International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **12(3)**(2024), 39–60
Available Online: http://ijmaa.in

On Mixed Type SP-iteration Schemes for Single-valued and Multi-valued Mappings in CAT(0) Spaces

Yongquan Liu[1,*], Suping Wang[1], Xing Huang[2]

[1] *School of Teacher Education, Ji'an Preschool Teachers College, Ji'an, China*
[2] *School of Sciences, Changzhou Institute of Technology, Changzhou, China*

Abstract

In this paper, we introduce a new mixed type SP-iteration process, which approximates the common fixed points of three single-valued non-expansive mappings and three multi-valued non-expansive mappings in CAT(0) spaces. We establish $\triangle-$convergence and strong convergence theorems for the new iterative process in CAT(0) spaces. Our results extend and improve the corresponding recent results announced by many authors.

Keywords: CAT(0) space; non-expansive mapping; mixed type SP- iteration; $\triangle-$ convergence; strong convergence.
2020 Mathematics Subject Classification: 47H09, 47H10, 49M09.

1. Introduction

A CAT(0) space plays a fundamental role in various fields of mathematics (see [1–3]). Moreover, there are applications in biology and computer science as well ([4,5]). A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. The complex Hilbert ball with a hyperbolic metric is a CAT(0) space ([1]).

The study of metric spaces without linear structure has played a vital roll in various branches of pure and applied sciences. In particular, existence and approximation results in CAT(0) spaces for classes of single-valued and multi-valued mappings have been studied extensively by many authors (see [6–11]). Iteration process for numerical reckoning fixed points of various classes of nonlinear operators are available in the literature. In this regard, the class of single-valued non-expansive mappings via iteration methods has extensively been studied ([12,13]). For multi-valued non-expansive mappings, Sastry and Babu [14] defined a Mann and Ishikawa iteration process in Hilbert spaces. Panyanak [15]

*Corresponding author (lyq60913016@hotmail.com)

and Song and Wang [16] (see also [17]) extended the result of Sastry and Babu [11] to uniformly convex Banach spaces. Shahzad and Zegeye [18] extended and improved results of (see [14,16,17]).

In 2008, Dhompongsa and Panyanak [19] established $\triangle$-convergence theorems for the Mann and Ishikawa iterations for non-expansive single-valued mappings in CAT(0) spaces. Inspired by Song and Wang [16], Laowang and Panyanak [7] extended the result of Dhompongsa and Panyanak [6] for multi-valued non-expansive mappings in a CAT(0) space.

In 2011, W. Phuengrattana and S. Suantai [20] introduced the SP-iterative process. The SP-iteration is defined by $x_1 \in K$ and

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n \\ y_n = (1-\beta_n)z_n + \beta_n T z_n \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n \end{cases} \tag{1}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$.

In 2015, R.P. Pathak et al. [21] introduce a Noor-type iteration process for non-expansive multi-valued mappings and prove strong convergence theorems for the proposed iterative process in CAT(0) spaces. Let K be a nonempty convex subset of a complete CAT(0) space X. The sequence of Noor-type iterates is defined by $x_1 \in K$,

$$\begin{cases} z_n = (1-\gamma_n)x_n \oplus \gamma_n w_n \\ y_n = (1-\beta_n)x_n \oplus \beta_n w'_n \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n w''_n \end{cases} \tag{2}$$

for all $n \geq 1$, where $w_n \in Tx_n$, $w'_n \in Tz_n$, $w''_n \in Ty_n$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$.

In 2018, K. Sokhuma [22] introduce a SP-iteration process for non-expansive multi-valued mappings and prove strong convergence theorems for the proposed iterative process in CAT(0) spaces. Let K be a nonempty convex subset of a complete CAT(0) space X. The sequence of SP-iteration is defined by $x_1 \in K$,

$$\begin{cases} z_n = (1-\gamma_n)x_n \oplus \gamma_n w_n \\ y_n = (1-\beta_n)z_n \oplus \beta_n w'_n \\ x_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n w''_n \end{cases} \tag{3}$$

for all $n \geq 1$, where $w_n \in Tx_n$, $w'_n \in Tz_n$, $w''_n \in Ty_n$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$.

In 2021, Y. Liu [23] introduce a mixed type iterative process for non-expansive single-valued and multi-valued mappings and prove strong convergence theorems for the proposed iterative process in Banach spaces. The sequence of mixed type iteration is defined by $x_1 \in K$,

$$x_{n+1} = \alpha_n S x_n + \beta_n y_n + \gamma_n z_n$$

for all $n \geq 1$, where $y_n \in T_1 x_n$, $z_n \in T_2 x_n$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$.

The purpose of this paper is to introduce the mixed type SP-iteration process for finding a common fixed point of the single-valued and multi-valued non-expansive mappings in the setting of CAT(0) spaces. Under suitable conditions some strong convergence and $\triangle-$ convergence theorems of the iterative sequence generated by the proposed scheme to approximate a common fixed point of single-valued and multi-valued non-expansive mappings are proved. The results presented in the paper extend and improve some recent results announced in the current literature [7]-[22].

