International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **12(2)**(2024), 11–18
Available Online: http://ijmaa.in

Applications on Laplace Transforms, Elzaki Transforms and Sumdu Transforms Using Initial and Boundary Conditions

F. A. Almostafa[1,*], Hekmat. M. Elnile[2], Manel Belkhamsa[1], Randa Eltayeb[1], Sana Hafdhi[3]

[1] *Department of MIS-Stat. and Quant. Methods Unit, Faculty of Business and Economics, University of Qassim, Buraidah, KSA*
[2] *Faculty of Technology, Mathematical Sciences and Statistics, Al-Neelain University, Sudan*
[3] *College of Business Administration of Yanbu, Taibah University, Saudi Arabia*

Abstract

The Laplace transform, Elzaki transform, and Sumudu transform methods are used in this paper to solve initial value problems directly without having to convert them to ordinary differential equations. One of the three methods can be used to solve initial value problems using these transformation techniques. By applying the relevant transform to the governing equation and initial conditions, initial value problems can be addressed using the Laplace, Elzaki, and Sumudu transform methods. By doing this, problems can be solved without having to convert them to ODE form. Transforming the initial value problems directly leads to solutions being obtained in one of the three proposed approaches: Laplace, Elzaki, or Sumudu. The added complexity of transforming problems to ODEs before using transforms to find solutions is avoided by this reformulation.

Keywords: Initial Value Problems; Laplace, Elzaki; Sumudu transform.

1. Introduction

In differential equations, a boundary value problem consists of a differential equation coupled with additional constraints known as boundary conditions. Abaker [1] defined a solution to a boundary value problem as one that meets both the governing differential equation and the specified boundary conditions [1]. The passage (2015) establishes that a boundary value problem is a combination of a partial differential equation and boundary conditions [5]. Furthermore, they presented a definition of a solution to such a problem that meets not only the differential relationship but also the boundary constraints. Boundary value problems commonly arise in various fields of physics, as physical differential equations inherently involve boundary conditions. For example, problems dealing with wave equations, such as finding normal modes, are often framed as boundary value problems. A boundary value problem must be well-posed to be applicable. A well-posed problem has a few key

*Corresponding author (F.Almostafa@qu.edu.sa)

properties: there exists a unique solution given the inputs, and the solution continuously depends on the inputs. A well-posed problem has a few key characteristics: there is a unique solution available based on the inputs, and the solution constantly relies on the inputs. Being well-posed is essential for a boundary value problem to be useful in practical applications., the solution to the problem could be unstable or non-existent. Therefore, boundary value problems regularly arise in physics due to their foundations in differential equations. It also emphasizes the importance of a problem being well-posed for solutions to be meaningful and relevant to application.

Integral transforms have solved boundary value problems for many authors by using double Alzaki transforms. Faiza and colleagues [2] utilized the double Alzaki transform technique to solve one-dimensional boundary value problems involving wave and heat equations without having to convert them to ordinary differential equations. By utilizing this transform approach, boundary value problems can be solved without determining the complete integral solution. The Double Alzaki Transform is a reliable and convenient way to address boundary value problems. Two examples from other sources were presented by the authors to demonstrate the scheme. By applying the transform to these test cases embodied in the wave and heat equations, while circumventing the intermediate step of reducing to ODE form, the study validated the appropriateness and usability of this method for boundary value problem-solving. This study, we use [7] Double Alzaki, Laplace [6, 12] and Sumudu Transforms [18] to solve initial and boundary value problem, that is Homogeneous & Inhomogeneous Wave Equation. Henceforth the different problems of initial and boundary value are solved without converting it into Ordinary Differential Equation. So, this method is very reliable & convenient for solving initial and boundary value problem, the scheme is tested through two different examples which are being referred from.

