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Perfect 2-colorings of 4-regular Circulant Graphs

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Abstract

In this study, we determine the perfect 2-colorings of $Ci_{2n}(1,n-1)$, a class of 4-regular circulant graphs with even order. We determine the parameter matrices that result to these perfect 2-colorings.

Keywords: graph colorings; perfect 2-colorings; circulant graphs; regular graphs.
2020 Mathematics Subject Classification: 05C78, 05C15.

1. Introduction

Let $G(V,E)$ be a finite undirected simple connected r-regular graph with vertex set $V(G)$ and edge set $E(G)$. A *perfect* 2*-coloring* of G or *perfect coloring* of G using 2 colors with 2×2 parameter matrix $M=\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{bmatrix}$ is an onto mapping $\mathcal{C}: V(G)\to\{\mathbf{1},\mathbf{2}\}$ of $V(G)$ to a set of 2 colors such that any vertex $v\in V(G)$ with $\mathcal{C}(v)=\mathbf{i}$ is adjacent to m_{ij} number of vertices assigned with color $\mathbf{j}$. When G has a perfect 2-coloring with parameter matrix M, we say that G is perfect 2-colorable with M.

In a perfect coloring of G using two colors, we let the first color as color **1** or yellow, and the second color as **2** or red. In a perfect 2-coloring of G, the parameter matrix $M=\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{bmatrix}$ means that every yellow vertex is adjacent to precisely m_{11} vertices assigned with color **1** or yellow and m_{12} vertices assigned with color **2** or red, and every red vertex is adjacent to precisely m_{21} vertices assigned with yellow and m_{22} vertices assigned with red. Whenever there exists a perfect 2-coloring of G with matrix M, we say that M is *admissible* for G, or that the perfect 2-coloring of G agrees with M.

Figure 1(a) shows a perfect 2-coloring of a graph with parameter matrix $M=\begin{bmatrix} 2 & 2 \\ 4 & 0\end{bmatrix}$. Every vertex colored yellow is adjacent to two vertices colored yellow and two vertices colored red. Moreover, every vertex colored red is adjacent only to vertices colored yellow. Another matrix admissible for G is $M'=\begin{bmatrix} 0 & 4 \\ 2 & 2\end{bmatrix}$. The perfect 2-coloring of G that agrees with M' is shown in Figure 1(b). In this coloring,

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every vertex colored yellow is adjacent only to vertices colored red, and every vertex colored red is adjacent to two vertices colored yellow and two vertices colored red.

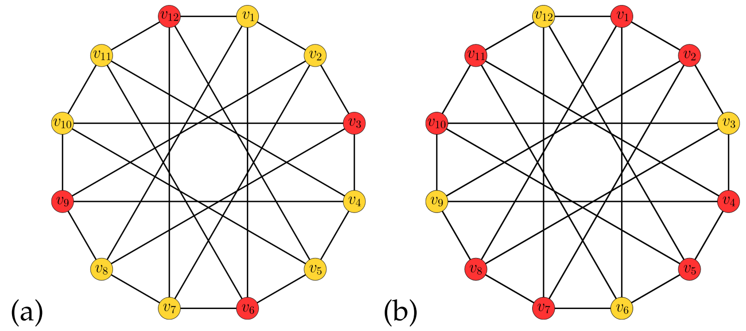

Figure 1: Perfect 2-coloring of $Ci_{12}(1,5)$ with parameter matrix (a) $\begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$ and (b) $\begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}$

In this work, we let $G := Ci_{2n}(1, n-1)$, a 4-regular circulant graph. In the next section we present related works on perfect 2-colorings. We describe the circulant graphs $Ci_{2n}(1, n-1)$ in section 3. Finally, in section 4 we present our main result, the enumeration of the perfect 2-colorings of the 4-regular circulant graphs $Ci_{2n}(1, n-1)$. Some results on perfect 2-colorings of circulant graphs $Ci_{2n}(1, k)$ are presented in section 5.

2. Related Works

Perfect k-coloring of a graph is closely related to a historical mathematics problem $-$ the existence of completely regular codes [17]. Perfect 2-coloring of a graph with parameter matrix $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ is a nonlinear completely regular code with intersection array $\{m_{12}; m_{21}\}$. A more detailed account between completely regular codes and perfect coloring is presented in [15], a survey paper on completely regular codes.

Over the years there had been several studies on determining the existence of perfect colorings on graphs; that is equivalent to listing all admissible matrices for particular family of graphs. Avgustinovich and colleagues investigated the perfect 2-colorings of Johnson graphs [12,13]. In [11], they also worked on perfect 2-colorings of transitive cubic graphs, as well as some 4-regular graphs: prism graphs, crossed prism graphs, chordal cycles and Mobius ladders. In [19], Gavrilyuk and Goryainov investigated perfect coloring using two colors of the Johnson graph $J(2n+1, 3)$.

Among the focus of the study of Alaeiyan and Karami are perfect 2-colorings of Platonic graphs $-$ graphs that form the skeleton of Platonic solids [7], and of the generalized Petersen graph $GP(n, 2)$, $n \geq 5$ [6]. Their other works with their colleagues focus on $GP(n, 3)$ [21], Johnson graph $J(9, 4)$ [4], cubic graph G with order $|G| \leq 10$ [8], quartic graph G with order $|G| \leq 8$ [20], and more recently on toroidal grid graphs in [5,25].

