1, \sigma > 0, \beta > 0; Re\left(\alpha\right) + \sigma \cdot \min_{l\leq j\leq l} Re\left(\frac{b_j}{B_j}\right) > \beta Re\left(\gamma\right)$

(ii) the H-function of one variable occurring in (8) *satisfies the conditions of analyticity similar to equation (1.1.10) [1] then,*

$$\int_0^\infty e^{-\alpha t}\left[\sinh(\beta t)\right]^\gamma H_{p,q}^{l,m}\left[ze^{-\sigma t}\left|\begin{matrix}((\ ))\\((\ ))\end{matrix}\right.\right]dt$$

$$=\frac{\beta^{-1}}{2^{1+\gamma}}\Gamma(1+\gamma)\,H_{p+1,q+1}^{l,m+1}\left[z\left|\begin{matrix}\left(1+\frac{\gamma}{2}-\frac{\alpha}{2\beta},\frac{\sigma}{2\beta}\right),((\ ))\\((\ )),\left(-\frac{\gamma}{2}-\frac{\alpha}{2\beta},\frac{\sigma}{2\beta}\right)\end{matrix}\right.\right] \tag{8}$$

Proof. To prove (8) expressing H-function in the L.H.S of (8) in contour integral form by equation (1.1.7) [1]; changing the order of integral and on evaluating the inner integrals with the help of (1), we get

$$=\frac{1}{2\pi i}\frac{\Gamma(1+\gamma)}{\beta\cdot 2^{\gamma+1}}\int_L \frac{\prod_{j=1}^{1}\Gamma(b_j-B_js)\prod_{j=1}^{m}\Gamma(1-a_j+A_js)\,\Gamma\left(\frac{\alpha}{2\beta}-\frac{\gamma}{2}+\frac{\sigma}{2\beta}s\right)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+B_js)\prod_{j=m+1}^{p}\Gamma(a_j-A_js)\,\Gamma\left(1+\frac{\alpha}{2\beta}+\frac{\gamma}{2}+\frac{\sigma}{2\beta}s\right)}z^s ds \tag{9}$$

Now on interpreting (9) into H-function with the help of equation (1.1.7) [1] the right hand side of (8) follows immediately. □

Theorem 2.2. *If the following conditions are satisfied:*

(i) $\sigma>0, Re(\beta)>0, \left\{Re(\gamma)+\sigma\min_{l\le j\le l}Re\left(\frac{b_j}{B_j}\right)\right\}>-1;$

$$Re\left(\frac{\alpha}{\beta}\right)>Re(\gamma)+\sigma\max_{l\le j\le m}Re\left(\frac{a_j-1}{A_j}\right)$$

and

(ii) *the H-function of one variable occurring in* (10) *satisfies the conditions of analyticity similar to equation (1.1.10) [1] then,*

$$\int_0^\infty e^{-\alpha t}\left[\sinh(\beta t)\right]^\gamma H_{p,q}^{l,m}\left[z\left\{2\sinh(\beta t)\right\}^\sigma\left|\begin{matrix}((\ ))\\((\ ))\end{matrix}\right.\right]dt$$

$$=\frac{1}{\beta\cdot 2^{1+\gamma}}H_{p+1,q+2}^{l+1,m+1}\left[z\left|\begin{matrix}(-\gamma,\sigma),((\ ))\\\left(\frac{\alpha}{2\beta}-\frac{\gamma}{2},\frac{\sigma}{2}\right),((\ )),\left(-\frac{\alpha}{2\beta}-\frac{\gamma}{2},\frac{\sigma}{2}\right)\end{matrix}\right.\right] \tag{10}$$

Proof. The proof is similar to that of (8). □

Theorem 2.3. *If the following conditions are satisfied:*

(i) $\lambda>0, Re(\mu)<1, Re(\sigma)+\lambda\min_{l\le j\le l}Re\left(\frac{b_j}{B_j}\right)>-1$ *and*