2. Preliminaries

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t\check{}t'|$ for all $t, t' \in [0, l]$.

In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. For any $x, y \in X$, we denote the point $z \in [x, y]$ by $z = (1-\alpha)x \oplus \alpha y$, where $0 \leq \alpha \leq 1$ if $d(x, z) = \alpha d(x, y)$ and $d(z, y) = (1-\alpha)d(x, y)$.

The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $K \subset X$ is called convex if K includes every geodesic segment joining any two of its points.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points of X (as the vertices of $\triangle$) and a geodesic segment between each pair of points (as the edges of $\triangle$). A comparison triangle for $\triangle(x_1, x_2, x_3)$ in (X, d) (denoted by $\triangle$) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(x_1, x_2, x_3)$ in Euclidean plane R^2 such that $d_{R2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$for $i, j \in \{1, 2, 3\}$. A point $\overline{x} \in [\overline{x}_1, \overline{x}_2]$ is said to be comparison point for $x \in [x_1, x_2]$ if $d(x_1, x) = d(\overline{x}_1, \overline{x})$. The comparison points on $[\overline{x}_2, \overline{x}_3]$ and $[\overline{x}_3, \overline{x}_1]$ are defined in same way.

A geodesic metric space X is called a CAT(0) space if all geodesic triangles satisfy the following comparison axiom (CAT(0) inequality):

Let $\triangle$ be a geodesic triangle in X and $\overline{\triangle}$ its comparison triangle in R^2. Then, $\triangle$ is said to satisfy CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{R^2}(\overline{x}, \overline{y}).$$

Finally, we observe that if x, y_1, y_2 are points of a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \tag{4}$$

The equality holds for the Euclidean metric. In fact (see [1]), a geodesic metric space is a CAT(0) space if and only if it satisfies inequality (2.1) (which is known as the CN inequality).

The following Lemma 2.1 can be found in [19].

Lemma 2.1. *Let (X,d) be a CAT(0) space.*

(i) For $x,y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that

$$d(x,z) = td(x,y) \quad \text{and} \quad d(y,z) = (1-t)d(x,y).$$

(ii) For $x,y,z \in X$ and $t \in [0,1]$, we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Lemma 2.2 ([27]). *Let (X,d) be a CAT(0) space, $x \in X$ be a given point and $\{t_n\}$be a sequence in $[b,c]$ with $b,c \in (0,1)$ and $0 < b(1-c) \leq \frac{1}{2}$. Let $\{x_n\}$ and$\{y_n\}$ be any sequences in X such that*

$$\begin{cases} \limsup_{n\to\infty} d(x_n,p) \leq r, \\ \limsup_{n\to\infty} d(y_n,p) \leq r \\ \lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r \end{cases}$$

for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Now, we recall some definitions.

Let K be the subset of CAT(0) space X. Then:

(i) The distance from $x \in X$ to K is defined by

$$\text{dist}(x, K) = \inf\{d(x, y) : y \in K\}.$$

(ii) The diameter of K is defined by

$$\text{diam}(K) = \sup\{d(u, v) : u, v \in K\}.$$

The set K is called proximinal if for each $x \in X$, there exists an element $y \in K$ such that $d(x,y) = dist(x,K)$. Let $CB(K)$, $C(K)$, and $P(K)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximinal subsets of K, respectively. The Hausdorff metric H on $CB(K)$ is defined by

$$H(U,V) = \max\{\sup_{x\in U} dist(x,V), \sup_{y\in V} dist(y,U)\}$$

for $U, V \in CB(K)$, where dist$(x,V) = \inf\{d(x, z), z \in V\}$.

Let $S : K \to K$ be a single-valued mapping. An element $x \in X$ is said to be a fixed point of S, if $x = Sx$. The set of fixed points will be denoted by $F(S)$.

Let $T : X \to 2^X$ be a multi-valued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points will be denoted by $F(T)$.

Definition 2.3.

(1) A single-valued mapping $S : K \to K$ is called non-expansive, if $d(Sx, Sy) \le d(x, y)$ for all $x, y \in K$;

(2) A multi-valued mapping $T : K \to CB(K)$ is called non-expansive, if $H(Tx, Ty) \le d(x, y)$ for all $x, y \in K$.

Let X be a complete CAT(0) space and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of x_n is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point ([28], Proposition 7). Also, every CAT(0) space has the Opial property, i.e., if $\{x_n\}$ is a sequence in K and $\triangle - \lim_{n\to\infty} x_n = x$, then for each $y(\neq x) \in K$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

In 2005, Khan and Fukhar-ud-din [29] introduced the condition (A'). In 2007, Fukhar-ud-din [30] gave an improved version for the condition (A'). In 2011, Abbas etc. [31] introduced a multi-valued version of condition (A').

Next, we introduce a mixed type version for the condition (A') of single-value and multi-valued as follows:

Three single-value mappings $S_1, S_2, S_3 : K \to K$ and three multi-valued mappings $T_1, T_2, T_3 : K \to CB(K)$ are said to satisfy condition (A') if there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, $g(t) > 0$ for all $t \in (0, \infty)$ for such that either $d(x, S_1x) \ge g(dist(x, F))$ or $d(x, S_2x) \ge g(dist(x, F))$ or $d(x, S_3x) \ge g(dist(x, F))$ or $dist(x, T_1x) \ge g(dist(x, F))$ or $dist(x, T_2x) \ge g(dist(x, F))$ for all $x \in K$ where $F = \bigcap_{i=1}^{3} F(S_i) \bigcap F(T_i)$.