Fon-Der-Flaass worked on perfect 2-colorings of hypercube graphs [18], De Winter and Metsch on

grassmann graph of planes [16], and Bespalov and colleagues on hamming graphs [14]. In [24], Parshina established results on perfect 2-colorings of infinite circulant graphs whose set of distances constitutes the segment of naturals $[1,n]$. Most recently, in [26] perfect colorings using two colors of special families of 4-regular circulant graphs: shadow graphs of cycles and total graphs of cycles.
Perfect 3-colorings of some graphs had also been studied by Alaeiyan and colleagues. These graphs include Johnson graphs [2,3], platonic graphs [10], and cubic graph G such that $|G| = 10$[9]. Lisitsyna also worked on perfect 3-colorings of prism graphs and Mobius ladders [22]. In [23], perfect 3-colorings of graph G such that $|G| = 9$ was investigated.

3. Circulant Graphs $Ci_{2n}(1, n-1)$

Let S be a subset of the nonzero elements of the ring $\mathbb{Z}_p$ of integers modulo p. The *circulant graph* $Ci_p(S)$ is a graph with vertex set $V(Ci_p(S)) = \{v_1, v_2, \ldots, v_p\}$ and edge set $E(Ci_p(S)) = \{v_i v_j : i - j \pmod{p} \in S\}$. The *order* of $Ci_p(S)$, denoted by $|V(Ci_p(S))|$, is the number of its vertices. The neigborhood of a vertex $v \in Ci_p(S)$, denoted by $N(v)$, is a set of vertices in $Ci_p(S)$ that are adjacent to v.
The circulant graphs $Ci_{2n}(1, n-1)$, are 4-regular graphs that are of order $2n$. Here $S = \{1, n-1\}$. Shown in Figures 2a, 2b and 2c are $Ci_{10}(1,4)$, $Ci_{12}(1,5)$ and $Ci_{14}(1,6)$, respectively. If we label the vertices in cyclic order (as in Figure 2), observe that $N(v_1) = N(v_{n+1}) = \{v_2, v_{2n}, v_n, v_{n+2}\}$. Generally, $N(v_i) = \{v_{i+1}, v_{i-1}, v_{n+i}, v_{n+i+2}\}$ where the indices of the vertices are under modulo $2n$.

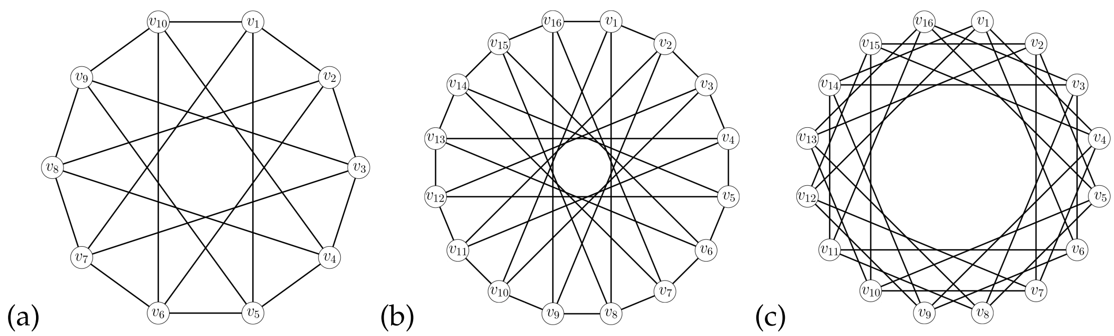

Figure 2: Circulant graphs $Ci_{2n}(1, n-1)$ of order (a)10, and (b)16; (c) circulant graph $Ci_{16}(3,5)$

The result in [1] allows us to determine two circulant graphs that are isomorphic.

Proposition 3.1 ([1]). *Let S be a subset of nonzero elements of the ring $\mathbb{Z}_p$ of integers modulo p. If $S' = \{ks \pmod{p} : s \in S, k$ is a unit in $\mathbb{Z}_p\}$, then $Ci_n(S) \cong Ci_n(S')$.*

Now consider $Ci_{2n}(1, n-1)$. Let k is a unit in $\mathbb{Z}_{2n}$, that is the greatest common divisor $gcd(2n, k) = 1$. Then $Ci_{2n}(1, n-1) \cong Ci_{2n}(k, kn-k \pmod{2n})$ by Proposition 3.1.
As an example, consider the circulant graph $Ci_{16}(1,7)$. The units in $\mathbb{Z}_{16}$ are $1,3,5,7,9,11,13,15$. We have $Ci_{16}(1,7) \cong Ci_{16}(3,5) \cong Ci_{16}(1,9) \cong Ci_{16}(3,11) \cong Ci_{16}(9,15)$. Figures 2(b) and 2(c) present $Ci_{16}(1,7) = Ci_{16}(7,15) = Ci_{16}(1,9) = Ci_{16}(9,15)$ and $Ci_{16}(3,5) = Ci_{16}(3,11)$, respectively.

4. Enumeration of Perfect 2-colorings of $Ci_{2n}(1, n-1)$

To enumerate all admissible matrices for graph $Ci_{2n}(1, n-1)$, we first present how to identify parameter matrices that correspond to equivalent perfect 2-colorings of $Ci_{2n}(1, n-1)$.