2. Main Result

Example 2.1. *Consider the homogeneous wave equation in the form:*

$$U_{tt} = U_{xx} - 3u, \tag{1}$$

With initial conditions

$$U(x,0) = 0, \qquad U_t(x,0) = 2\cos x, \tag{2}$$

and boundary conditions

$$U(0,t) = \sin 2t, \qquad U_x(0,t) = 0, \tag{3}$$

By taking the double Laplace transform of equation (19)*, we get*

$$s^2F(p,s) - sF(p,0) - \frac{\partial F(p,0)}{\partial t} = p^2F(p,s) - pF(0,s) - \frac{\partial F(0,s)}{\partial x} - 3F(p,s) \tag{4}$$

The single Laplace transform of initial conditions gives

$$F(p,0)=0, \qquad \frac{\partial F(p,0)}{\partial t}=\frac{2p}{p^2+1} \tag{5}$$

The single Laplace transform of boundary conditions

$$F(0,s)=\frac{2}{s^2+4}, \qquad \frac{\partial F(0,s)}{\partial x}=0 \tag{6}$$

By substituting (23) *&* (24) *into equation* (22), *we get*

$$\begin{aligned} s^2F(p,s)-sF(p,0)-\frac{2p}{p^2+1} &= p^2F(p,s)-\frac{2p}{s^2+4}-3F(p,s) \\ \Rightarrow s^2F(p,s)-p^2F(p,s)+3F(p,s) &= \frac{2p}{p^2+1}-\frac{2p}{s^2+4} \\ \Rightarrow \left(s^2-p^2+3\right)F(p,s) &= \frac{2p(s^2+3-p^2)}{(p^2+1)\,(s^2+4)} \end{aligned}$$

That is

$$F(p,s)=\frac{p}{p^2+1}.\frac{2}{s^2+4} \tag{7}$$

Applying inverse double Laplace transform of equation (25) *gives the solution of wave equation* (19) *in the form*

$$U\,(x,t)=\cos x\sin 2t. \tag{8}$$

By taking the double Elzaki transform of equation (19), *we get*

$$\frac{T(u,v)}{v^2}-T(u,0)-v\frac{\partial T(u,0)}{\partial t}=\frac{T(u,v)}{u^2}-T(0,v)-u\frac{\partial T(0,v)}{\partial x}-3T(u,v) \tag{9}$$

The single Elzaki transform of initial conditions gives

$$T(u,0)=0, \quad \frac{\partial T(u,0)}{\partial t}=\frac{2u^2}{u^2+1}, \tag{10}$$

The single Elzaki transform of boundary conditions gives

$$T(0,v)=\frac{2v^3}{4v^2+1}, \quad \frac{\partial T(0,v)}{\partial x}=0, \tag{11}$$

By substituting (28) *&* (29) *into equation* (27), *we get*

$$\begin{aligned} \frac{T(u,v)}{v^2}-v\frac{2u^2}{u^2+1} &= \frac{T(u,v)}{u^2}-\frac{2v^3}{4v^2+1}-3T(u,v) \\ \Rightarrow \left(\frac{u^2-v+3v^2u^2}{v^2u^2}\right)T(u,v) &= \frac{2vu^2}{u^2+1}-\frac{2v^3}{4v^2+1} \\ \Rightarrow \left(u^2-v+3v^2u^2\right)T(u,v) &= \frac{2v^3u^4}{u^2+1}-\frac{2u^2v^5}{4v^2+1} \end{aligned}$$

$$\begin{aligned} \Rightarrow \left(u^2 - v + 3v^2u^2\right) T(u,v) &= \frac{2u^2v^3(\ u^2 + 3v^2u^2 - v^2)}{(u^2+1)\,(4v^2+1)} \\ T(u,v) &= \frac{u^2}{u^2+1}.\frac{2v^3}{4v^2+1} \end{aligned} \tag{12}$$

Applying inverse double Elzaki transform of equation (30) *gives the solution of wave equation* (19) *in the form*

$$U(x,t) = \cos x \sin 2t. \tag{13}$$

By taking the double Sumudu transform of equation (19) *we get*

$$\frac{1}{v^2}M(u,v) - \frac{1}{v^2}M(u,0) - \frac{1}{v}\frac{\partial M(u,0)}{\partial t} = \frac{1}{u^2}M(u,v) - \frac{1}{u^2}M(0,v) - \frac{1}{u}\frac{\partial M(0,v)}{\partial x} - 3M(u,v) \tag{14}$$

