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# Matrix Representations of Group Algebras of $A_4$ and $S_4$

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**Abstract**

In this paper we extend the results of [1] by using semidirect products to find the matrix representations of group algebras of the alternating group $A_4$ and the symmetric group $S_4$, when the groups $A_4$ and $S_4$ are presented by more than two generators.

**Keywords:** Semidirect Product; Group Algebra; Circulant Matrix.

## 1. Preliminaries

Let $F$ be a field. A ring $A$ is an algebra over $F$ (briefly $F$-algebra) if $A$ is a vector space over $F$ and the following compatibility condition holds $(sa).b = s(a.b) = a.(sb)$ for any $a, b \in A$ and any $s \in F$. $A$ is also called associative algebra (over $F$). The dimension of the algebra $A$ is the dimension of $A$ as a vector space over $F$.

**Theorem 1.1** ([3]). *Let $A$ be a $n$-dimensional algebra over a field $F$. Then there is a one to one algebra homomorphism from $A$ into $M_n(F)$, the algebra of $n$-matrices over $F$.*

Let $G = \{g_1 = 1, g_2, \ldots, g_n\}$ be a finite group of order $n$ and $F$ a field. Define $FG = \{a_1g_1 + a_2g_2 + \cdots + a_ng_n : a_i \in F\}$. $FG$ is $n$-dimensional vector space over $F$ with basis $G$. Multiplication of $G$ can be extended linearly to $FG$. Thus $FG$ becomes an algebra over $F$ of dimension $n$. $FG$ is called group algebra. The following identifications should be realized.

(i) $0_Fg_G = 0_{FG} = 0$ for any $g \in G$.

(ii) $1_Fg_G = g_{FG} = g$ for any $g \in G$. In particular $1_F1_G = 1_{FG} = 1$.

(iii) $a_F1_G = a_{FG}$ for any $a \in F$.

Let $G$ be a group. Assume that $H \lhd G$, $K \leq G$, $H \cap K = \{1\}$, and $G = HK$. Suppose that $K$ acts on $H$ by automorphisms of $H$. Then there exists a homomorphism $\phi : K \to Aut(H)$. Assume the action is

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by conjugation. Then for $k \in K$ and $h \in H$ we have

$$k.h = \phi(k)(h) = khk^{-1}$$

$G$ is a semidirect product of $H$ and $K$ by $\phi$ and is denoted by $G = H \rtimes_{\phi} K$ [2]. A group $G$ is metacyclic group if it has a cyclic normal subgroup $N$ such that $G/N$ is cyclic. Equivalently $G$ has cyclic subgroups $H$ and $K$ such that $H \lhd G$ and $G = HK$. If $H \cap K = \{1\}$ also, then $G$ is called a split metacyclic group. A circulant matrix $M$ on parameters $a_0, a_1, \ldots, a_{n-1}$ is defined as follows

$$M(a_0, a_1, \ldots, a_{n-1}) = \begin{bmatrix} a_0 & a_{n-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix}$$

This matrix may be denoted in terms of its columns by $[col(a_0)\,|col(a_{n-1})| \ldots |col(a_1)]$. $M$ is said to be circulant block matrix if it is of the form $M(M_1, M_2, \cdots, M_n)$ i.e it is circulant blockwise on the blocks $M_1, M_2, \cdots, M_n$. Thus

$$M = \begin{bmatrix} M_1 & M_n & \cdots & M_2 \\ M_2 & M_1 & \cdots & M_3 \\ \vdots & \vdots & \ddots & \vdots \\ M_n & M_{n-1} & \cdots & M_1 \end{bmatrix}$$

## 2. Main Results

**Theorem 2.1** ([1]). *Let $F$ be a field and $G = \langle \alpha : \alpha^n = 1 \rangle$ a cyclic group of order $n$. Then any element $a_0 1 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}$ of $FG$ can be represented with respect to the ordered basis $\{1, \alpha, \ldots, \alpha^{n-1}\}$ by the circulant matrix $M(a_0, a_1, \ldots, a_{n-1})$.*

Note that if the order of the basis elements is changed we obtain a different matrix of representation. The new matrix is obtained by suitable interchanging of the columns of the matrix $M(a_0, a_1, \ldots, a_{n-1})$ with the same notations used in [1] we have

**Theorem 2.2** ([1]). *Let $F$ be a field and $G$ a split metacyclic group. the representation of the general element $\sum_{j=0}^{m-1}\sum_{i=0}^{n-1} a_{ij}\alpha^i\beta^j$ in $FG$ is given by the circulant block matrix $M\left(M(a_{i0}), M^{\beta}(a_{i1}), \ldots, M^{\beta^{m-1}}(a_{i(m-1)})\right)$; $i = 0, 1, \ldots, n-1. a_{ij} \in F$.*

For more complicated finite groups we use the circulant block matrices to do the required representations. Now, we extend this method by using semidirect products to do the representations for some finite groups involving more than two generators.