Definition 2.4 ([32,33]). *A sequence $\{x_n\}$ in a CAT(0) space X is said to be $\triangle$-convergent to $x \in X$ if x is the* *unique asymptotic center of* $\{u_n\}$ *for every subsequence* $\{u_n\}$ *of* $\{x_n\}$*. In this case, we write* $\triangle - \lim_{n\to\infty} x_n = x$ *and* x *is called the* $\triangle$ *-limit of* $\{x_n\}$*.*

The notion of $\triangle$ -convergence in a general metric space was introduced by Lim [33]. In 2008, Kirk and Panyanak [32] used the concept of $\triangle$ -convergence introduced by Lim [33] to prove on the CAT(0) space analogous of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [19] obtained $\triangle$ -convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space.

Lemma 2.5 ([34])**.** *Let X be a complete CAT(0) space, K be a closed convex subset of X. If* $\{x_n\}$ *is a bounded sequence in K, then the asymptotic center of* $\{x_n\}$ *is in K.*

Lemma 2.6 ([32])**.** *Every bounded sequence in a complete CAT(0) space always has a* $\triangle$ *-convergent subsequence.*

Lemma 2.7 ([32])**.** *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let* $S : K \to X$ *be a single-valued non-expansive mapping. If* $\triangle - \lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} d(x_n, Sx_n) = 0$*, then x is a fixed point of S.*

Lemma 2.8 ([34])**.** *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let* $T : K \to C(K)$ *be a multi-valued non-expansive mapping. If* $\triangle - \lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} dist(x_n, Tx_n) = 0$*, then x is a fixed point of T.*

3. Main Results

Now we introduce the notion of the proposed mixed type version of the SP iteration process for three single-valued non-expansive mappings and three multi-valued non-expansive mappings. Let K be a nonempty convex subset of a complete CAT(0) space X. The sequence of mixed type SP iterates is defined by $x_1 \in K$,

$$\begin{cases} z_n = (1-\gamma_n)S_1x_n \oplus \gamma_n w_n \\ y_n = (1-\beta_n)S_2z_n \oplus \beta_n w'_n \\ x_{n+1} = (1-\alpha_n)S_3y_n \oplus \alpha_n w''_n \end{cases} \tag{5}$$

for all $n \geq 1$, where $w_n \in T_1x_n$, $w'_n \in T_2z_n$, $w''_n \in T_3y_n$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$.
If $S_1 = S_2 = S_3 = I$ is a identity mapping, then the iterative process (5) reduces to the sequences as follows:

$$\begin{cases} z_n = (1-\gamma_n)x_n \oplus \gamma_n w_n \\ y_n = (1-\beta_n)z_n \oplus \beta_n w'_n \\ x_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n w''_n \end{cases} \tag{6}$$

for all $n \geq 1$, where $w_n \in T_1x_n$, $w'_n \in T_2z_n$, $w''_n \in T_3y_n$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$.

If $T_1 = T_2 = T_3 = T$ is a multi-valued non-expansive mapping, then the iterative process (5) reduces to the sequences (3).

Lemma 3.1. *Let K be a nonempty closed convex subset of a complete CAT(0) space X. Let $S_i : K \to K$ be single-valued non-expansive mappings and $T_i : K \to CB(K)$ be multi-valued non-expansive mappings with $F = \bigcap_{i=1}^{3} F(S_i) \bigcap F(T_i) \neq \emptyset$ with $T_i p = \{p\}$ for each $p \in \bigcap_{i=1}^{3} F(T_i)$ for all $i = 1, 2, 3$. Let $\{x_n\}$ be the mix type SP-iterates is defined by (5). Then $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.*

Proof. For $p \in F$, in view of Lemma 2.2 (ii) and Equation (5)

$$\begin{aligned} d(z_n, p) &= d((1-\gamma_n)S_1x_n \oplus \gamma_n w_n, p) \\ &\leq (1-\gamma_n)d(S_1x_n, p) + \gamma_n d(w_n, p) \\ &\leq (1-\gamma_n)d(S_1x_n, p) + \gamma_n dist(w_n, T_1p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n H(T_1x_n, T_1p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(x_n, p) \\ &= d(x_n, p) \end{aligned} \tag{7}$$

Also, we have

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)S_2z_n \oplus \beta_n w'_n, p) \\ &\leq (1-\beta_n)d(S_2z_n, p) + \beta_n d(w'_n, p) \\ &\leq (1-\beta_n)d(S_2z_n, p) + \beta_n dist(w'_n, T_2p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n H(T_2z_n, T_2p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(z_n, p) \\ &= d(z_n, p) \end{aligned} \tag{8}$$