Remark 4.1 ([11]). *Two 2×2 admissible matrices M_1 and M_2 for graph G are called equivalent if M_2 is a permutation of rows and columns of M_1. The permutation of rows and columns in the parameter matrix corresponds to the reassignment of colors or permutation of colors.*

Now consider the admissible matrices $M = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$ and $M' = \begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}$ for $Ci_{12}(1, 5)$. The corresponding perfect 2-colorings that agree with these matrices are presented in Figure 1. M and M' are equivalent. The permutation of rows and columns of M to obtain M' corresponds to interchanging the colors (red to yellow, and yellow to red) in the perfect 2-coloring of $Ci_{12}(1, 5)$ with parameter matrix M.
The next result, which also appears in [5,20,26] and are equivalent to the list in [25], list the possible admissible matrices for $Ci_{2n}(1, n-1)$. Given an admissible matrix $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ for a 4-regular graph, such as $Ci_{2n}(1, n-1)$, we have $m_{11} + m_{12} = 4$ and $m_{21} + m_{22} = 4$. Furthermore, note that $m_{12} \neq 0$ or $m_{21} \neq 0$; otherwise we have a coloring of $Ci_{2n}(1, n-1)$ with single color.

Lemma 4.2 ([5,20,25,26]). *In a perfect 2-coloring of a 4-regular graph, the only possible parameter matrices up to equivalence are the following:*

$$M_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad M_2 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \quad M_3 = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \quad M_4 = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} \quad M_5 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$M_6 = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \quad M_7 = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix} \quad M_8 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad M_9 = \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} \quad M_{10} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$$

Lemma 4.3. *[11] Consider a perfect 2-coloring of G with parameter matrix $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$.*

(i) The ratio of the number of vertices of G colored yellow and the number of vertices of G colored is $\frac{m_{21}}{m_{12}}$;

(ii) The order G of G is divisible by $\frac{m_{12} + m_{21}}{gcd(m_{12}, m_{21})}$, where $gcd(m_{12}, m_{21})$ is the greatest common divisor of m_{12} and m_{21}.

Theorem 4.4. *Consider the circulant graph $Ci_{2n}(1, n-1), n \geq 2$ and the parameter matrices in Lemma 4.2. The following holds:*

(i) for any n, $Ci_{2n}(1, n-1)$ is not perfect 2-colorable with M_1;

(ii) for any n, $Ci_{2n}(1, n-1)$ is not perfect 2-colorable with M_2;

(iii) for n divisible by 4, $Ci_{2n}(1, n-1)$ is perfect 2-colorable with M_3;

(iv) for any n, $Ci_{2n}(1, n-1)$ is not perfect 2-colorable with M_4;

(v) for any n, $Ci_{2n}(1, n-1)$ is perfect 2-colorable with M_5;

(vi) for any n, $Ci_{2n}(1, n-1)$ is not perfect 2-colorable with M_6;

(vii) for n divisible by 3, $Ci_{2n}(1, n-1)$ is perfect 2-colorable with M_7;

(viii) for any n, $Ci_{2n}(1, n-1)$ is not perfect 2-colorable with M_8;

(ix) for any n, $Ci_{2n}(1, n-1)$ is not perfect 2-colorable with M_9;

(x) for n divisible by 2, $Ci_{2n}(1, n-1)$ is perfect 2-colorable with M_{10}.

Proof. In $Ci_{2n}(1, n-1)$, for any vertex v_i, its neighborhood $N(v_i) = \{v_j : j \equiv i+1, i-1, i+n-1, i-n+1 \pmod{2n}\}$.

(i) Suppose $M_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is admissible to $Ci_{2n}(1, n-1)$. Let $\mathcal{C}(v_1) = \mathbf{1}$ (yellow). We have four subcases:

(a) Suppose $\mathcal{C}(v_2) = \mathbf{2}$ (red). Then $\mathcal{C}(v_{2n}) = \mathcal{C}(v_n) = \mathcal{C}(v_{n+2}) = \mathbf{1}$ (Figure 3(a)). Now, $N(v_2) = \{v_1, v_3, v_{n+1}, v_{n+3}\}$. Moreover, $N(v_{n+1}) = \{v_n, v_{n+2}, v_{2n}, v_2\}$. By M_1, vertex v_{n+1} is both assigned with color yellow and red. It was assigned with yellow since three of its adjacent vertices are assigned with color yellow. Moreover, it was also assigned with red since v_2 is colored red adjacent to v_1 colored yellow; the remaining adjacent vertices of v_1 should be assigned with color red. Hence, a contradiction.

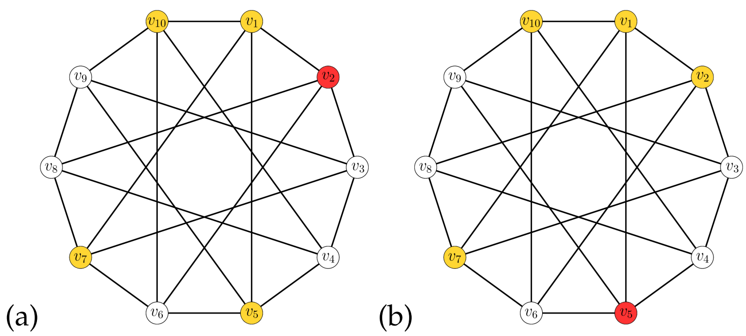

Figure 3: Initial coloring of vertices of $Ci_{10}(1, 4)$ with parameter matrix M_1

(b) Suppose $\mathcal{C}(v_{2n}) = \mathbf{2}$ (red). We obtain a proof analogous to (a).

