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Common Fixed Point Theorem for a New Type of $F-$Contraction in $b-$Metric Spaces

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Abstract

In this paper, firstly, we introduce the notion of contraction of $F-$type function and the concept of α_s-orbital admissible mapping. Secondly, we establish a fixed point theorem for this type contraction in the setting of complete $b-$metric spaces and substantiate our theoretical findings with a supportive example. This approach not only broadens the spectrum of fixed point theory but also demonstrates the practical applicability of our results in the analysis of $b-$metric spaces.

Keywords: $b-$metric spaces; fixed point; α_s-orbital admissible mapping; $F-$type function.
2020 Mathematics Subject Classification: 47H09, 47H10.

1. Introduction

The Banach [5] contraction principle, originally established by Banach in 1922, stands as a seminal work in the realm of nonlinear analysis, serving as a fundamental tool to ascertain the existence and uniqueness of fixed points in the framework of complete metric spaces. This theorem has been inspired by a multitude of scholars to extend its scope by altering the underlying space settings or the contraction conditions (see reference [3,7,11,13]). Czerwik [6], in 1993, made a notable contribution by amending the third condition of the standard metric space and provided the concept of the $b-$metric spaces. In this modified context, Czerwik explored a new class of fixed point theorems for contractive mappings. Recently, the research on $b-$metric spaces and fixed point theory flourishes and several eminent studies in these fields emerge to advance the theory (see references [1,2,4,9,12]). A noteworthy development in this field was established by Wardowski [14] in 2012, who defined a new class of contraction, referred to as $F-$type contraction in the framework of complete metric spaces, and provided essential conditions to guarantee the existence and uniqueness of fixed points for such mappings. Building upon the concept of $F-$type contraction in $b-$metric spaces, Goswami [8] presented rigorous proofs to substantiate related theorems (see references [10]). In this paper, we

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introduce the concept of contraction of $F-$type function and α_s-orbital admissible mapping. In the setting of complete $b-$metric spaces, we prove a fixed point theorem for this type contraction. Furthermore, we provide a supportive example to demonstrate the effectiveness of the result.

2. Preliminaries

Definition 2.1. *Suppose $s \geq 1$ and X is a nonempty set. A function $d : X \times X \to [0, +\infty)$ is said to be a $b-$metric if for $x, y, z \in X$, the following conditions hold:*

(1). $d(x, y) = 0$ if and only if $x = y$;

(2). $d(x, y) = d(y, x)$;

(3). $d(x, y) \leq s[d(x, y) + d(y, z)]$.

A couple (X, d) is called a $b-$metric space with parameter s.

Definition 2.2. *Suppose (X, d) is a $b-$metric space with parameter s, $x \in X$ and $\{x_n\}$ is a sequence in X.*

(a). $\{x_n\}$ is convergent to x, if for each $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$, satisfying $d(x_n, x) < \varepsilon$ for all $n > n_\varepsilon$. We denote this as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ when $n \to \infty$.

(b). $\{x_n\}$ is a Cauchy sequence, if for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > n_\varepsilon$.

Definition 2.3. *The mappings $f, g : X \to X$ are called α_s-orbital admissible mappings, if the following conditions hold:*

$$\alpha(x, fx) \geq s^p \Rightarrow \alpha(fx, gfx) \geq s^p,$$
$$\alpha(x, gx) \geq s^p \Rightarrow \alpha(gx, fgx) \geq s^p$$

for a constant $p \geq 2$.

Definition 2.4. *Let (X, d) be a complete $b-$metric space with parameter $s \geq 1$ and let $\alpha_s : X \times X \to \mathbb{R}^+$ be a function. Then,*

(H_{s^p}) If $\{x_n\}$ is a sequence in X such that $gx_n \to gx$ as $n \to \infty$, then there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ with $\alpha_s(gx_{n_k}, gx) \geq s^p$ for all $k \in \mathbb{N}$.

(V_{s^p}) For all $u, v, w \in X$, $\alpha_s(u, v) \geq s^p, \alpha_s(v, w) \geq s^p$, we have $\alpha_s(u, w) \geq s^p$.