Similarly, we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)S_3y_n \oplus \alpha_n w''_n, p) \\ &\leq (1-\alpha_n)d(S_3y_n, p) + \alpha_n d(w''_n, p) \\ &\leq (1-\alpha_n)d(S_3y_n, p) + \alpha_n dist(w''_n, T_3p) \\ &\leq (1-\alpha_n)d(y_n, p) + \alpha_n H(T_3y_n, T_3p) \\ &\leq (1-\alpha_n)d(y_n, p) + \alpha_n d(y_n, p) \\ &= d(y_n, p) \end{aligned} \tag{9}$$

By Equation (7), (8) and (9), we have

$$d(x_{n+1}, p) \leq d(y_n, p) \leq d(z_n, p) \leq d(x_n, p) \tag{10}$$

This implies that the sequence $\{d(x_n, p)\}$ is decreasing and bounded below, and so $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F$. The conclusion is proved. □

Lemma 3.2. *Let K be a nonempty closed convex subset of a complete CAT(0) space X. Let $S_i : K \to K$ be single-valued non-expansive mappings and $T_i : K \to CB(K)$ be multi-valued non-expansive mappings with $F = \bigcap_{i=1}^{3} F(S_i) \bigcap F(T_i) \neq \emptyset$ with $T_i p = \{p\}$ for each $p \in \bigcap_{i=1}^{3} F(T_i)$ for all $i = 1, 2, 3$. Let $\{x_n\}$ be the mix type SP-iterates is defined by* (5). *Assume that*

(i) there exist constants $b, c \in (0, 1)$ and $0 < b(1-c) \leq \frac{1}{2}$ such that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [b, c]$;

(ii) $d(x, u) \leq d(S_i x, u)$ for all $x, y \in K$, $u \in T_i y$.

Then (1) $\lim_{n\to\infty} d(x_n, S_i x_n) = 0, i = 1, 2, 3$; *(2)* $\lim_{n\to\infty} dist(x_n, T_i x_n) = 0, i = 1, 2, 3$.

Proof. (1) By Lemma 3.1, for each given $p \in F$, $\lim_{n\to\infty} d(x_n, p)$ exists, without loss of generality, we can assume that $\lim_{n\to\infty} d(x_n, p) = r \geq 0$. By Equation (7) and (8), we have

$$d(y_n, p) \leq d(z_n, p) \leq d(x_n, p).$$

Taking limsup on both sides, we can obtain

$$\limsup_{n\to\infty} d(y_n, p) \leq r \tag{11}$$

Since

$$d(S_3 y_n, p) \leq d(y_n, p)$$

and

$$d(w''_n, p) \leq H(T_3 y_n, T_3 p) \leq d(y_n, p),$$

It follows from Equation (11) that

$$\limsup_{n\to\infty} d(S_3 y_n, p) \leq r,$$

and

$$\limsup_{n\to\infty} d(w''_n, p) \leq r.$$

Notice that

$$\lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d((1-\alpha_n) S_3 y_n \oplus \alpha_n w''_n, p) = r,$$

by Lemma 2.2, we have

$$\lim_{n\to\infty} d(S_3 y_n, w''_n) = 0. \tag{12}$$

By condition (ii), $d(y_n, w''_n) \leq d(S_3 y_n, w''_n)$, and Equation (12), we have

$$\lim_{n\to\infty} d(y_n, w''_n) = 0. \tag{13}$$

Notice that $dist(y_n, T_3 y_n) \leq d(y_n, w''_n)$, so we have

$$\lim_{n\to\infty} dist(y_n, T_3 y_n) = 0. \tag{14}$$

Notice that

$$d(y_n, S_3 y_n) \leq d(y_n, w''_n) + d(S_3 y_n, w''_n), w''_n \in T_3 y_n,$$

by Equation (12), (13), we have

$$\lim_{n\to\infty} d(y_n, S_3 y_n) = 0. \tag{15}$$

Again,

$$\begin{aligned}
d(x_{n+1}, p) &= d((1-\alpha_n)S_3 y_n \oplus \alpha_n w''_n, p) \\
&\leq (1-\alpha_n)d(S_3 y_n, p) + \alpha_n d(w''_n, p) \\
&\leq (1-\alpha_n)d(S_3 y_n, p) + \alpha_n d(w''_n, S_3 y_n) + \alpha_n d(S_3 y_n, p) \\
&\leq d(y_n, p) + \alpha_n d(S_3 y_n, w''_n)
\end{aligned}$$

Taking liminf on both sides, by Equation (12), we can obtain

$$\liminf_{n\to\infty} d(y_n, p) \geq r.$$

It follows from Equation (11) that

$$\lim_{n\to\infty} d(y_n, p) = \lim_{n\to\infty} d((1-\beta_n)S_2 z_n \oplus \beta_n w'_n, p) = r. \tag{16}$$