**Theorem 2.3.** *Let $F$ be a field and $G$ be a group, and suppose that $G$ is an internal semidirect product of $H$*

*and cyclic group $K = \langle \gamma \rangle$ by $\phi$. Then the matrix representation $[u]$ of the general element $u$ in $FG$ is given as follows:* $[u] = \begin{bmatrix} [u_1] & [u_m]^{\gamma^{m-1}} & \cdots & [u_2]^{\gamma} \\ [u_2]^{\gamma} & [u_1] & \cdots & [u_m]^{\gamma^2} \\ \vdots & \vdots & \ddots & \vdots \\ [u_m]^{\gamma^{m-1}} & [u_m]^{\gamma^{m-2}} & \cdots & [u_1] \end{bmatrix}.$

*Proof.* Let $G$ be an internal semidirect product of $H$ and a cyclic group $K = \langle \gamma \rangle$ by $\phi$. $G = H \rtimes_\phi K$, $\phi : K \to Aut(H)$ is homomorphism, $\phi(\gamma)(h) = \gamma h \gamma^{-1}$. Suppose that $H = \{h_1, h_2, \ldots, h_n\}$, $K = C_m \langle \gamma \rangle = \{1, \gamma, \ldots, \gamma^{m-1}\}$ then the general element $u$ in $FG$ is

$$u = a_1 h_1 1 + a_2 h_2 1 + \cdots + a_n h_n 1 + a_{n+1} h_1 \gamma + a_{n+2} h_2 \gamma + \ldots$$
$$\cdots + a_{2n} h_n \gamma + a_{2n+1} h_1 \gamma^2 + \cdots + a_{3n} h_n \gamma^2 + \cdots + a_{mn} h_n \gamma^{m-1}$$

Now we can write $u$ as $u = u_1 + u_2 + \cdots + u_m$, where

$$\begin{aligned} u_1 &= a_1 h_1 1 + a_2 h_2 1 + \cdots + a_n h_n 1 \\ u_2 &= a_{n+1} h_1 \gamma + a_{n+2} h_2 \gamma + \cdots + a_{2n} h_n \gamma \\ \vdots & \quad \vdots \\ u_m &= a_{(m-1)(n+1)} h_1 \gamma^{m-1} + \cdots + a_{mn} h_n \gamma^{m-1}. \end{aligned}$$

The matrix representation $[u]$ of $u$:

$$[u] = M([u_1], [u_2]^{\gamma}, \ldots, [u_m]^{\gamma^{m-1}})$$

Where $\gamma^i : H \to H$ is the automorphism $\gamma^i = \phi(\gamma^i)(h) = \gamma^i h \gamma^{-i}$ and

$$\begin{aligned} [u_i] &= [col(h_1) | col(h_2) | \ldots | col(h_n)], \\ [u_i]^{\gamma^i} &= \left[col\left(\gamma^i(h_1)\right) | col\left(\gamma^i(h_2)\right) | \ldots | col(\gamma^i(h_n))\right] \end{aligned}$$

Then the matrix representation $[u]$ of $u$ is given as follows

$$[u] = \begin{bmatrix} [u_1] & [u_m]^{\gamma^{m-1}} & \cdots & [u_2]^{\gamma} \\ [u_2]^{\gamma} & [u_1] & \cdots & [u_m]^{\gamma^2} \\ \vdots & \vdots & \ddots & \vdots \\ [u_m]^{\gamma^{m-1}} & [u_m]^{\gamma^{m-2}} & \cdots & [u_1] \end{bmatrix}$$

□

## 3. Applications

We use Theorem 2.3, to compute the matrix representations of $FA_4$ and $FS_4$, where $A_4$ and $S_4$ are presented as follows.

(1) $A_4 = \langle \alpha, \beta, \gamma : \alpha^2 = \beta^2 = \gamma^3 = 1, \beta\alpha = \alpha\beta, \gamma\alpha = \beta\gamma, \gamma\beta = \alpha\beta\gamma \rangle$

$= \{1, \alpha, \beta, \alpha\beta, \gamma, \alpha\gamma, \beta\gamma, \alpha\beta\gamma, \gamma^2, \alpha\gamma^2, \beta\gamma^2, \ \alpha\beta\gamma^2\}$

$A_4 = (C_2 \langle\alpha\rangle \times C_2 \langle\beta\rangle) \rtimes_\phi C_3 \langle\gamma\rangle$.