The single Sumudu transform of initial conditions gives

$$M(u,0) = 0, \qquad \frac{\partial M(u,0)}{\partial t} = \frac{2}{1+u^2} \tag{15}$$

The single Laplace transform of boundary conditions

$$M(0,v) = \frac{2v}{1+4v^2}, \qquad \frac{\partial M(0,v)}{\partial x} = 0 \tag{16}$$

By substituting (33) *&* (34) *into equation* (32), *we get*

$$\begin{aligned} \frac{1}{v^2}M(u,v) - \frac{1}{v}\frac{2}{1+u^2} &= \frac{1}{u^2}M(u,v) - \frac{1}{u^2}\frac{2v}{1+4v^2} - 3M(u,v) \\ \Rightarrow \frac{1}{v^2}M(u,v) - \frac{1}{u^2}M(u,v) + 3M(u,v) &= \frac{1}{v}\frac{2}{1+u^2} - \frac{1}{u^2}\frac{2v}{1+4v^2} \\ \Rightarrow \left(\frac{u^2 - v^2 + 3v^2u^2}{v^2u^2}\right) M(u,v) &= \frac{1}{v}\frac{2}{1+u^2} - \frac{1}{u^2}\frac{2v}{1+4v^2} \\ \Rightarrow \left(u^2 - v^2 + 3v^2u^2\right) M(u,v) &= \frac{1}{v}\frac{2v^2u^2}{1+u^2} - \frac{1}{u^2}\frac{2v^3u^2}{1+4v^2} \\ &= \frac{2v\left(u^2 - v^2 + 3v^2u^2\right)}{(1+u^2)\,(1+4v^2)} \\ M(u,v) &= \frac{1}{1+u^2}.\frac{2v}{1+4v^2} \end{aligned} \tag{17}$$

Applying inverse double Sumudu transform of equation (36) *gives the solution of wave equation* (19) *in the form*

$$U(x,t) = \cos x \sin 2t. \tag{18}$$

Example 2.2. *Solve the inhomogeneous wave equation*

$$U_{tt} - U_{xx} + U = 2\sin x \tag{19}$$

With initial conditions

$$u(x,0)=\sin x,\quad u_t(x,0)=1 \tag{20}$$

and boundary conditions

$$u(0,t)=\sin t,\quad u_x(0,t)=1 \tag{21}$$

By taking the double Laplace transform of equation (19), *we get*

$$s^2F(p,s)-sF(p,0)-\frac{\partial F(p,0)}{\partial t}-p^2F(p,s)+pF(0,s)+\frac{\partial F(0,s)}{\partial x}+F(p,s)=\frac{1}{s}\frac{2}{p^2+1^2} \tag{22}$$

The single Laplace transform of initial conditions gives

$$F(p,0)=\frac{1}{p^2+1},\qquad \frac{\partial F(p,0)}{\partial t}=\frac{1}{p} \tag{23}$$

The single Laplace transform of boundary conditions

$$F(0,s)=\frac{1}{s^2+1},\qquad \frac{\partial F(0,s)}{\partial x}=\frac{1}{s} \tag{24}$$

By substituting (23) *&* (24) *into equation* (22), *we get*

$$\begin{aligned} s^2F(p,s)-s\frac{1}{p^2+1}-\frac{1}{p}-p^2F(p,s)+p\frac{1}{s^2+1}+\frac{1}{s}+F(p,s)&=\frac{1}{s}\frac{2}{p^2+1^2}\\ \Rightarrow \left(s^2-p^2+1\right)F(p,s)&=\frac{2}{s\left(p^2+1\right)}+\frac{s}{p^2+1}-\frac{1}{s}+\frac{1}{p}-\frac{p}{s^2+1}\\ F(p,s)&=\frac{1}{s(p^2+1^2)}+\frac{1}{p(s^2+1)} \end{aligned} \tag{25}$$