Similarly, by Equation (7), we have

$$d(z_n, p) \leq d(x_n, p).$$

Taking limsup on both sides, we can obtain

$$\limsup_{n\to\infty} d(z_n, p) \leq r \tag{17}$$

Since

$$d(S_2 z_n, p) \leq d(z_n, p)$$

and

$$d(w_n', p) \leq H(T_2 z_n, T_2 p) \leq d(z_n, p),$$

It follows from Equation (17) that

$$\limsup_{n\to\infty} d(S_2 z_n, p) \leq r,$$

and

$$\limsup_{n\to\infty} d(w_n', p) \leq r.$$

By Equation (16) and Lemma 2.2, we have

$$\lim_{n\to\infty} d(S_2 z_n, w_n') = 0. \tag{18}$$

By condition (ii), $d(z_n, w_n') \leq d(S_2 z_n, w_n')$, and Equation (18), we have

$$\lim_{n\to\infty} d(z_n, w_n') = 0. \tag{19}$$

Notice that $dist(z_n, T_2 z_n) \leq d(z_n, w_n')$, so we have

$$\lim_{n\to\infty} dist(z_n, T_2 z_n) = 0. \tag{20}$$

Notice that

$$d(z_n, S_2 z_n) \leq d(z_n, w_n') + d(S_2 z_n, w_n'), w_n' \in T_2 z_n,$$

by Equation (18) and (19), we have

$$\lim_{n\to\infty} d(z_n, S_2 z_n) = 0. \tag{21}$$

Again,

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n) S_2 z_n \oplus \beta_n w_n', p) \\ &\leq (1-\beta_n) d(S_2 z_n, p) + \beta_n d(w_n', p) \\ &\leq (1-\beta_n) d(S_2 z_n, p) + \beta_n d(w_n', S_2 z_n) + \beta_n d(S_2 z_n, p) \\ &\leq d(z_n, p) + \beta_n d(S_2 z_n, w_n') \end{aligned}$$

Taking liminf on both sides, by Equation (18), we can obtain

$$\liminf_{n\to\infty} d(z_n, p) \geq r.$$

It follows from Equation (17) that

$$\lim_{n\to\infty} d(z_n, p) = \lim_{n\to\infty} d((1-\gamma_n)S_1x_n \oplus \gamma_n w_n, p) = r. \tag{22}$$

Since

$$d(S_1x_n, p) \le d(x_n, p)$$

and

$$d(w_n, p) \le H(T_1x_n, T_1p) \le d(x_n, p),$$

By $\lim_{n\to\infty} d(x_n, p) = r$, we have

$$\limsup_{n\to\infty} d(S_1x_n, p) \le r,$$

and

$$\limsup_{n\to\infty} d(w_n, p) \le r.$$

By Equation (22) and Lemma 2.2, we have

$$\lim_{n\to\infty} d(S_1x_n, w_n) = 0. \tag{23}$$

By condition (ii), $d(x_n, w_n) \le d(S_1x_n, w_n)$, and Equation (23), we have

$$\lim_{n\to\infty} d(x_n, w_n) = 0. \tag{24}$$

Notice that $dist(x_n, T_1x_n) \le d(x_n, w_n)$, so we have

$$\lim_{n\to\infty} dist(x_n, T_1x_n) = 0. \tag{25}$$

Notice that

$$d(x_n, S_1x_n) \le d(x_n, w_n) + d(S_1x_n, w_n), w_n \in T_1x_n,$$

by (23) and (24), we have

$$\lim_{n\to\infty} d(x_n, S_1x_n) = 0. \tag{26}$$

Since

$$\begin{aligned} d(x_{n+1}, y_n) &= d((1-\alpha_n)S_3y_n \oplus \alpha_n w_n'', y_n) \\ &\le (1-\alpha_n)d(S_3y_n, y_n) + \alpha_n d(w_n'', y_n) \end{aligned}$$

by Equation (13) and (15), we have

$$\lim_{n\to\infty} d(x_{n+1}, y_n) = 0. \tag{27}$$

Similarly, since

$$\begin{aligned} d(y_n, z_n) &= d((1-\beta_n)S_2 z_n \oplus \beta_n w'_n, z_n) \\ &\leq (1-\beta_n)d(S_2 z_n, z_n) + \beta_n d(w'_n, z_n) \end{aligned}$$

by Equation (19) and (21), we have

$$\lim_{n\to\infty} d(y_n, z_n) = 0. \tag{28}$$

Since

$$\begin{aligned} d(z_n, x_n) &= d((1-\gamma_n)S_1 x_n \oplus \gamma_n w_n, x_n) \\ &\leq (1-\gamma_n)d(S_1 x_n, x_n) + \gamma_n d(w_n, x_n) \end{aligned}$$

by Equation (24) and (26), we have

$$\lim_{n\to\infty} d(z_n, x_n) = 0. \tag{29}$$

Notice that $d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$, by Equation (27) and (28), we have

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{30}$$

Notice that $d(x_{n+1}, x_n) \leq d(x_{n+1}, y_n) + d(y_n, z_n) + d(y_n, x_n)$, by Equation (27), (27) and (28), we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{31}$$

Since $d(y_{n+1}, y_n) \leq d(y_{n+1}, x_{n+1}) + d(x_{n+1}, y_n)$, by Equation (27) and (28), we have