The general element of $FA_4$ is $u = a1 + b\alpha + c\beta + d\alpha\beta + e\gamma + f\alpha\gamma + g\beta\gamma + h\alpha\beta\gamma + i\gamma^2 + j\alpha\gamma^2 + k\beta\gamma^2 + l\alpha\beta\gamma^2$; $a, b, \ldots, l \in F$. Let $u_1 = a1 + b\alpha + c\beta + d\alpha\beta$, $u_2 = e\gamma + f\alpha\gamma + g\beta\gamma + h\alpha\beta\gamma$, $u_3 = i\gamma^2 + j\alpha\gamma^2 + k\beta\gamma^2 + l\alpha\beta\gamma^2$. Then $u = u_1 + u_2 + u_3$.
By Theorem 2.3, matrix representation of $u$ is

$$[u] = \begin{bmatrix} [u_1] & [u_3]^{\gamma^2} & [u_2]^{\gamma} \\ [u_2]^{\gamma} & [u_1] & [u_3]^{\gamma^2} \\ [u_3]^{\gamma^2} & [u_2]^{\gamma} & [u_1] \end{bmatrix}$$

$$[u_1] = \begin{bmatrix} a & b & \vdots & c & d \\ b & a & \vdots & d & c \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c & d & \vdots & a & b \\ d & c & \vdots & b & a \end{bmatrix}$$

$A_4 = (C_2 \langle\alpha\rangle \times C_2 \langle\beta\rangle) \rtimes_\phi C_3 \langle\gamma\rangle$, $\phi C_3 \langle\gamma\rangle :\to Aut(H)$ is a homomorphism, $\phi(\gamma) : C_2 \langle\alpha\rangle \times C_2 \langle\beta\rangle \to C_2 \langle\alpha\rangle \times C_2 \langle\beta\rangle$ an automorphism, $\phi(\gamma)(1) = \gamma 1 \gamma^{-1} = \gamma\gamma^2 = 1$, $\phi(\gamma)(\alpha) = \gamma\alpha\gamma^{-1}, = \gamma\alpha\gamma^2 = \beta\gamma\gamma^2 = \beta$, $\phi(\gamma)(\beta) = \gamma\beta\gamma^{-1} = \gamma\beta\gamma^2 = \alpha\beta\gamma\gamma^2 = \alpha\beta$, $\phi(\gamma)(\alpha\beta) = \gamma\alpha\beta\gamma^{-1} = \gamma\alpha\beta\gamma^2 = \beta\gamma\beta\gamma^2 = \beta\alpha\beta\gamma\gamma^2 = \alpha\beta\beta = \alpha$.

$$[u_2] = [col(1)\,|col(\alpha)|\,col(\beta)\,|col(\alpha\beta)]$$

$$[u_2]^{\gamma} = [col(1)\,|col(\beta)\,|col(\alpha\beta)\,|col(\alpha)]$$

$$[u_2]^{\gamma} = \begin{bmatrix} e & g & \vdots & h & f \\ f & h & \vdots & g & e \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g & e & \vdots & f & h \\ h & f & \vdots & e & g \end{bmatrix}$$

$\phi(\gamma^2) : C_2 \langle\alpha\rangle \times C_2 \langle\beta\rangle \to C_2 \langle\alpha\rangle \times C_2 \langle\beta\rangle$ an automorphism, $\phi(\gamma^2)(1) = \gamma^2 1 (\gamma^2)^{-1} = 1$, $\phi(\gamma^2)(\alpha) =$

$\gamma^2\alpha(\gamma^2)^{-1} = \alpha\beta, \phi\left(\gamma^2\right)(\beta) = \gamma^2\beta(\gamma^2)^{-1} = \alpha, \phi\left(\gamma^2\right)(\alpha\beta) = \gamma^2\alpha\beta(\gamma^2)^{-1} = \beta.$

$$[u_3] = [col\,(1)\,, col\,(\alpha)\,, col\,(\beta)\,, col\,(\alpha\beta)]$$

$$[u_3]^{\gamma^2} = [col\,(1)\,, col\,(\alpha\beta)\,, col\,(\alpha)\,, col(\beta)]$$

$$[u_3]^{\gamma^2} = \begin{bmatrix} i & l & \vdots & j & k \\ j & k & \vdots & i & l \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ k & j & \vdots & l & i \\ l & i & \vdots & k & j \end{bmatrix}$$