Applying inverse double Laplace transform of equation (25) *gives the solution of Klein Gordon equation* (19) *in the form*

$$U(x,t)=\sin x+\sin t. \tag{26}$$

By taking the double Elzaki, transform of equation (19) *we get,*

$$\frac{T(u,v)}{v^2}-T(u,0)-v\frac{\partial T(u,0)}{\partial t}-\frac{T(u,v)}{u^2}+T(0,v)+u\frac{\partial T(0,v)}{\partial x}+T(u,v)=2\frac{v^2u^3}{1+u^2} \tag{27}$$

The single Elzaki transform of initial conditions gives

$$T(u,0)=\frac{u^3}{u^2+1},\qquad \frac{\partial T(u,0)}{\partial t}=u^2 \tag{28}$$

The single Elzaki transform of boundary conditions gives

$$T(0,v)=\frac{v^3}{1+v^2},\qquad \frac{\partial T(0,v)}{\partial x}=v^2 \tag{29}$$

By substituting (28) *&* (29) *into equation* (27), *we get*

$$\begin{aligned}
\frac{T(u,v)}{v^2} - \frac{u^3}{u^2+1} - vu^2 - \frac{T(u,v)}{u^2} + \frac{v^3}{1+v^2} + uv^2 + T(u,v) = 2\frac{v^2u^3}{1+u^2} \\
\Rightarrow \left(u^2 - v^2 + u^2v^2\right) T(u,v) &= \frac{2v^4u^5}{1+u^2} + \frac{u^5v^2}{u^2+1} - u^3v^4 + v^3u^4 - \frac{u^2v^5}{1+v^2} \\
\Rightarrow \left(u^2 - v^2 + u^2v^2\right) T(u,v) &= \frac{u^3v^2\left(u^2 - v^2 + u^2v^2\right)}{(1+u^2)} \\
&\quad + \frac{u^3v^2\left(u^2 - v^2 + u^2v^2\right)}{(1+v^2)} \\
T(u,v) &= \frac{u^3v^2}{1+u^2} + \frac{u^2v^3}{1+v^2} \qquad (30)
\end{aligned}$$

Applying inverse double Elzaki transform of equation (30) *gives the solution of Klein Gordon equation* (19) *in the form*

$$U(x,t) = \sin x + \sin t. \qquad (31)$$

By taking the double Sumudu transform of equation (19) *we get*

$$\frac{1}{v^2}M(u,v) - \frac{1}{v^2}M(u,0) - \frac{1}{v}\frac{\partial M(u,0)}{\partial t} - \frac{1}{u^2}M(u,v) + \frac{1}{u^2}M(0,v) + \frac{1}{u}\frac{\partial M(0,v)}{\partial x} + M(u,v) = \frac{2u}{(1+u^2)} \qquad (32)$$

The single Sumudu transform of initial conditions gives

$$M(u,0) = \frac{u}{u^2+1}, \qquad \frac{\partial M(u,0)}{\partial t} = 1, \qquad (33)$$

The single Laplace transform of boundary conditions

$$M(0,v) = \frac{v}{v^2+1}, \qquad \frac{\partial M(0,v)}{\partial x} = 1 \qquad (34)$$

By substituting (33) *&* (34) *into equation* (32), *we get*

$$\begin{aligned}
\frac{1}{v^2}M(u,v) - \frac{1}{v^2}\frac{u}{u^2+1} - \frac{1}{v} - \frac{1}{u^2}M(u,v) + \frac{1}{u^2}\frac{v}{v^2+1} + \frac{1}{u} + M(u,v) = \frac{2u}{(1+u^2)} \\
\Rightarrow \left(u^2 - v^2 + v^2u^2\right) M(u,v) &= \frac{2v^4u^3 + u^3v^2}{v^2\left(1+u^2\right)} + \frac{u^2v^2}{v} \qquad (35) \\
&\quad - \frac{1}{u^2}\frac{v^3u^2}{v^2+1} - \frac{u^2v^2}{u} \\
\Rightarrow \left(u^2 - v^2 + v^2u^2\right) M(u,v) &= \frac{2v^2u^3 + u^3}{(1+u^2)} + u^2v - \frac{v^3}{v^2+1} - uv^2 \\
M(u,v) &= \frac{u}{u^2 + 1\frac{v}{v^2+1}} \qquad (36)
\end{aligned}$$