(ii) *the H-function of one variable occurring in* (11) *satisfies the conditions of analyticity similar to equation (1.1.10) [1] then,*

$$\int_0^1 x^\sigma\left(1-x^2\right)^{-\mu/2}P_v^\mu(x)\,H_{p,q}^{l,m}\left[zx^\lambda\left|\begin{matrix}((\ ))\\((\ ))\end{matrix}\right.\right]dx$$

$$= 2^{\mu-1} H^{l,\, m+2}_{p+2,\, q+2}\left[z \,\middle|\, \begin{array}{c} \left(-\frac{\sigma}{2}-\frac{1}{2},\frac{\lambda}{2}\right),\left(-\frac{\sigma}{2},\frac{\lambda}{2}\right),\left((a_p,A_p)\right) \\ \left((b_q,B_q)\right),\left(\frac{v+\mu-\sigma}{2},\frac{\lambda}{2}\right),\left(\frac{\mu-v-\sigma-1}{2},\frac{\lambda}{2}\right) \end{array} \right] \tag{11}$$

Proof. The proof is similar to that of (10) except that result (2) is used. □

Theorem 2.4. *If the following conditions are satisfied:*

(i) $\eta > 0, Re(\mu) < 1, Re\left(\rho+\mu+v\right) - \eta \max_{l\le j\le m} Re\left(\frac{a_j-1}{A_j}\right) > 0;\ Re\left(\rho+\mu-v\right) - \eta \max_{l\le j\le m} Re\left(\frac{a_j-1}{A_j}\right) > 1$ *and*

(ii) the H-function of one variable occurring in (12) *satisfies the conditions of analyticity similar to equation (1.1.10) [1] then,*

$$\int_1^\infty x^{-\rho}\left(x^2-1\right)^{-\mu/2} P_v^{\mu}(x)\, H^{l,\, m}_{p,\, q}\left[zx^{\eta} \,\middle|\, \begin{array}{c} ((\ \ )) \\ ((\ \ )) \end{array} \right] dx$$

$$= \frac{2^{\rho+\mu-2}}{\sqrt{\pi}} H^{l+2,\, m}_{p+1,\, q+2}\left[\frac{z}{2^{\eta}} \,\middle|\, \begin{array}{c} ((\ \ )),(\rho,\eta) \\ \left(\frac{\rho}{2}+\frac{\mu}{2}+\frac{v}{2},\frac{\eta}{2}\right),\left(\frac{\rho+\mu-v-1}{2},\frac{\eta}{2}\right),((\ \ )) \end{array} \right] \tag{12}$$

Proof. Proof of (12) can be established in a similar way, as above, except that result (3) has been used here instead of (2). □

Theorem 2.5. *If the following conditions are satisfied:*

(i) $\lambda > 0, Re(\alpha) + \lambda \min_{l\le j\le l} Re\left(\frac{b_j}{B_j}\right) > -1;\ n = 0,1,2,\ldots$

(ii) the H-function of one variable occurring in (13) *satisfies the conditions of analyticity similar to equation (1.1.10) [1] then,*

$$\int_{-1}^1 (1-x)^{\frac{1}{2}}(1+x)^{\alpha}\, U_n(x)\, H^{l,m}_{p,q}\left[z\left\{\frac{1+x}{2}\right\}^{\lambda} \,\middle|\, \begin{array}{c} ((\ \ )) \\ ((\ \ )) \end{array} \right] dx$$

$$= \frac{[(n+1)!]^2}{(2n+2)!} 2^{\alpha+2n+3/2}\sqrt{\pi} H^{l,\, m+2}_{p+2,\, q+2}\left[z \,\middle|\, \begin{array}{c} \left(\frac{1}{2}-\alpha,\lambda\right),(-\alpha,\lambda),((\ \ )) \\ ((\ \ )),\left(-\frac{3}{2}-\alpha-n,\lambda\right),\left(\frac{1}{2}+n-\alpha,\lambda\right) \end{array} \right] \tag{13}$$

Proof. The proof can be established in a similar way, as above, by using result (4) instead of (3). □

Theorem 2.6. *If the following conditions are satisfied:*

(i) $\sigma > 0, Re(\lambda) + \sigma \min_{l\le j\le l} Re\left(\frac{b_j}{B_j}\right) > -1;\ r = 0,1,2,3,\ldots.$

