International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **12(2)**(2024), 51–59
Available Online: http://ijmaa.in

Multiplicative Neutrosophic Metric Space and Fixed point Result for Multiplicative Neutrosophic Contraction on MNMS

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Abstract

M. Kirişci, N. Simsek, M. Akyigit [21] have established fixed point result for neutrosophic banach contraction. The aim of this paper is to put the notion of multiplicative netrosophic metric space, to define topology on MNMS and to prove a fixed point theorem for a multiplicative neutrosophic contraction mapping.

Keywords: Fixed Point; Neutrosophic Contraction; Generalized Neutrosophic Contraction; Neutrosophic Metric Space.

1. Introduction

The concept of Fuzzy Sets introduced by Zadeh [1] has attracted all the scientific fields since its starting. It is seen that this concept remained failed for real-life situations, to provide enough solution to some problems in time.Atanassov [2] put the idea of Intuitionistic fuzzy sets for such cases. Neutrosophic set (NS) is a new version of the idea of the classical set which is defined by Smarandache [3]. Some of other generalizations are FS [1] interval-valued FS [4], IFS [2], interval-valued IFS [5], the sets paraconsistent, dialetheist, paradoxist, and tautological [6], Pythagorean fuzzy sets [7].
Combining the concepts Probabilistic metric space and fuzziness, fuzzy metric space (FMS) is introduced in [8]. Kaleva and Seikkala [9] have defined the fuzz metric as the nearness between two points with respect to a real number to be a non-negative fuzzy number. In [10] some basic properties of FMS studied and the Baire Category Theorem for FMS proved. Further, some properties of metric structure like separability, countabilityetc are given and Uniform Limit Theorem is proved in [11]. Afterward, FMS has used in the applied sciences such as fixed point theory, image and signal processing, medical imaging, decision-making et al. After itroduction of the intuitionistic fuzzy set (IFS), it was used in all areas where FS theory was studied. Park [12] defined IF metric space (IFMS), which is a generalization of FMSs. Park used George and Veeramani's [10] idea of applying t-norm

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and t-conorm to the FMS meanwhile defining IFMS and studying its basic features. Fixed point theorem for fuzzy contraction mappings is initiated by Heilpern [13]. Bose and Sahani [14] extended the Heilpern's study. Alaca et al. [15] are given fixed point theorems related to intuitionistic fuzzy metric spaces(IFMSs). Fixed point results for fuzzy metric spaces and IFMSs are studied by many researchers [16, 17, 18, 19, 20]. Kirisci et al. [21, 23] defined neutrosophic contractive mapping and gave a fixed point results in complete neutrosophic metric spaces. In [22], Mohamad studied fixed point aprroach in intuitionistic fuzzy metric spaces. In this paper, we introduce the notion of Multiplicative Neutrosophic Metric Space (MNMS) and define topology on it and at last having defined multiplicative neutrosophic contraction we prove some fixed point result on MNMS.

2. Preliminaries

Triangular norms (t-norms) (TN) were initiated by Menger [27]. In the problem of computing the distance between two elements in space, Menger offered using probability distributions instead of using numbers of distance. TNs are used to generalize with the probability distribution of triangle inequality in metric space conditions. Triangular conforms (t-conorms) (TC) know as dual operations of TNs. TNs and TCs are very significant for fuzzy operations (intersections and unions).

Definition 2.1. *Give an operation $\odot : [0,1] \times [0,1] \to [0,1]$. If the operation $\odot$ is satisfying the following conditions, then it is called that the operation $\odot$ is continuous TN (CTN): For $s, t, u, \in [0,1]$,*

(i) $s \odot 1 = s$,

(ii) *If $s \leq u$ and $t \leq v$, than $s \odot t \leq u \odot v$,*

(iii) *$\odot$ is commutative and associate,*

(iv) *$\odot$ is continuous.*

Definition 2.2. *Give an operation $\boxdot : [0,1] \times [0,1] \to [0,1]$. If the operation $\boxdot$ is satisfying the following conditions, then it is called that the operation $\boxdot$ is continuous TC (CTC):*

(i) $s \boxdot 0 = s$,

(ii) *If $s \leq u$ and $t \leq v$, than $s \boxdot t \leq u \boxdot v$,*

(iii) *$\boxdot$ is commutative and associate,*

(iv) *$\boxdot$ is continuous.*

Remark 2.3 ([23]). *Take $\odot$ and $\boxdot$ are CTN and CTC, respectively. For $s, t, v, \in [0,1]$,*

(a) *If $s > t$, then there are u, such that $s \odot u \geq t$ and $s \geq t \boxdot v$.*