Then $[u] = \begin{bmatrix} a & b & c & d & \vdots & i & l & j & k & \vdots & e & g & h & f \\ b & a & d & c & \vdots & j & k & i & l & \vdots & f & h & g & e \\ c & d & a & b & \vdots & k & j & l & i & \vdots & g & e & f & h \\ d & c & b & a & \vdots & l & i & k & j & \vdots & h & f & e & g \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ e & g & h & f & \vdots & a & b & c & d & \vdots & i & l & j & k \\ f & h & g & e & \vdots & b & a & d & c & \vdots & j & k & i & l \\ g & e & f & h & \vdots & c & d & a & b & \vdots & k & j & l & i \\ h & f & e & g & \vdots & d & c & b & a & \vdots & l & i & k & j \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ i & l & j & k & \vdots & e & g & h & f & \vdots & a & b & c & d \\ j & k & i & l & \vdots & f & h & g & e & \vdots & b & a & d & c \\ k & j & l & i & \vdots & g & e & f & h & \vdots & c & d & a & b \\ l & i & k & j & \vdots & h & f & e & g & \vdots & d & c & b & a \end{bmatrix}.$

(2) $S_4 = \left\langle \alpha, \beta, \gamma, \delta : \alpha^2 = \beta^2 = \gamma^3 = \delta^2 = 1, \beta\alpha = \alpha\beta, \gamma\alpha = \beta\gamma, \gamma\beta = \alpha\beta\gamma, \delta\alpha = \alpha\delta, \delta\beta = \alpha\beta\delta, \delta\gamma = \gamma^2\delta \right\rangle$

$S_4 = A_4 \rtimes_\phi C_2 \langle\delta\rangle$, where $C_2\langle\delta\rangle$ acts on Aut $(A_4)$ by conjugation.

Let $u_{S_4}$ be the general element of $FS_4$, $u_{S_4} \in FS_4$, $u_{S_4} = u_{A_4}(a_1, a_2, \ldots, a_{12}) + u_{A_4}(b_1, b_2, \ldots, b_{12})\delta$,

$[u_{S_4}] = M([u_{A_4}(a_1, a_2, \ldots, a_{12})], [u_{A_4}(b_1, b_2, \ldots, b_{12})]^{\delta}).$

$$[u_{A_4}(a_1, \ldots, a_{12})] = \begin{bmatrix}
a_1 & a_2 & a_3 & a_4 & | & a_9 & a_{12} & a_{10} & a_{11} & | & a_5 & a_7 & a_8 & a_6 \\
a_2 & a_1 & a_4 & a_3 & | & a_{10} & a_{11} & a_9 & a_{12} & | & a_6 & a_8 & a_7 & a_5 \\
a_3 & a_4 & a_1 & a_2 & | & a_{11} & a_{10} & a_{12} & a_9 & | & a_7 & a_5 & a_6 & a_8 \\
a_4 & a_3 & a_2 & a_1 & | & a_{12} & a_9 & a_{11} & a_{10} & | & a_8 & a_6 & a_5 & a_7 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a_5 & a_7 & a_8 & a_6 & | & a_1 & a_2 & a_3 & a_4 & | & a_9 & a_{12} & a_{10} & a_{11} \\
a_6 & a_8 & a_7 & a_5 & | & a_2 & a_1 & a_4 & a_3 & | & a_{10} & a_{11} & a_9 & a_{12} \\
a_7 & a_5 & a_6 & a_8 & | & a_3 & a_4 & a_1 & a_2 & | & a_{11} & a_{10} & a_{12} & a_9 \\
a_8 & a_6 & a_5 & a_7 & | & a_4 & a_3 & a_2 & a_1 & | & a_{12} & a_9 & a_{11} & a_{10} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a_9 & a_{12} & a_{10} & a_{11} & | & a_5 & a_7 & a_8 & a_6 & | & a_1 & a_2 & a_3 & a_4 \\
a_{10} & a_{11} & a_9 & a_{12} & | & a_6 & a_8 & a_7 & a_5 & | & a_2 & a_1 & a_4 & a_3 \\
a_{11} & a_{10} & a_{12} & a_9 & | & a_7 & a_5 & a_6 & a_8 & | & a_3 & a_4 & a_1 & a_2 \\
a_{12} & a_9 & a_{11} & a_{10} & | & a_8 & a_6 & a_5 & a_7 & | & a_4 & a_3 & a_2 & a_1
\end{bmatrix}$$