(ii) the H-function of one variable occurring in (14) *satisfies the conditions of analyticity similar to equation (1.1.10) [1] then,*

$$\int_0^1 x^{\lambda} P_{2r}(x)\, H^{l,\, m}_{p,\, q}\left[zx^{\sigma} \,\middle|\, \begin{array}{c} ((\ \ )) \\ ((\ \ )) \end{array} \right] dx$$

$$= \frac{(-1)^r}{2} H^{l+1,\, m+1}_{p+2,\, q+2}\left[z \middle| \begin{array}{c} \left(\frac{1}{2}-\frac{1}{2}\lambda, \frac{1}{2}\sigma\right), ((\ \ )), \left(-\frac{1}{2}\lambda, \frac{1}{2}\sigma\right) \\ \left(r-\frac{\lambda}{2}, \frac{1}{2}\sigma\right), ((\ \ )), \left(-r-\frac{\lambda}{2}-\frac{1}{2}, \frac{\sigma}{2}\right) \end{array} \right] \tag{14}$$

Proof. The theorem can be proved in the same way, as above, by using the result (5). □

Theorem 2.7. *If the following conditions are satisfied:*

(i) $\eta > 0, Re(\lambda) + \eta \min_{l \le j \le l} Re\left(\frac{b_j}{B_j}\right) > -2$, *$\nu$ is non negative integer.*

(ii) the H-function of one variable occurring in (15) *satisfies the conditions of analyticity similar to equation (1.1.10) [1] then,*

$$\int_0^1 x^\lambda P_{2\nu+1}(x)\, H^{l,\, m}_{p,\, q}\left[zx^\eta \middle| \begin{array}{c} ((\ \ )) \\ ((\ \ )) \end{array} \right] dx$$

$$= H^{l+1,\, m+1}_{p+2,\, q+2}\left[z \middle| \begin{array}{c} \left(-\frac{1}{2}\lambda, \frac{1}{2}\eta\right), ((\ \ )), \left(\frac{1}{2}-\frac{1}{2}\lambda, \frac{1}{2}\eta\right) \\ \left(\frac{1}{2}-\frac{1}{2}\lambda+\nu, \frac{1}{2}\eta\right), ((\ \ )), \left(-1-\nu-\frac{1}{2}\lambda, \frac{1}{2}\eta\right) \end{array} \right] \tag{15}$$

Proof. The proof can be established in the same way, as above, by using result (6). □

Theorem 2.8. *If the following conditions are satisfied:*

(i) δ *is positive integer,* $Re(\sigma) + \delta \min_{l \le j \le m} Re\left(\frac{b_j}{B_j}\right) > -1$

(ii) the H-function of one variable occurring in (16) *satisfies the conditions of analyticity similar to equation (1.1.10) [1] then,*

$$\int_0^\infty x^\sigma e^{-x} L_k^{(\alpha)}(x)\, H^{m,\, n}_{p,\, q}\left[z(2x)^\delta \middle| \begin{array}{c} ((\ \ )) \\ ((\ \ )) \end{array} \right] dx = \frac{1}{k!} H^{m+1,\, n+1}_{p+2,\, q+1}\left[2^\delta z \middle| \begin{array}{c} (-\sigma, \delta), ((\ \ )), (\alpha-\sigma, \delta) \\ (k+\alpha-\sigma, \delta), ((\ \ )) \end{array} \right] \tag{16}$$

Proof. The proof is similar to that of integrals, as above, excepting that the result (7) has been used here. □

3. Special Cases of (2) for G-functions

Taking A_j, B_j each equal to unity in (8) to (16), we get the corresponding integrals for Meijer's G-function. These are given below:

(i) Putting $\sigma = 2\beta$ in (8), we get

$$\int_0^\infty e^{-\alpha t}\, [\sinh(\beta t)]^\gamma\, G^{l,\, m}_{p,\, q}\left[ze^{-2\beta t} \middle| \begin{array}{c} ((a_p)) \\ ((b_q)) \end{array} \right] dt$$

$$= \frac{\Gamma(1+\gamma)}{\beta \cdot 2^{\gamma+1}} G^{l,\, m+1}_{p+1,\, q+1}\left[z \middle| \begin{array}{c} \left(1+\frac{1}{2}\gamma-\frac{1}{2}\frac{\alpha}{\beta}\right), ((a_p)) \\ ((b_q)), \left(-\frac{1}{2}-\frac{1}{2}\frac{\alpha}{\beta}\right) \end{array} \right] \tag{17}$$