Applying inverse double Sumudu transform of equation (36) *gives the solution of Klein Gordon equation* (19) *in the form*

$$U(x,t) = \sin x + \sin t. \qquad (37)$$

3. Conclusion

In this paper, the method of Laplace transform, Elzaki transform and Sumudu transform has been successfully applied to find the solutions of the homogeneous and inhomogeneous wave equation with initial conditions and boundary conditions by applying double and single Laplace transform, Elzaki transforms and Sumudu transform and inverse double Laplace transform, Elzaki transform and Sumudu transform.The paper concludes by summarizing the methodology and the solutions obtained using Laplace, Elzaki, and Sumudu transforms for the given examples. Overall, the paper demonstrates the utility of Laplace, Elzaki, and Sumudu transforms in solving initial and boundary value problems directly without converting them into ODEs, providing a reliable and convenient approach for solving such problems.

References

[1] Abaker A. Hassab Alla and Yagoub A. Salih, *On Double Elzaki Transform and Double Laplace Transform*, IOSR Journal of Mathematics, (2015).

[2] K. S. Aboodh, F. A. Al Mostafa and H. M. Elnile, *Applications of Double Elzaki Transform to Boundary Value Problem*, International Journal of Mathematics And its Applications, 10(4)(2022), 17-19.

[3] G. Prem Kiran, Bhadane V. H. Pradhan and Satish V. Desale, *Elzaki transform solution of a one dimensional groundwater Recharge through spreading*, International Journal of Engineering Research and applications, 3(6)(2013).

[4] R. Bullough and P. Caudrey, *Solitons Vol 17 in Topics in Current Physics*, Springer, Berlin, (1980).

[5] S. J. Farlow, *Partial Differential Equations for Scientists and engineers*, Dover, New York, (1993).

[6] Thomas John and L'Anson Bromwich, *Inverse Laplace transform*, Duke University, (2008).

[7] Tarig M. Elzaki, Salih M. Elzaki and Elsayed A. Elnour, *On the New Integral Transform "ELzaki Transform" Fundamental Properties Investigations and Applications*, Global Journal of Mathematical Sciences: Theory and Practical, 4(1)(2012), 1-13.

[8] L. Lam, *Nonlinear Physics for Beginners*, World Scientific, Singapore, (1998).

[9] Lokenath Debnath and Dambaru Bhatta, *Integral transforms and their applications*, Second Editition, Chapman & Hall, CRC, (2007).

[10] M. J. Ablowitz and P. A. Clarkson, *Solitons: Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, (1991).

[11] P. G. Drazin and R. S. Johnson, *Solitons: an Introduction*, Cambridge University Press, Cambridge, (1996).

[12] Pmar Ozel and L Ozlem Bayar, *The double Laplace transform*, (2012).

[13] J. Boussinesq, *Theorie des ondes et des remous qui se propagent le long d Run canal rectangulaire horizontal, en communiquant au liquide continu dans ce canal des vitesses sensiblement pareilles de la surface au fond*, J.Math. Pures Appl., 17(1872), 55-108.

[14] J. Boyd, *Weakly Nonlocal Solitary Waves and Beyond-All-Orders Asymptotics*, Kluwer, Boston, (1998).

[15] J. D. Logan, *An Introduction to Nonlinear Partial Differential Equations*, John Wiley, New York, (1994).

[16] J. H. He, *A variational iteration method-a kind of nonlinear analytical technique: Some examples*, Int. J. Nonlinear Mech., 34(1999), 699-708.

[17] J. H. He, *Variational iteration method: some recent results and new interpretations*, J. Comput. and Appl. Math., 207(1)(2007), 3-17.

[18] Jean M. Tchuenche and Nyimvua S. Mbare, *An Application of the double Sumudu transform*, Mathematics Department, University of Dar es Salaam, (2007).

[19] Kiryl Tsishchanka, *Improper Integrals*, Notes, (2010).