$\phi(\delta) : A_4 \to A_4$ an automorphism, $\delta 1 \delta^{-1} = \delta 1 \delta = 1$, $\delta\alpha\delta^{-1} = \delta\alpha\delta = \alpha$, $\delta\beta\delta^{-1} = \delta\beta\delta = \alpha\beta$, $\delta\alpha\beta\delta^{-1} = \delta\alpha\beta\delta = \beta$, $\delta\gamma\delta^{-1} = \delta\gamma\delta = \gamma^2$, $\delta\alpha\gamma\delta^{-1} = \delta\alpha\gamma\delta = \alpha\gamma^2$, $\delta\beta\gamma\delta^{-1} = \delta\beta\gamma\delta = \alpha\beta\gamma^2$, $\delta\alpha\beta\gamma\delta^{-1} = \delta\alpha\beta\gamma\delta = \beta\gamma^2$, $\delta\gamma^2\delta^{-1} = \delta\gamma^2\delta = \gamma$, $\delta\alpha\gamma^2\delta^{-1} = \delta\alpha\gamma^2\delta = \alpha\gamma$, $\delta\beta\gamma^2\delta^{-1} = \delta\beta\gamma^2\delta = \alpha\beta\gamma$, $\delta\alpha\beta\gamma^2\delta^{-1} = \delta\alpha\beta\gamma^2\delta = \beta\gamma$.

$$\begin{aligned}
[u_{A_4}(b_1, \ldots, b_{12})] = {} & [col(1), col(\alpha), col(\beta), col(\alpha\beta), col(\gamma), col(\alpha\gamma), \\
& col(\beta\gamma), col(\alpha\beta\gamma), col(\gamma^2), col(\alpha\gamma^2), col(\beta\gamma^2), col(\alpha\beta\gamma^2)] \\
[u_{A_4}(b_1, \ldots, b_{12})]^{\delta} = {} & [col(1), col(\alpha), col(\alpha\beta), col(\beta), col(\gamma^2), col(\alpha\gamma^2), \\
& col(\beta\gamma^2), col(\beta\gamma^2), col(\gamma), col(\alpha\gamma), col(\alpha\beta\gamma), col(\beta\gamma)]
\end{aligned}$$

$$[u_{A_4}(b_1,\dots,b_{12})]^{\delta}=\begin{bmatrix}
b_1 & b_2 & b_4 & b_3 & | & b_5 & b_7 & b_6 & b_8 & | & b_9 & b_{12} & b_{11} & b_{10}\\
b_2 & b_1 & b_3 & b_4 & | & b_6 & b_8 & b_5 & b_7 & | & b_{10} & b_{11} & b_{12} & b_9\\
b_3 & b_4 & b_2 & b_1 & | & b_7 & b_5 & b_8 & b_6 & | & b_{11} & b_{10} & b_9 & b_{12}\\
b_4 & b_3 & b_1 & b_2 & | & b_8 & b_6 & b_7 & b_5 & | & b_{12} & b_9 & b_{10} & b_{11}\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
b_5 & b_7 & b_6 & b_8 & | & b_9 & b_{12} & b_{11} & b_{10} & | & b_1 & b_2 & b_4 & b_3\\
b_6 & b_8 & b_5 & b_7 & | & b_{10} & b_{11} & b_{12} & b_9 & | & b_2 & b_1 & b_3 & b_4\\
b_7 & b_5 & b_8 & b_6 & | & b_{11} & b_{10} & b_9 & b_{12} & | & b_3 & b_4 & b_2 & b_1\\
b_8 & b_6 & b_7 & b_5 & | & b_{12} & b_9 & b_{10} & b_{11} & | & b_4 & b_3 & b_1 & b_2\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
b_9 & b_{12} & b_{11} & b_{10} & | & b_1 & b_2 & b_4 & b_3 & | & b_5 & b_7 & b_6 & b_8\\
b_{10} & b_{11} & b_{12} & b_9 & | & b_2 & b_1 & b_3 & b_4 & | & b_6 & b_8 & b_5 & b_7\\
b_{11} & b_{10} & b_9 & b_{12} & | & b_3 & b_4 & b_2 & b_1 & | & b_7 & b_5 & b_8 & b_6\\
b_{12} & b_9 & b_{10} & b_{11} & | & b_4 & b_3 & b_1 & b_2 & | & b_8 & b_6 & b_7 & b_5
\end{bmatrix}$$

Thus

$$[u_{S_4}]=\begin{bmatrix}[u_{A_4}(a_1,\dots,a_{12})] & [u_{A_4}(b_1,\dots,b_{12})]^{\delta}\\ [u_{A_4}(b_1,\dots,b_{12})]^{\delta} & [u_{A_4}(a_1,\dots,a_{12})]\end{bmatrix}$$

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