(b) *There are p, such that $t \odot t \geq s$ and $s \geq p \boxdot p$.*

Definition 2.4 ([28]). *Take F be an arbitrary set, $\Omega = \{\langle a, HU(a), MU(a), SU(a)\rangle : a \in F\}$ be a NS such that $\Omega : F \times F \times \mathbb{R}+ [0,1]$. Let $\odot$ and $\boxdot$ show the CTN and CTC, respectively. The four tuple $V = (F, \Omega, \odot, \boxdot)$ is called Multiplicative Neutrosophic Metric Space(MNMS) when the following conditions are satisfied. $\forall a, b, c \in F$,*

(i) $0 \leq H(a,b,\lambda) \leq 1, 0 \leq M(a,b,\lambda) \leq 1, 0 \leq S(a,b,\lambda) \leq 1 \forall \lambda \in \mathbb{R}^+$,

(ii) $H(a,b,\lambda) + M(a,b,\lambda) + S(a,b,\lambda) \leq 3$, (*for* $\lambda \in \mathbb{R}+$),

(iii) $H(a,b,\lambda) = 1$(*for* $\lambda > 0$) *if and only if* $a = b$,

(iv) $H(a,b,\lambda) = H(b,a,\lambda)$(*for* $\lambda > 0$),

(v) $H(a,b,\lambda) \odot H(b,c,\mu) \leq H(a,c,\lambda+\mu)$(*for* $\lambda, \mu > 0$),

(vi) $H(a,b,.) : [0,\infty) \to [0,1]$ *is continuous,*

(vii) $\lim \lambda \to \infty H(a,b,\lambda) = 1 (\forall \lambda > 0)$,

(viii) $M(a,b,\lambda) = 0$(*for* $\lambda > 0$) *if and only if* $a = b$,

(ix) $M(a,b,\lambda) = M(b,a,\lambda)$(*for* $\lambda > 0$),

(x) $M(a,b,\lambda) \boxdot M(b,c,\mu) \geq M(a,c,\lambda+\mu)$(*for* $\lambda, \mu > 0$)

(xi) $M(a,b,.) : [0,\infty) \to [0,1]$ *is continuous,*

(xii) $\lim \lambda \to \infty M(a,b,\lambda) = 0 (\forall \lambda > 0)$,

(xiii) $M(a,b,\lambda) = 0$ *(for* $\lambda > 0$) *if and only if* $a = b$

(xiv) $M(a,b,\lambda) = S(b,a,\lambda)$(*for* $\lambda > 0$),

(xv) $S(a,b,\lambda) \boxdot S(b,c,\mu) \geq S(a,c,\lambda+\mu)$(*for* $\lambda, \mu > 0$),

(xvi) $S(a,b,.) : [0,\infty) \to [0,1]$ *is continuous,*

(xvii) $\lim \lambda \to \infty S(a,b,\lambda) = 0 (\forall \lambda > 0)$

(xviii) *If* $\lambda \leq 0$, *then* $H(a,b,\lambda) = 0, M(a,b,\lambda) = 1, S(a,b,\lambda) = 1$.

Then $\Omega = (H, M, S)$ is called Multiplicative Neutrosophic Metric (MNM) on F.

Definition 2.5 ([31]). *Let X be a non empty set. A mapping $d : X \times X \to R$ is said to be a multiplicative metric on X if it satisfies the following condition*

(1) $d(x,y) \geq 1$ *for all* $x, y \in X$

(2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$

(3) $d(x,y) = d(x,z)d(z,y)$ *for all* $x, y, z \in X$.

3. Main Result

Now we define multiplicative neutrosophic metric space (MNMS) and define a topology on it. Then we define multiplicative neutrosophic contraction (MNC) and prove fixed point result for it.

Definition 3.1 ([28]). *Take F be an arbitrary set, $\Omega = \{\langle a, HU(a), MU(a), SU(a)\rangle : a \in F\}$ be a NS such that $\Omega : F \times F \times \mathbb{R}+ \to [0,1]$. Let $\odot$ and $\boxdot$ show the CTN and CTC, respectively. The four tuple $V = (F, \Omega, \odot, \boxdot)$ is called Multiplicative Neutrosophic Metric Space(MNMS) when the following conditions are satisfied. $\forall a, b, c \in F$,*

(i) $0 \leq H(a,b,\lambda) \leq 1, 0 \leq M(a,b,\lambda) \leq 1, 0 \leq S(a,b,\lambda) \leq 1 \ \forall \ \lambda \in \mathbb{R}^+$,

(ii) $H(a,b,\lambda) + M(a,b,\lambda) + S(a,b,\lambda) \leq 3$, *(for $\lambda \in \mathbb{R}^+$),*

(iii) $H(a,b,\lambda) = 1$ *(for $\lambda > 0$) if and only if $a = b$,*

(iv) $H(a,b,\lambda) = H(b,a,\lambda)$ *(for $\lambda > 0$),*

(v) $H(a,b,\lambda) \odot H(b,c,\mu) \leq H(a,c,\lambda\mu)$ *(for $\lambda, \mu > 1$),*