(c) Suppose $\mathcal{C}(v_n) = \mathbf{2}$ (red). Then $\mathcal{C}(v_2) = \mathcal{C}(v_{2n}) = \mathcal{C}(v_{n+2}) = \mathbf{1}$ (Figure 3(b)). Now, $N(v_n) = \{v_{n-1}, v_{n+1}, v_1, v_{2n-1}\}$. Moreover, $N(v_{n+1}) = \{v_n, v_{n+2}, v_{2n}, v_2\}$. By M_1, vertex v_{n+1} is both assigned with color yellow and red. It was assigned with yellow since three of its adjacent vertices are assigned with color yellow. Moreover, it was assigned with red since v_n is colored red adjacent to v_1 colored yellow; the remaining adjacent vertices of v_n should be assigned with color red. Hence, a contradiction.

(d) Suppose $\mathcal{C}(v_{n+2}) = \mathbf{2}$ (red). We obtain a proof analogous to (c).

(ii) Suppose $M_2 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ is admissible to $Ci_{2n}(1, n-1)$. Let $\mathcal{C}(v_1) = \mathbf{2}$ (red). We have six subcases: (a) $\mathcal{C}(v_2) = \mathcal{C}(v_{2n}) = \mathbf{2}$ (red) (Figure 4a), (b) $\mathcal{C}(v_n) = \mathcal{C}(v_{n+2}) = \mathbf{2}$ (Figure 4b), (c) $\mathcal{C}(v_2) = \mathcal{C}(v_n) = \mathbf{2}$ (Figure 4c), (d) $\mathcal{C}(v_{2n}) = \mathcal{C}(v_{n+2}) = \mathbf{2}$ (Figure 4d), (e) $\mathcal{C}(v_{2n}) = \mathcal{C}(v_n) = \mathbf{2}$ (Figure 4e), and (f) $\mathcal{C}(v_2) = \mathcal{C}(v_{n+2}) = \mathbf{2}$ (Figure 4f).

In each case, the other two adjacent vertices of of v_1 is assigned with color yellow. Now, observed that vertex v_{n+1} is both assigned with color red and yellow, a contradiction. Vertex v_{n+1} is assigned with color red since it is adjacent to two yellow and two red vertices. Moreover, vertex v_{n+1} is assigned with color yellow since it is adjacent to a yellow vertex that is adjacent to a red vertex. For instance in Figure 4a, vertex v_{n+1} is adjacent to yellow vertex v_n that is adjacent to red vertex v_1.

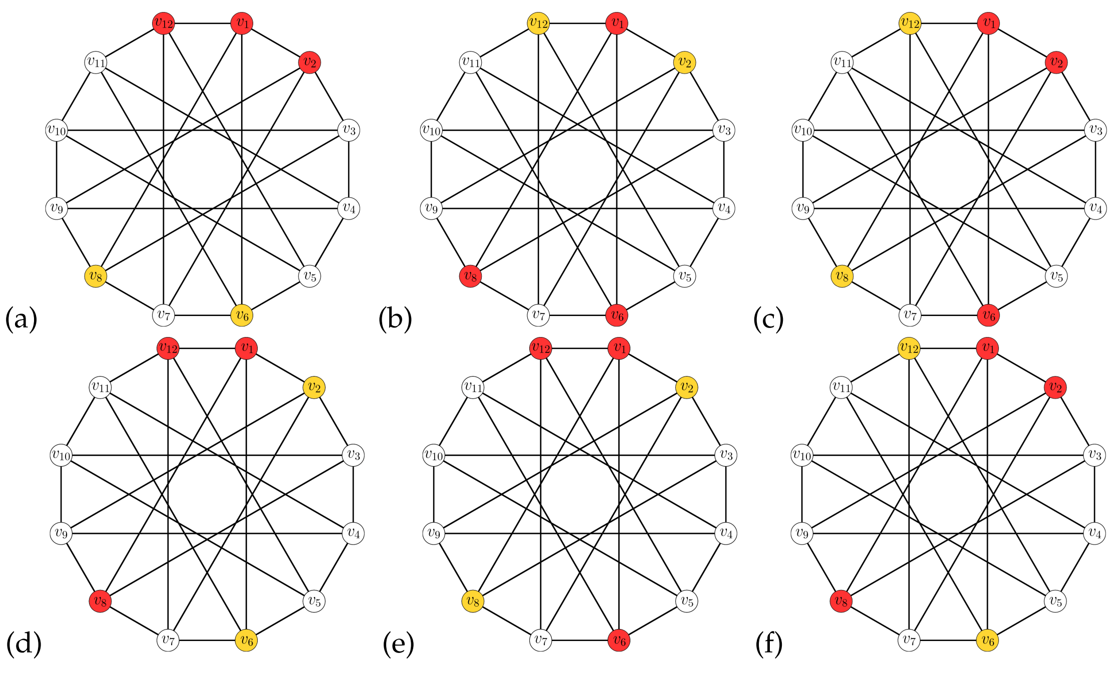

Figure 4: Initial coloring of vertices of $Ci_{12}(1,5)$ with parameter matrix M_2

(iii) By Lemma 4.3, if $|V(Ci_{2n}(1, n-1))| = 2n$ is not divisible by 4; that is, n is not divisible by 2 ($n \equiv 1 \pmod 2$), then $M_3 = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ is not admissible for $Ci_{2n}(1, n-1)$. Now suppose $n \equiv 0 \pmod 2$. We have two subcases:

(a) Suppose $n \equiv 2 \pmod 4$. Suppose M_3 is admissible to $Ci_{2n}(1, n-1)$. Note that in this coloring, every red vertex is adjacent to exactly one red vertex. Moreover, each vertex is adjacent to three yellow vertices. By Lemma 4.3, there are $2t+1$ red vertices and $6t+3$ yellow vertices where $2n = 8t+4$. There is an odd number of red vertices, a contradiction since every red vertex is adjacent to exactly one red vertex.