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = 0. \tag{32}$$

Since $d(z_{n+1}, z_n) \leq d(z_{n+1}, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, z_n)$, by Equation (28) and (29), we have

$$\lim_{n\to\infty} d(z_{n+1}, z_n) = 0. \tag{33}$$

Since

$$\begin{aligned} d(x_n, S_2x_n) &\le d(x_n, z_n) + d(z_n, S_2z_n) + d(S_2z_n, S_2x_n) \\ &\le 2d(x_n, z_n) + d(z_n, S_2z_n) \end{aligned}$$

by Equation (21) and (28), we have

$$\lim_{n\to\infty} d(x_n, S_2x_n) = 0. \tag{34}$$

Similarly, since

$$\begin{aligned} d(x_n, S_3x_n) &\le d(x_n, y_n) + d(y_n, S_3y_n) + d(S_3y_n, S_3x_n) \\ &\le 2d(x_n, y_n) + d(y_n, S_3y_n) \end{aligned}$$

by Equation (15) and (28), we have

$$\lim_{n\to\infty} d(x_n, S_3x_n) = 0. \tag{35}$$

Hence

$$\lim_{n\to\infty} d(x_n, S_ix_n) = 0, i = 1, 2, 3.$$

(2) By Equation (25), we have

$$\lim_{n\to\infty} dist(x_n, T_1x_n) = 0.$$

Notice that

$$\begin{aligned} dist(x_n, T_2x_n) &\le d(x_n, z_n) + dist(z_n, T_2z_n) + H(T_2z_n, T_2z_n) \\ &\le 2d(x_n, z_n) + dist(z_n, T_2x_n) \end{aligned}$$

by Equation (20) and (28), we have

$$\lim_{n\to\infty} dist(x_n, T_2x_n) = 0. \tag{36}$$

Similarly, since

$$\begin{aligned} dist(x_n, T_3x_n) &\le d(x_n, y_n) + dist(y_n, T_3y_n) + H(T_3y_n, T_3x_n) \\ &\le 2d(x_n, y_n) + dist(y_n, T_3y_n) \end{aligned}$$

by Equation (14) and (28), we have

$$\lim_{n\to\infty} dist(x_n, T_3x_n) = 0. \tag{37}$$

Hence

$$\lim_{n\to\infty} dist(x_n, T_ix_n) = 0, i = 1,2,3.$$

The conclusion is proved. □

Now, we find two mappings, $S_1 = S_2 = S_3 = S$ and $T_1 = T_2 = T_3 = T$, satisfying the condition (ii) in Lemma 3.1 as follows.

Example 3.3. *Let $X = (-\infty, \infty)$ with the usual norm $|.|$ and let $K = [-1,1]$. Define the single-valued mapping $S : K \to K$, the multi-valued mapping $T : K \to C(K)$ by*

$$Sx = \begin{cases} -x, x \in [0,1], \\ x, x \in [-1,0), \end{cases} \quad Tx = \begin{cases} [0,x], x \in [0,1], \\ 0, x \in [-1,0), \end{cases}$$

Now, we show that $S : K \to K$ is single-valued non-expansive mapping. In fact, if $x, y \in [0,1]$, then we have

$$|Sx - Sy| = |-x+y| = |x-y|.$$

If $x, y \in [-1,0)$, then we have

$$|Sx - Sy| = |x-y|.$$

If $x \in [0,1]$, $y \in [-1,0)$, then we have

$$|Sx - Sy| = |-x-y| = |x+y| \le |x-y|.$$

If $x \in [-1,0)$, $y \in [0,1]$, then we have

$$|Sx - Sy| = |x+y| \le |x-y|.$$

This implies that S is non-expansive.

Next, we show that $T : K \to C(K)$ is multi-valued non-expansive mapping. In fact, if $x, y \in [-1,0)$, then we have

$$H(Tx, Ty)| = 0 \le |x-y|.$$

If $x \in [-1,0], y \in [0,1]$, then we have

$$H(Tx, Ty)| = \max\{0, |y|\} = |y| \le |x-y|.$$

If $x \in [0,1], y \in [-1,0)$, then we have

$$H(Tx,Ty)| = \max\{0,|x|\} = |x| \le |x-y|.$$

If $x, y \in [0,1]$, without loss of generality, let $x \le y$,

$$H(Tx,Ty) = \max\{\sup_{a\in Tx} d(a,Ty), \sup_{b\in Ty} d(b,Tx)\}.$$

$\forall a \in Tx$, $d(a,Ty) = \inf\{|a-c| : c \in Ty\} = 0$, then

$$\sup_{a\in Tx} d(a,Ty) = 0.$$

$\forall b \in Ty$, $d(b,Tx) = \inf\{|b-c'| : c' \in Tx\}$, then

$$d(b,Tx) = 0, \quad \forall b \le x,$$

$$d(b,Tx) = |b-x|, \quad \forall b > x,$$

so, we have

$$\sup_{b\in Ty} d(b,Tx) = \begin{cases} 0, b < x, \\ |y-x|, b \ge x, \end{cases}$$

So, If $x, y \in [0,1]$, then we have

$$H(Tx,Ty) = \max\{\sup_{a\in Tx} d(a,Ty), \sup_{b\in Ty} d(b,Tx)\} \le |x-y|.$$

This implies that T is multi-valued non-expansive.