(vi) $H(a,b,.) : [0,\infty) \to [0,1]$ *is continuous,*

(vii) $\lim \lambda \to \infty H(a,b,\lambda) = 1$ *($\forall \ \lambda > 0$),*

(viii) $M(a,b,\lambda) = 0$ *(for $\lambda > 0$) if and only if $a = b$,*

(ix) $M(a,b,\lambda) = M(b,a,\lambda)$ *(for $\lambda > 0$),*

(x) $M(a,b,\lambda) \boxdot M(b,c,\mu) \geq M(a,c,\lambda\mu)$ *(for $\lambda, \mu > 1$),*

(xi) $M(a,b,.) : [0,\infty) \to [0,1]$ *is continuous,*

(xii) $\lim \lambda \to \infty M(a,b,\lambda) = 0$ *($\forall \ \lambda > 0$),*

(xiii) $S(a,b,\lambda) = 0$ *(for $\lambda > 0$) if and only if $a = b$,*

(xiv) $S(a,b,\lambda) = S(b,a,\lambda)$ *(for $\lambda > 0$),*

(xv) $S(a,b,\lambda) \boxdot S(b,c,\mu) \geq S(a,c,\lambda\mu)$ *(for $\lambda, \mu > 1$),*

(xvi) $S(a,b,.) : [0,\infty) \to [0,1]$ *is continuous,*

(xvii) $\lim \lambda \to \infty S(a,b,\lambda) = 0$ *($\forall \ \lambda > 0$),*

(xviii) *If $\lambda \leq 0$, then $H(a,b,\lambda) = 0, M(a,b,\lambda) = 1, S(a,b,\lambda) = 1$.*

Then $\Omega = (H, M, S)$ is called Multiplicative Neutrosophic Metric (MNM) on F.

The functions $H(a,b,\lambda), M(a,b,\lambda), S(a,b,\lambda)$ denote the degree of nearness, the degree of neutralness and the degree of nonnearness between a and b with respect to λ, respectively.

Definition 3.2. *Give V be a MNMS, $0<\varepsilon<1, \lambda>0$ and $\in F$. The set $D(a,\varepsilon,\lambda)=\{b \in F : H(a,b,\lambda) > 1-\varepsilon, M(a,b,\lambda)<\varepsilon, S(a,b,\lambda)<\varepsilon$ } is said to be the open ball (OB) (center a and radius ε with respect to λ).*

Lemma 3.3. *Every $OBD(a,\varepsilon,\lambda)$ is an open set (OS).*

Definition 3.4. *Let $\{a_n\}$ be a sequence in $V=(F,\Omega,\odot,\square)$. Then the sequence converges to a point $a \in F$ if and only if for given $\varepsilon \in (0,1), \lambda>0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

$$H(a_n,a,\lambda) > 1-\varepsilon, M(a_n,a,\lambda)<\varepsilon, S(a_n,a,\lambda)<\varepsilon,$$

or

$$\lim_{n\to\infty} H(a_n,a,\lambda)=1, \lim_{n\to\infty} M(a_n,a,\lambda)=0, \lim_{n\to\infty} S(a_n,a,\lambda)=0. \tag{1}$$

Definition 3.5. *Take V to be a MNMS. A sequence $\{a_n\}$ in F is called Cauchy if for each $\varepsilon>0$ and each $\lambda>0$, there exist $n_0 \in \mathbb{N}$ such that*

$$H(a_n,a_m,\lambda) > 1-\varepsilon, M(a_n,a_m,\lambda)<\varepsilon, S(a_n,a_m,\lambda)<\varepsilon,$$

or

$$\lim_{n,m\to\infty} H(a_n,a_m,\lambda)=1, \lim_{n,m\to\infty} M(a_n,a_m,\lambda)=0, \lim_{n,m\to\infty} S(a_n,a_m,\lambda)=0$$

for all $n,m \geq n_0$. V is called complete if every Cauchy sequence is convergent.

4. Multiplicative Neutrosophic Contractive Mapping

Definition 4.1. *Let V be a MNMS. The mapping $f: F \to F$ is called neutrosophic contraction (MNC) if there exists $k \in (0,1)$ such that*

$$\frac{1}{H(f(a),f(b),\gamma)}-1 \leq \left(\frac{1}{H(a,b,\gamma)}-1\right)^k$$
$$M(f(a),f(b),\lambda) \leq (M(a,b,\lambda))^k,$$
$$S(f(a),f(b),\lambda) \leq (S(a,b,\lambda))^k$$

for each $a,b \in F$ and $\lambda>0$.