(b) Suppose $n \equiv 0 \pmod 4$. We obtain a perfect 2-coloring (Figure 5) given by the mapping below

$$\mathcal{C}(v_i) = \begin{cases} \mathbf{1}\text{ (red)} & \text{if } i \equiv 1,2 \pmod 4, 1 \le i \le n \\ \mathbf{2}\text{ (yellow)} & \text{if } n+1 \le i \le 2n \text{ or if } i \equiv 0,3 \pmod 4, 1 \le i \le n \end{cases}$$

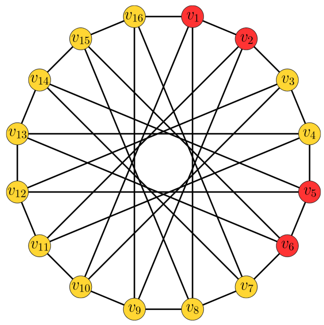

Figure 5: Perfect 2-coloring of $Ci_{16}(1,7)$ with parameter matrix M_3

(iv) Suppose $M_4 = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$ is admissible to $Ci_{2n}(1, n-1)$. Let $\mathcal{C}(v_1) = \mathbf{2}$ (red). Then $\mathcal{C}(v_2) = \mathcal{C}(v_{2n}) = \mathcal{C}(v_n) = \mathcal{C}(v_{n+2}) = \mathbf{1}$ (yellow) (Figure 6(a)). By M_4, vertex v_{n+1} is both assigned with color red and yellow, a contradiction. It was assigned with red since its adjacent vertices are assigned with color yellow. Moreover, it was assigned with yellow since v_2 (colored yellow) is adjacent to v_1 colored red.

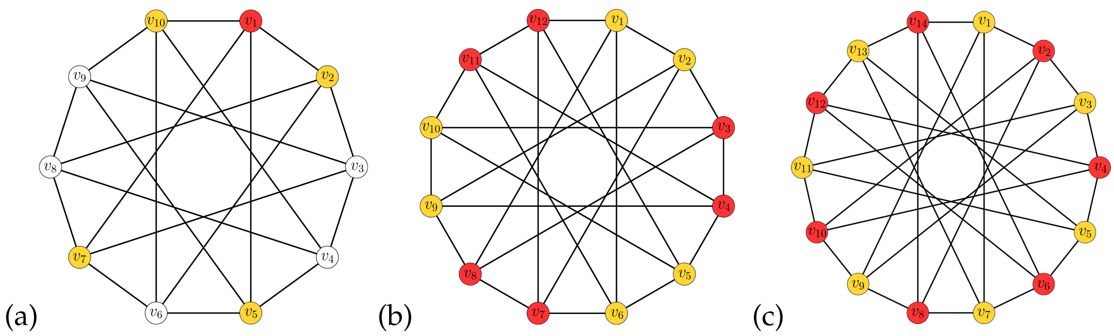

Figure 6: (a) Initial coloring of vertices of $Ci_{10}(1,4)$ with parameter matrix M_4; (b)-(c) perfect 2-colorings of $Ci_{10}(1,4)$ with parameter matrix M_5

(v) Consider $M_5 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. We have two subcases.

(i) Suppose $n \equiv 0 \pmod 2$. We obtain a perfect 2-coloring (Figure 6(b)) of $Ci_{2n}(1, n-1)$ given by the mapping

$$\mathcal{C}(v_i) = \begin{cases} \mathbf{1} \text{ (yellow)} & \text{if } i \equiv 1,2 \pmod 4 \\ \mathbf{2} \text{ (red)} & \text{if } i \equiv 0,3 \pmod 4 \end{cases}$$

(ii) Suppose $n \equiv 1 \pmod 2$. We obtain a perfect 2-coloring (Figure 6(c)) of $Ci_{2n}(1, n-1)$ given by the mapping

$$\mathcal{C}(v_i) = \begin{cases} \mathbf{1} \text{ (yellow)} & \text{if } i \equiv 1 \pmod 2 \\ \mathbf{2} \text{ (red)} & \text{if } i \equiv 0 \pmod 2 \end{cases}$$

(vi) By Lemma 4.3, if $|V(Ci_{2n}(1, n-1))| = 2n$ is not divisible by 5; that is, n is not divisible by 5, then $M_6 = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$ is not admissible for $Ci_{2n}(1, n-1)$. Suppose n is divisible by 5. Further, suppose $M_6 = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$ is admissible to $Ci_{2n}(1, n-1)$. In this coloring, every red vertex is adjacent to exactly one red vertex. Moreover every yellow vertex is adjacent to two vertices colored yellow and two vertices colored red. By Lemma 4.3, in the coloring there are $3t$ yellow vertices and $2t$ red vertices where $2n = 3t + 2t$ for some positive integer t. Hence, if the graph has $2n$, then the number of red vertices should be $\frac{4n}{5}$.