Definition 2.5. *Let Δ denote the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties:*

(F_1) F is strictly increasing;

(F_2) for each sequence $\{x_n\}_{n=1}^\infty$ of positive numbers, we have

$$\lim_{n\to\infty} x_n = 0, \lim_{n\to\infty} F(x_n) = -\infty;$$

(F_3) *there exists* $k \in (0,1)$ *such that* $\lim_{x \to 0^+} x^k F(x) = 0$;

(F_4) *If* $\forall \ n \in \mathbb{N}, \tau + F(sx_n) \leq F(x_{n-1})$, *we have* $\tau + F(s^n x_n) \leq F(s^{n-1} x_{n-1})$.

Lemma 2.6. *Let* (X,d) *be a* $b-$*metric space with parameter* $s \geq 1$. *Assume that* $\{x_n\}$ *and* $\{y_n\}$ *are* $b-$*convergent to* x *and* y *respectively. Then,*

$$s^{-2} d(x,y) \leq \lim \inf_{n \to \infty} d(x_n, y_n) \leq \lim \sup_{n \to \infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, *then we have* $\lim_{n \to \infty} d(x_n, y_n) = 0$. *Moreover, for eachz* $\in X$,

$$s^{-1} d(x,z) \leq \lim \inf_{n \to \infty} d(x_n, z) \leq \lim \sup_{n \to \infty} d(x_n, z) \leq s d(x,z).$$

3. Main Results

Theorem 3.1. *Let* (X,d) *be a complete* $b-$*metric space with parameter* $s \geq 1$ *and* $f, g : X \to X$ *be two given mappings. If the following conditions hold:*

(1). there is $x_0 \in X$ *with satisfying* $\alpha_s(x_0, fx_0) \geq s^p$;

(2). f, g *are* α_s-*orbital admissible mapping in* X;

(3). if for $x, y \in X$, *we have* $\tau + F(\alpha_s(x,y) d(fx, gy)) \leq F(N(x,y))$,

$$N(x,y) = \max \left\{ d(x,y), d(x,fx), d(y,gy), \frac{1}{2s} d(x,gy), \frac{1}{2s} d(y,fx), \frac{d(x,fx)d(y,gy)\min\{d(x,fx), d(y,gy)\}}{1 + d^2(x,y)} \right\}, \tag{1}$$

where $\tau > 0, \alpha_s : X \times X \to \mathbb{R}, \alpha_s(x,y) = \alpha_s(y,x) \geq s^p, p \geq 2$, $F : \mathbb{R}^+ \to \mathbb{R}$ *is a function that satisfies condition* $F_1 - F_4$ *and properties* (H_{s^p}) *and* (V_{s^p}) *hold, then* f *and* g *possess a common fixed point in* X.

Proof. For each x_0 satisfying condition (1), define sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$ for $n \in \mathbb{N}$. According to conditions (2), we have

$$\begin{aligned} &\alpha_s(x_0, fx_0) \geq s^p \\ &\Rightarrow \alpha_s(fx_0, gfx_0) \geq s^p = \alpha_s(x_1, gx_1) \geq s^p, \\ &\Rightarrow \alpha_s(gx_1, fgx_1) \geq s^p = \alpha_s(x_2, fx_2) \geq s^p. \end{aligned}$$

Hence

$$\alpha_s(x_{2n}, fx_{2n}) \geq s^p, \alpha_s(x_{2n+1}, gx_{2n+1}) \geq s^p,$$

and

$$\alpha_s(x_{2n}, x_{2n+1}) \geq s^p, \alpha_s(x_{2n+1}, x_{2n+2}) \geq s^p.$$

Replacing x by x_{2n} and y by x_{2n+1} in (1), we obtain

$$\tau + F(\alpha_s(x_{2n}, x_{2n+1})d(fx_{2n}, gx_{2n+1})) \leq F(N(x_{2n}, x_{2n+1})).$$

It follows that

$$\tau + F(\alpha_s(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})) \leq F(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}).$$

If $d(x_{2n}, x_{2n+1}) < d(x_{2n+1}, x_{2n+2})$, then

$$\tau + F(\alpha_s(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n+1}, x_{2n+2})),$$

which is a contradiction. Thus, $d(x_{2n}, x_{2n+1}) > d(x_{2n+1}, x_{2n+2})$, and the inequality becomes

$$\tau + F(\alpha_s(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n}, x_{2n+1})).$$