Definition 4.2. *Let V be a MNMS and let $f: F \to F$ be a NC mapping. If there exists $c \in F$ such that $f(c)=c$. Then c is called multiplicative neutrosophic fixed point (MNFP) of f.*

Theorem 4.3. *Let V be a complete NMS with (2) in which a NC sequence is a Cauchy sequence. Let $f: F \to F$ is a generalized neutrosophic contraction satisfying conditions of Definition 3.9. Then f has a unique fixe point in V.*

Proof. Let $a \in V$ and $a_n = f^n(a)$ for all $n \in \mathbb{N}$. For each $\gamma > 0$,

$$\begin{aligned}
\frac{1}{H(a_n, a_{n+1}, \gamma)} - 1 &= \frac{1}{H(f(a_{n-1}), f(a_n), \gamma)} - 1 \\
&\leq \left(\frac{1}{H(a_{n-1}, a_n, \gamma)} - 1\right)^k \\
\vdots \quad & \quad \vdots \\
&\leq \left(\frac{1}{H(a_0, a_1, \gamma)} - 1\right)^{k^n}
\end{aligned}$$

which implies that in the same way

$$\begin{aligned}
M(a_n, a_{n+1}, \gamma) &\leq (M(a_0, a_1, \gamma))^{k^n}, \\
S(a_n, a_{n+1}, \gamma) &\leq (S(a_0, a_1, \gamma))^{k^n} \\
\frac{1}{H(a_n, a_{n+p}, \gamma)} - 1 &\leq \frac{1}{*_{i=n}^{p} H\left(a_i, a_{i+1}, \gamma^{\frac{1}{2^{i+1-n}}}\right)} - 1 \\
&\leq *_{i=n}^{p} \left(\frac{1}{H\left(a_i, a_{i+1}, \gamma^{\frac{1}{2^{i+1-n}}}\right)}\right) - 1 \\
&\leq *_{i=n}^{p} \left(\frac{1}{H\left(a_0, a_1, \gamma^{\frac{1}{2^{i+1-n}}}\right)}\right)^{k^i} - 1
\end{aligned}$$

Which tends to 0 as $n \rightarrow \infty$. So that $\lim_{n\rightarrow\infty} H(a_n, a_{n+p}, \gamma) = 1$. In the same way $\lim_{n\rightarrow\infty} M(a_n, a_{n+p}, \gamma) = 0$. And $\lim_{n\rightarrow\infty} S(a_n, a_{n+p}, \gamma) = 0$. Therefore it is a Cauchy sequence in complete NMS V.Hence $\{a_n\}$ is convergent and converges to some $c \in V$. Now we show that this point c is a neutrosophic fixed point of f. For

$$\begin{aligned}
\frac{1}{H(a_{n+1}, f(c), \gamma)} - 1 &= \frac{1}{H(f(a_n), f(c), \gamma)} - 1 \\
&\leq \left(\frac{1}{H(a_n, c, \gamma)} - 1\right)^k \rightarrow 0 \text{ as } n \rightarrow \infty
\end{aligned}$$

So that $\frac{1}{H(c,f(c),\gamma)} - 1 = 0$ and thus $H(c, f(c), \gamma) = 1$. In the same way, we can have

$$M(c, f(c), \gamma) = 0 \quad \text{and} \quad S(c, f(c), \gamma) = 0$$

Therefore $f(c) = c$. To show the uniqueness, let $f(b) = b$ for some $b \in V$. Then for all $\gamma > 0$, we have

$$\begin{aligned}
\frac{1}{H(c, b, \gamma)} - 1 &= \frac{1}{H(f(c), f(b), \gamma)} - 1 \\
&\leq \left(\frac{1}{H(c, b, \gamma)} - 1\right)^k
\end{aligned}$$

Which on repeating yields

$$\frac{1}{H(c,b,\gamma)} - 1 \leq \left(\frac{1}{H(c,b,\gamma)} - 1\right)^{k^n} \to 0 \quad \text{as } n \to \infty$$

Also $M(c,b,\gamma) = M(f(c), f(b), \gamma) \leq (M(c,b,\gamma))^k \leq (M(c,b,\gamma))^{k^n} \to 0$ as $n \to \infty$ so that $S(c,b,\gamma) = 0$. $S(c,b,\gamma) = S(f(c), f(b), \gamma) \leq (S(c,b,\gamma))^k \leq (S(c,b,\gamma))^{k^n} \to 0$ as $n \to \infty$ so that $S(c,b,\gamma) = 0$. Thus $H(c,b,\gamma) = 1$ and $M(c,b,\gamma) = S(c,b,\gamma) = 0$ and hence $c = b$. □

5. Conclusion

The above theorem is an extension to the oneKirisci et al [21] on MNMS. Also it opens an era to establish a fixed point theory on MNMS.

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