Now suppose $\mathcal{C}(v_1) = \mathcal{C}(v_2) = \mathbf{2}$ (red). Then all vertices in $(N(v_1) \cup N(v_2)) - \{v_1, v_2\} = \{v_{2n}, v_3, v_n, v_{n+1}, v_{n+2}, v_{n+3}\}$ are assigned with color $\mathbf{1}$ (yellow) (Figure 7(a)). Hence for every adjacent red vertices there are six yellow vertices. The assignment of colors to these eight vertices is completely determined by the assignment of color red to two adjacent vertices. The assignment of colors to the other uncolored vertices are not completely determined by these previous assignments of colors.

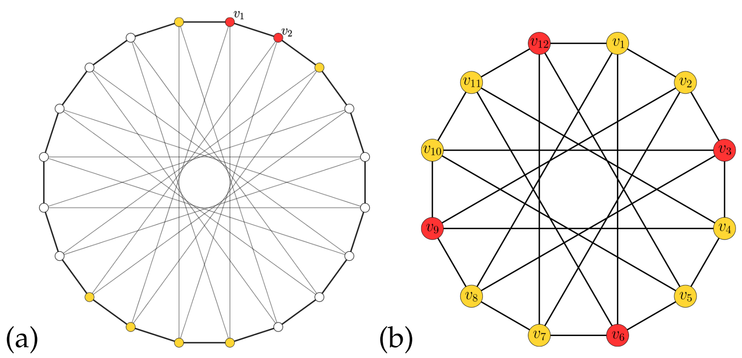

Figure 7: (a) Initial coloring of vertices of $Ci_{20}(1, 9)$ with parameter matrix M_6; (b) perfect 2-coloring of $Ci_{12}(1, 5)$ with parameter matrix M_7

Assigning two adjacent vertices (where one of them is adjacent to one of the yellow vertices) with color red determines the assignment of colors of six vertices (including the two red vertices). This time the number of red vertices is 4.

Continuing the same process until almost all vertices or all vertices are assigned with a color, we can observed that there is a largest positive integer k such that $2n \geq 8 + 6k$, and the number of vertices assigned with red color is $2 + 2k$. Now, note that $\frac{4n}{5} > \frac{4n}{6} \geq \frac{16 + 12k}{6} > 2 + 2k$. That is, a contradiction to the Lemma 4.3. Note that when $2n > 8 + 6k$ for some largest positive integer k, the assignment of colors to the uncolored vertices arrive with a contradiction to Lemma 4.3.

(vii) By Lemma 4.3, if $|V(Ci_{2n}(1, n-1))| = 2n$ is not divisible by 3; that is, n is not divisible by 3, then $M_7 = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$ is not admissible for $Ci_{2n}(1, n-1)$. Let n be divisible by 3. Consider the coloring

$$\mathcal{C}(v_i) = \begin{cases} \mathbf{1} \text{ (yellow)} & \text{if } i \equiv 1, 2 \pmod 3 \\ \mathbf{2} \text{ (red)} & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

This perfect 2-coloring (Figure 7(b)) agrees with parameter matrix M_7.

(viii) Suppose $M_8 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is admissible for $Ci_{2n}(1, n-1)$. Note that in this coloring, every yellow vertex is adjacent to exactly one yellow vertex. Similarly, every red vertex is adjacent to exactly one red vertex. We have two subcases:

(a) Suppose $n \equiv 1 \pmod 2$. By Lemma 4.3, there are n yellow vertices and n red vertices. There is an odd number of yellow vertices, a contradiction since every yellow vertex is adjacent to exactly one yellow vertex.

(b) Suppose $n \equiv 0 \pmod 2$. Let $\mathcal{C}(v_1) = \mathbf{1}$ (yellow). We have four subcases:

(1) Suppose $\mathcal{C}(v_2) = \mathbf{1}$ (yellow). Then $\mathcal{C}(v_{2n}) = \mathcal{C}(v_n) = \mathcal{C}(v_{n+2}) = \mathbf{2}$ (red). Now, $N(v_2) = \{v_1, v_3, v_{n+1}, v_{n+3}\}$. Moreover, $N(v_{n+1}) = \{v_n, v_{n+2}, v_{2n}, v_2\}$ (Figure 8(a)). By M_8, vertex v_{n+1} is both assigned with color yellow and red, a contradiction. It was assigned with yellow since three of its adjacent vertices are assigned with color red. Moreover, it was assigned with red since v_2 is colored yellow adjacent to v_1 colored yellow.

(2) Suppose $\mathcal{C}(v_{2n}) = \mathbf{1}$ (yellow). The proof is analogous to (a).

(3) Suppose $\mathcal{C}(v_n) = \mathbf{1}$ (yellow). Then $\mathcal{C}(v_2) = \mathcal{C}(v_{2n}) = \mathcal{C}(v_{n+2}) = \mathbf{2}$ (red). Now, $N(v_n) = \{v_{n-1}, v_{n+1}, v_1, v_{2n-1}\}$. Moreover, $N(v_{n+1}) = \{v_n, v_{n+2}, v_{2n}, v_2\}$ (Figure 8(b)). By M_8, vertex v_{n+1} is both assigned with color yellow and red, a contradiction. It was assigned with yellow since three of its adjacent vertices are assigned with color red. Moreover, it was assigned with red since v_n is colored yellow adjacent to v_1 colored yellow.

(4) Suppose $\mathcal{C}(v_{n+2}) = \mathbf{1}$ (yellow). The proof is analogous to (c).