Since $sd(x_{2n+1}, x_{2n+2}) \leq \alpha_s(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})$, we have

$$\tau + F(sd(x_{2n+1}, x_{2n+2})) \leq \tau + F(\alpha_s(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})) \leq F(N(x_{2n}, x_{2n+1})),$$

where

$$N(x_{2n}, x_{2n+1}) = \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1}), \frac{1}{2s}d(x_{2n}, gx_{2n+1}), \frac{1}{2s}d(x_{2n+1}, fx_{2n}), \frac{d(x_{2n}, fx_{2n})d(x_{2n+1}, gx_{2n+1})\min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}}{1 + d^2(x_{2n}, x_{2n+1})}\right\}.$$

It follows that

$$\tau + F(sd(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n}, x_{2n+1})). \tag{2}$$

Similarly, it can be concluded that

$$\tau + F(sd(x_{2n+2}, x_{2n+3})) \leq F(d(x_{2n+1}, x_{2n+2})). \tag{3}$$

Consequently,

$$\tau + F(sd(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})). \tag{4}$$

In view of (F_4), we get

$$\tau + F(s^n d(x_n, x_{n+1})) \leq F(s^{n-1}d(x_{n-1}, x_n)).$$

By calculation,

$$\tau + F(s^n d(x_n, x_{n+1})) \leq F(s^{n-1} d(x_{n-1}, x_n)),$$
$$\tau + F(s^{n-1} d(x_{n-1}, x_n)) \leq F(s^{n-2} d(x_{n-2}, x_{n-1})),$$
$$\vdots$$
$$\tau + F(sd(x_1, x_2)) \leq F(d(x_0, x_1)),$$

which implies that

$$F(s^n d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau. \tag{5}$$

In (5), letting $n \to \infty$, we have $\lim_{n\to\infty} F(s^n d(x_n, x_{n+1})) = -\infty$. Thus we obtain

$$\lim_{n\to\infty} s^n d(x_n, x_{n+1}) = 0.$$

In light of (F_3), one can get that there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (s^n d(x_n, x_{n+1}))^k F(s^n d(x_n, x_{n+1})) = 0.$$

In (5), multiplicating $(s^n d(x_n, x_{n+1}))^k$ at both ends, we have

$$(s^n d(x_n, x_{n+1}))^k F(s^n d(x_n, x_{n+1})) - (s^n d(x_n, x_{n+1}))^k F(d(x_0, x_1)) \leq -(s^n d(x_n, x_{n+1}))^k n\tau. \tag{6}$$

In (6), taking the limit as $n \to \infty$, we deduce

$$\lim_{n\to\infty} n(s^n d(x_n, x_{n+1}))^k = 0.$$

Hence, there exists $n_1 \in \mathbb{N}$, for any $n \geq n_1$, we arrive at

$$n(s^n d(x_n, x_{n+1}))^k \leq 1.$$

Then,

$$s^n d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}.$$

Next we shall prove $\{x_n\}$ is Cauchy. For ease of use, let $d_n = d(y_n, y_{n+1})$. So

$$d(x_n, x_{n+i}) \leq (sd_n + s^2 d_{n+1} + \cdots + s^{i-1} d_{n+i-2} + s^{i-1} d_{n+i-1})$$

and

$$sd_n + s^2 d_{n+1} + \cdots + s^{i-1} d_{n+i-2} + s^{i-1} d_{n+i-1} \leq s^n d_n + s^{n+1} d_{n+1} + \cdots + s^{n+i-2} d_{n+i-2} + s^{n+i-2} d_{n+i-1}$$

$$\begin{aligned} &\leq s^n d_n + s^{n+1} d_{n+1} + \cdots + s^{n+i-2} d_{n+i-2} + s^{n+i-1} d_{n+i-1} \\ &= \sum_{i=n}^{n+i-1} s^i d_i \leq \sum_{i=n}^{\infty} s^i d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Since $k \in (0,1)$, and $\frac{1}{k} > 1$, $\sum\limits_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} = 0$, then

$$\lim_{n\to\infty} (sd_n + s^2 d_{n+1} + \cdots + s^{i-1} d_{n+i-2} + s^{i-1} d_{n+i-1}) = 0, \lim_{n\to\infty} d(x_n, x_{n+i}) = 0.$$

We obtain $\lim\limits_{n\to\infty} d(y_n, y_{n+i}) = 0$ from the completeness of $b-$metric space. Therefore, the sequence is convergent. Then, we can choose $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*, \lim_{n\to\infty} gx_{2n+1} = x^*, \lim_{n\to\infty} fx_{2n} = x^*.$$