Next, we show that two mappings S, T satisfy the condition (ii) in Lemma 3.2.

Case 1: Let $x, y \in [-1,0)$. Then we have, $Ty = 0$, that is $u = 0$, then

$$|x-u| = |Sx| = |Sx-u|.$$

Case 2: Let $x \in [-1,0), y \in [0,1]$. Then we have, $Ty = [0,y]$, that is $u \in [0,y]$, then

$$|x-u| = |Sx-u|.$$

Case 3: Let $x \in [0,1], y \in [-1,0)$. Then we have, $Ty = 0$, that is $u = 0$, then

$$|x-u| = |Sx| = |Sx-u|.$$

Case 4: Let $x, y \in [0,1]$. Then we have, $Ty = [0, y]$, that is $u \in [0, y] \subset [0,1]$, then

$$|x - u| \leq |-x - u| = |Sx - u|.$$

Therefore, the condition (ii) in Lemma 3.2 is satisfied.

Now, we give the $\triangle$-convergence theorem of the mixed type SP-iteration on a CAT(0) space.

Theorem 3.4. *Let K be a nonempty closed convex subset of a complete CAT(0) space X. Let $S_i : K \to K$ be single-valued non-expansive mappings and $T_i : K \to CB(K)$ be multi-valued non-expansive mappings with $F = \bigcap_{i=1}^{3} F(S_i) \bigcap F(T_i) \neq \emptyset$ with $T_i p = \{p\}$ for each $p \in \bigcap_{i=1}^{3} F(T_i)$ for all $i = 1,2,3$. Let $\{x_n\}$ be the mix type SP-iterates is defined by* (5). *Assume that*

(i) there exist constants $b, c \in (0,1)$ and $0 < b(1-c) \leq \frac{1}{2}$ such that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [b, c]$;

(ii) $d(x,u) \leq d(S_i x, u)$ for all $x, y \in K$, $u \in T_i y$, $i = 1,2,3$.

Then $\{x_n\}$ $\triangle$-converges to a common fixed point of S_1, S_2, S_3, T_1, T_2 and T_3.

Proof. By Lemma 3.1 and Lemma 3.2 , we have $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$, so that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, S_i x_n) = 0 = \lim_{n\to\infty} dist(x_n, T_i x_n)$, $i = 1,2,3$.

Let $W_w(\{x_n\}) =: \bigcup A(\{u_n\})$, where union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. To show that the $\triangle$-convergence of $\{x_n\}$ to a common fixed point of S_1, S_2, S_3, T_1, T_2 and T_3, firstly we will prove $W_w(\{x_n\}) \subset F$ and thereafter argue that $W_w(\{x_n\})$ is a singleton set. To show $W_w(\{x_n\}) \subset F$, let $y \in W_w(\{x_n\})$. Then, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $A(y_n) = y$. By Lemmas 2.5 and 2.6, there exists a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\triangle - \lim_{n\to\infty} z_n = z$ and $z \in K$. Since $\lim_{n\to\infty} d(z_n, S_i z_n) = 0$. In view of Lemma 2.7, we have $z = S_i z, i = 1,2,3$.

Similarly, by Lemma 2.8, we can show that $z \in T_i z$, $i = 1,2,3$, hence $z \in F$. Now, we claim that $z = y$. Let on contrary that $z \neq y$, then we have

$$\begin{aligned}
\limsup_{n\to\infty} d(z_n, z) &< \limsup_{n\to\infty} d(z_n, y) \\
&\leq \limsup_{n\to\infty} d(y_n, y) \\
&< \limsup_{n\to\infty} d(y_n, z) \\
&= \limsup_{n\to\infty} d(x_n, z) \\
&= \limsup_{n\to\infty} d(z_n, z)
\end{aligned}$$

which is a contradiction and hence $z = y \in F$.

To show that $W_w(\{x_n\})$ is a singleton, let $\{y_n\}$ be a subsequence of $\{x_n\}$. In view of Lemmas 2.5 and 2.6, there exists a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\triangle - \lim_{n\to\infty} z_n = z$. Let $A(\{y_n\}) = \{y\}$ and $A(\{x_n\}) = \{x\}$. Earlier, we have shown that $y = z$; therefore, it is enough to show $z = x$. If $z \neq x$ then by Lemma 3.1, $\{d(x_n,z)\}$ is convergent. By uniqueness of asymptotic centers

$$\begin{aligned}\limsup_{n\to\infty} d(z_n,z) &< \limsup_{n\to\infty} d(z_n,x)\\ &\le \limsup_{n\to\infty} d(x_n,x)\\ &< \limsup_{n\to\infty} d(x_n,z)\\ &= \limsup_{n\to\infty} d(z_n,z)\end{aligned}$$

which is a contradiction. Hence the conclusion follows. □

Now, we prove a strong convergence theorem which extends Theorem 1 of [19] for the mixed type SP-iteration in CAT(0) spaces

Theorem 3.5. *Let K be a nonempty closed convex subset of a complete CAT(0) space X. Let $S_i : K \to K$ be single-valued non-expansive mappings and $T_i : K \to CB(K)$ be multi-valued non-expansive mappings with $F = \bigcap_{i=1}^{3} F(S_i) \bigcap F(T_i) \neq \emptyset$ with $T_i p = \{p\}$ for each $p \in \bigcap_{i=1}^{3} F(T_i)$ for all $i = 1,2,3$. Let $\{x_n\}$ be the mix type SP-iterates is defined by (3.1). Assume that*

(1) there exist constants $b, c \in (0,1)$ and $0 < b(1-c) \le \frac{1}{2}$ such that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [b,c]$;

(2) $d(x,u) \le d(S_i x,u)$ for all $x,y \in K$, $u \in T_i y$, $i = 1,2,3$.