(ii) Taking $\lambda = 2$ in (11), we get

$$\int_0^1 x^{\sigma}(1-x^2)^{-\mu/2}P_v^{\mu}(x)\; G_{p,\,q}^{l,\,m}\left[zx^2 \,\middle|\, \begin{matrix}((a_p))\\((b_q))\end{matrix}\right]dx$$

$$= 2^{\mu-1}\; G_{p+2,\,q+2}^{\mathrm{l},\,m+2}\left[z \,\middle|\, \begin{matrix}\left(-\frac{1}{2}\sigma-\frac{1}{2}\right),\left(-\frac{1}{2}\sigma\right),((a_p))\\((b_q)),\left(\frac{v+\mu-\sigma}{2}\right),\left(\frac{\mu-v-\sigma-1}{2}\right)\end{matrix}\right] \tag{18}$$

(iii) Taking $\lambda = 1$ in (13), we get

$$\int_{-1}^1 (1-x)^{\frac{1}{2}}(1+x)^{\alpha}\,\mathrm{U}_n(x)\; G_{p,\,q}^{l,\,m}\left[z\left(\frac{1+x}{2}\right) \,\middle|\, \begin{matrix}((a_p))\\((b_q))\end{matrix}\right]dx$$

$$= \frac{[(n+1)!]^2}{(2n+2)!}2^{\alpha+2n+3/2}\sqrt{\pi}G_{p+2,\,q+2}^{l,\,m+2}\left[z \,\middle|\, \begin{matrix}\left(\frac{1}{2}-\alpha\right),(-\alpha),((a_p))\\((b_q)),\left(-\frac{3}{2}-\alpha-n\right),\left(\frac{1}{2}+n-\alpha\right)\end{matrix}\right] \tag{19}$$

(iv) If we put $\sigma = 2$ in (14), we get

$$\int_0^1 x^{\lambda}P_{2m}(x)\; G_{p,\,q}^{l,\,n}\left[zx^2 \,\middle|\, \begin{matrix}((a_p))\\((b_q))\end{matrix}\right]dx$$

$$= \frac{(-1)^m}{2}\; G_{p+2,\,q+2}^{l+1,\,n+1}\left[z \,\middle|\, \begin{matrix}\left(-\frac{1}{2}-\frac{1}{2}\lambda\right),((a_p)),\left(-\frac{1}{2}\lambda\right),\\\left(m-\frac{1}{2}\lambda\right),((b_q)),\left(-m-\frac{1}{2}\lambda-\frac{1}{2}\right)\end{matrix}\right] \tag{20}$$

(v) If we put $\eta = 2$ in (15), we get

$$\int_0^1 x^{\lambda}P_{2m+1}(x)\; G_{p,\,q}^{l,\,n}\left[zx^2 \,\middle|\, \begin{matrix}((a_p))\\((b_q))\end{matrix}\right]dx$$

$$= G_{p+2,\,q+2}^{l+1,\,n+1}\left[z \,\middle|\, \begin{matrix}\left(-\frac{1}{2}\lambda\right),((a_p)),\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\\\left(\frac{1}{2}+m-\frac{1}{2}\lambda\right),((b_q)),\left(-1-m-\frac{1}{2}\lambda\right)\end{matrix}\right] \tag{21}$$

(vi) Taking $\delta = 1$ in (16), we get

$$\int_0^{\infty} x^{\sigma}e^{-x}L_k^{(\alpha)}(x)\; G_{p,\,q}^{m,\,n}\left[z(2x) \,\middle|\, \begin{matrix}((a_p))\\((b_q))\end{matrix}\right]dx = \frac{1}{k!}\,G_{p+2,\,q+1}^{m+1,\,n+1}\left[2z \,\middle|\, \begin{matrix}(-\sigma),((a_p)),(\alpha-\sigma)\\(k+\alpha-\sigma),((b_q))\end{matrix}\right] \tag{22}$$

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