In view of the property (H_{s^p}), one can get a subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$ with $\alpha(fx_{2n_k}, x^*) \geq s^p$, for all $k \in \mathbb{N}$. Next, we will prove that $gx^* = x^*$. Replacing x by x_{2n_k} and y by x^* in (1), we have

$$\begin{aligned} \tau + F(\alpha_s(x_{2n_k}, x^*) d^2(fx_{2n_k}, gx^*)) \leq F \Bigg(\max \Bigg\{ & d(x_{2n_k}, x^*), d(x_{2n_k}, fx_{2n_k}), d(x^*, gx^*), \frac{1}{2s} d(x_{2n_k}, gx^*), \\ & \frac{1}{2s} d(x^*, fx_{2n_k}), \frac{d(x_{2n_k}, fx_{2n_k}) d(x^*, gx^*) \min\{d(x_{2n_k}, fx_{2n_k}), d(x^*, gx^*)\}}{1 + d^2(x_{2n_k}, x^*)} \Bigg\} \Bigg). \end{aligned} \tag{7}$$

In (7) , letting $n \to \infty$, and from Lemma 2.6, we obtain

$$\begin{aligned} & s^p \frac{1}{s} d(x^*, gx^*)) \leq \limsup_{n\to\infty} (\alpha_s(x_{2n_k}, x^*) d(fx_{2n_k}, gx^*)) \\ & \leq \max \Bigg\{ \limsup_{n\to\infty} d(x_{2n_k}, x^*), \limsup_{n\to\infty} d(x_{2n_k}, fx_{2n_k}), \limsup_{n\to\infty} d(x^*, gx^*), \limsup_{n\to\infty} \frac{1}{2s} d(x_{2n_k}, gx^*), \\ & \limsup_{n\to\infty} \frac{1}{2s} d(x^*, fx_{2n_k}), \limsup_{n\to\infty} \frac{d(x_{2n_k}, fx_{2n_k}) d(x^*, gx^*) \min\{d(x_{2n_k}, fx_{2n_k}), d(x^*, gx^*)\}}{1 + d^2(x_{2n_k}, x^*)} \Bigg\}. \end{aligned}$$

We have $d(x^*, gx^*) = 0$ and $gx^* = x^*$. Similarly, it can be concluded that $fx^* = x^*$. Therefore, $fx^* = x^* = gx^*$. Then, f and g possess a common fixed point in X. □

Example 3.2. *Let $X = [0, \frac{1}{8}]$ and $d : X \times X \to [0, +\infty)$ be a mapping defined by $d(x, y) = |x - y|^2$, for all $x, y \in X$. Then (X, d) is a complete $b-$metric space with $s = 2$. Define mappings $f, g : X \to X$ by $f(x) = x^2$, $g(x) = \frac{1}{256}$, $\forall\ x \in X$, and $\alpha_s(x, y) = 2^4$, $\forall\ x, y \in X$. It is easy to show that*

$$\begin{aligned} \alpha_s(x, y) d(fx, gy) &= 2^4 \left| fx - gy \right|^2 \\ &= 2^4 \left| x^2 - \frac{1}{256} \right|^2 = 16 \left| x + \frac{1}{16} \right|^2 \left| x - \frac{1}{16} \right|^2 \\ &\leq 16 \left| \frac{1}{8} + \frac{1}{16} \right|^2 \left| x - \frac{1}{16} \right|^2 \\ &= \frac{9}{16} \left| x - \frac{1}{16} \right|^2 \end{aligned}$$

$$\begin{aligned}&\leq \frac{9}{16}d(x,y)\\&\leq \frac{9}{16}N(x,y).\end{aligned}$$

The above equation can be written as

$$\ln\frac{16}{9}+\ln(\alpha_s(x,y)d(fx,gy))\leq \ln(N(x,y)).$$

Let $F(x)=\ln x$, $\tau=\ln\frac{16}{9}$. It follows that all the conditions of Theorem 3.1 are satisfied. Hence we can conclude that f and g possess a common fixed point in X.

4. Conclusion

In the paper, we have established fixed point theorems for double mappings exhibiting $F-$type contractions in the setting of $b-$metric spaces through the introduction of the concept of α_s-orbital admissible mappings. Furthermore, we present a detailed example to elucidate the practical implications and validate the applicability of our derived results. This contribution not only enhances the theoretical landscape of fixed point theorems but also paves the way for future research in the complexity of $b-$metric spaces.

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