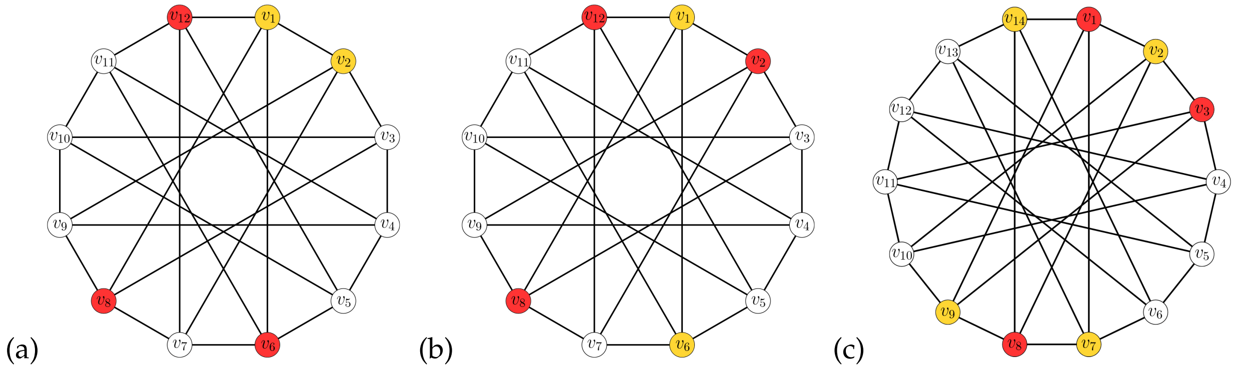

Figure 8: (a)-(b) Initial coloring of vertices of $Ci_{12}(1,5)$ with parameter matrix M_8; (c) Initial coloring of vertices of $Ci_{14}(1,6)$ with parameter matrix M_9

(ix) Suppose $M_9 = \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix}$ is admissible to $Ci_{2n}(1,n-1)$. Let $\mathcal{C}(v_1) = \mathbf{2}$ (red). Then $\mathcal{C}(v_2) = \mathcal{C}(v_{2n}) = \mathcal{C}(v_n) = \mathcal{C}(v_{n+2}) = \mathbf{1}$ (yellow) (Figure 8(c)). Now, observe v_3, adjacent to more than one, in particular two, yellow vertices v_2 and v_{n+2}. Hence, v_3 should be assigned with color red. By M_9, vertex v_{n+3} is both assigned with color yellow and red, a contradiction. Vertex v_{n+2} is adjacent to three red vertices v_1, v_3, v_{n+1}, so its remaining adjacent vertex v_{n+3} should be assigned with color yellow. At the same time, v_{n+3} is adjacent to more than one, in particular two, yellow vertices v_2 and v_{n+2} implying that v_{n+3} should be assigned with color red.

(x) Consider $M_{10} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$. We describe the possible coloring of $Ci_{2n}(1,n-1)$. In this coloring, every red (respectively yellow) vertex is adjacent only to yellow (red) vertex. Let $\mathcal{C}(v_1) = \mathbf{1}$. Under this initial assignment of colors, the assignment of colors to other vertices can be determined. In particular, $\mathcal{C}(v_2) = \mathcal{C}(v_{2n}) = \mathcal{C}(v_n) = \mathcal{C}(v_{-n+2}) = \mathbf{2}$. Consequently, the assignment of colors to the uncolored vertices are also determined based on the previous assignments. That is, the vertices in the cycle C_{2n} are colored alternately. We have two subcases.

(a) Suppose $n \equiv 1 \pmod 2$. Vertex v_n is both adjacent to v_1 (colored yellow) and v_{n-1} (colored red) (Figure 9(a)). In other words, v_n is assigned with both color yellow and red, a contradiction.

(b) Suppose $n \equiv 0 \pmod 2$. We obtain a perfect 2-coloring (Figure 9(b)) given by the mapping below

$$\mathcal{C}(v_i) = \begin{cases} \mathbf{1} \text{ (yellow)} & \text{if } i \equiv 1 \pmod 2 \\ \mathbf{2} \text{ (red)} & \text{if } i \equiv 0 \pmod 2 \end{cases}$$

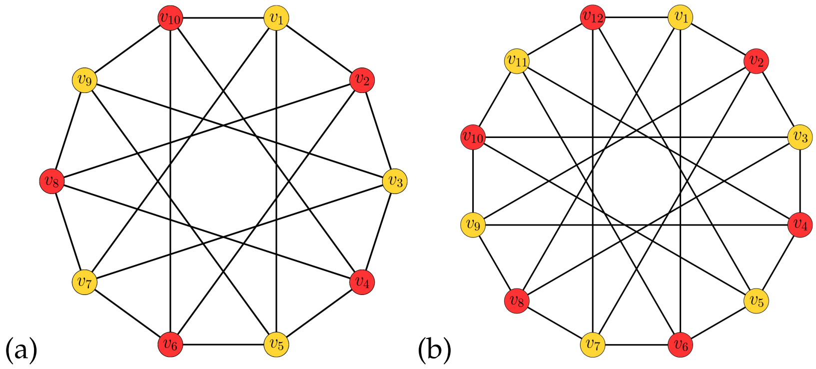

Figure 9: (a) Assignment of colors to $V(Ci_{10}(1,4))$ where the vertices are colored alternately, and (b) perfect 2-coloring of $Ci_{12}(1,5)$ with M_{10}

$\square$

5. Perfect 2-colorings of Circulant Graph $Ci_{2n}(1,k)$

This section enumerates further results on perfect 2-colorings of 4-regular circulant graphs. It should be noted that the perfect 2-colorings of antiprism graphs $Ci_{2n}(1,2)$ are presented in [26]. The next theorems present results on the perfect 2-colorings of 4-regular graphs $Ci_{2n}(1,s)$. Observe that in the next results, we assume that $s \neq n$, otherwise $Ci_{2n}(1,s)$ is a 3-regular graph.

Theorem 5.1. *Consider the circulant graph $Ci_{2n}(1,s)$ where $s \equiv 1 \pmod 2, s \neq n$. For any n, $Ci_{2n}(1,s)$ is perfect 2-colorable with $M_{10} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$.*

Proof. Consider M_{10}. In this coloring with M_{10}, every red (respectively yellow) vertex is adjacent only to yellow (red) vertex. Moreover, in this graph, $N(v_i) = \{v_{i+1}, v_{i-1}, v_{i+s}, v_{i-s}\}$ where the indices are under modulo $2n$. Let $\mathcal{C}(v_1) = \mathbf{1}$. Under this initial assignment of colors, the assignment of colors to other vertices can be determined. In particular, $\mathcal{C}(v_2) = \mathcal{C}(v_{2n}) = \mathcal{C}(v_{1+s}) = \mathcal{C}(v_{1-k}) = \mathbf{2}$. Since 1 and s are odd numbers then $1+s$ and $1-s$ are even numbers. Consequently, the assignment of colors to the uncolored vertices are also determined based on the previous assignments. That is, the vertices in the cycle C_{2n} are colored alternately. We have two subcases.

We obtain a perfect 2-coloring (Figure 10(a)) given by the mapping below

$$\mathcal{C}(v_i) = \begin{cases} \mathbf{1} \text{ (yellow)} & \text{if } i \equiv 1 \pmod 2 \\ \mathbf{2} \text{ (red)} & \text{if } i \equiv 0 \pmod 2 \end{cases}$$

$\square$

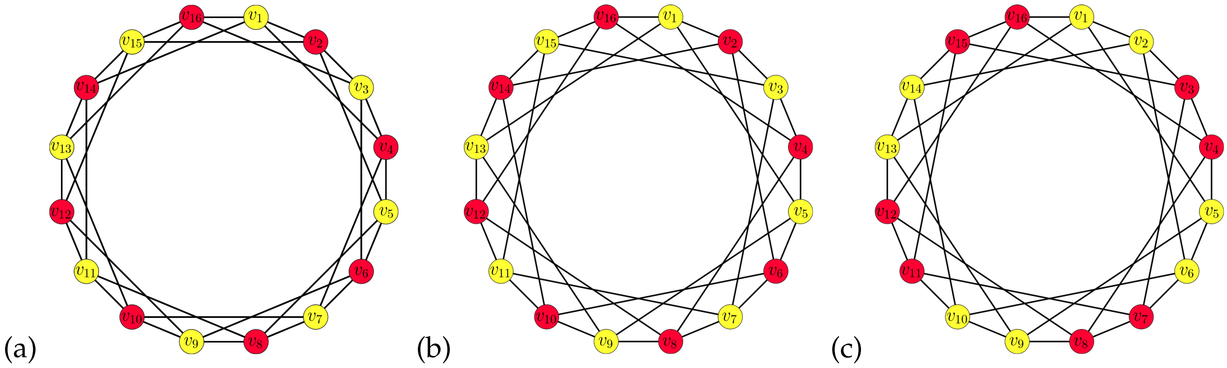

Figure 10: Perfect 2-coloring (a) of $Ci_{16}(1,3)$ with M_{10}, (b) of $Ci_{16}(1,4)$ with M_5, (c) of $Ci_{16}(1,4)$ with M_1

Theorem 5.2. *Consider the circulant graph $Ci_{2n}(1,s)$ where s divides $2n, s \neq n$, and $s \equiv 0 \pmod 2$. For any n, $Ci_{2n}(1,s)$ is perfect 2-colorable with $M_5 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.*

Proof. Consider M_5. We obtain a perfect 2-coloring (Figure 10(b)) of $Ci_{2n}(1,k)$ using the same mapping of colors to $V(Ci_{2n}(1,k))$ as in Theorem 5.1. □

Theorem 5.3. *Consider the circulant graph $Ci_{2n}(1,4)$. For any $n \equiv 0 \pmod 2$, $Ci_{2n}(1,4)$ is perfect 2-colorable with $M_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.*

Proof. Consider M_1. We obtain a perfect 2-coloring (Figure 10(c)) of $Ci_{2n}(1,4)$ using the mapping below

$$\mathcal{C}(v_i) = \begin{cases} \mathbf{1} \text{ (yellow)} & \text{if } i \equiv 1,2 \pmod 4 \\ \mathbf{2} \text{ (red)} & \text{if } i \equiv 0,3 \pmod 4 \end{cases}$$

□

6. Summary

This work determines the perfect 2-colorings of 4-regular circulant graphs $Ci_{2n}(1,n-1)$. The results obtained appear to be different from the perfect 2-colorings of 4-regular graphs such as those in [20,25]. The perfect 2-colorings of antiprism graphs $Ci_{2n}(1,2)$ are presented in [26]. Results presented in section 5 are perfect 2-colorings of 4-regular circulant graphs $Ci_{2n}(1,k)$ where the number of vertices assigned with color yellow and the number of vertices assigned with color red are equal.

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