Then $\{x_n\}$ strong converges to a common fixed point of S_1, S_2, S_3, T_1, T_2 and T_3 if and only if $\liminf_{n\to\infty} dist(x,F) = 0$.

Proof. The necessity is obvious. We only prove the sufficiency, suppose that $\liminf_{n\to\infty} dist(x_n,F) = 0$. By (9) we have,

$$d(x_{n+1},p) \le d(x_{n+1},p).$$

This gives

$$dist(x_{n+1},F) \le dist(x_n,F).$$

Hence $\lim_{n\to\infty} dist(x_n,F)$ exists. By hypothesis, $\liminf_{n\to\infty} dist(x_n,F) = 0$, therefore we must have $\lim_{n\to\infty} dist(x_n,F) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence in K. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} dist(x_n,F) = 0$, therefore there exists a constant n_0 such that for all $n \ge n_0$, we have

$$dist(x_n,F) < \frac{\epsilon}{4}.$$

In particular, $dist(x_{n_0},F) < \frac{\epsilon}{4}$. That is $\inf\{d(x_{n_0},p) : p \in F\} < \frac{\epsilon}{4}$. So there must exist a $p^* \in F$ such that

$$d(x_{n_0},p^*) < \frac{\epsilon}{2}.$$

Now for $m, n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(p^*, x_n) \\ &\leq 2d(x_{n_0}, p^*) \\ &< 2 \times \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space X, and therefore it must converge in K. Let $\lim_{n\to\infty} x_n = q$. Now, for $i = 1, 2, 3$,

$$\begin{aligned} d(q, S_i q) &\leq d(q, x_n) + d(x_n, S_i x_n) + d(S_i x_n, q) \\ &\leq d(q, S_i x_n) + d(x_n, S_i x_n) + d(x_n, q) \\ &\to 0 \quad as \quad n \to 0 \end{aligned}$$

gives that $d(q, S_i q) = 0$ which implies that $q = S_i q, i = 1, 2, 3$. For $i = 1, 2, 3$,

$$\begin{aligned} dist(q, T_i q) &\leq d(q, x_n) + dist(x_n, T_i x_n) + H(T_i x_n, T_i q) \\ &\leq d(q, x_n) + dist(x_n, T_i x_n) + d(x_n, q) \\ &\to 0 \quad as \quad n \to 0 \end{aligned}$$

gives that $dist(q, T_i q) = 0$ which implies that $q \in T_i q, i = 1, 2, 3$. Consequently, $q \in F$. □

As an application of Theorem 3.5, we can get the following result:

Theorem 3.6. *Let K be a nonempty closed convex subset of a complete CAT(0) space X. Let $S_1, S_2, S_3 : K \to K$ be three single-valued non-expansive mappings, $T_1, T_2, T_3 : K \to C(K)$ be three multi-valued non-expansive mappings satisfying condition (A'). Assume that $F = \bigcap_{i=1}^{3} F(S_i) \bigcap F(T_i) \neq \emptyset$ and $T_i(p) = p, (i = 1, 2, 3)$ for each $p \in F$. Let $\{x_n\}$ be the mix type SP-iterates is defined by (3.1), then $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, S_3, T_1, T_2 and T_3.*

Proof. As proof in Theorem 3.5, we know $\lim_{n\to\infty} dist(x_n, F)$ exists. So by condition(A'), then

$$\lim_{n\to\infty} f(dist(x_n, F)) \leq \lim_{n\to\infty} d(x_n, S_1 x_n) = 0,$$

or

$$\lim_{n\to\infty} f(dist(x_n, F)) \leq \lim_{n\to\infty} d(x_n, S_2 x_n) = 0,$$

or

$$\lim_{n\to\infty} f(dist(x_n, F)) \leq \lim_{n\to\infty} d(x_n, S_3 x_n) = 0,$$

or

$$\lim_{n\to\infty} f(dist(x_n, F)) \le \lim_{n\to\infty} dist(x_n, T_1 x_n) = 0,$$

or

$$\lim_{n\to\infty} f(dist(x_n, F)) \le \lim_{n\to\infty} dist(x_n, T_2 x_n) = 0.$$

or

$$\lim_{n\to\infty} f(dist(x_n, F)) \le \lim_{n\to\infty} dist(x_n, T_3 x_n) = 0.$$

we have

$$\lim_{n\to\infty} f(dist(x_n, F)) = 0.$$

Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function and $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, there we have $\lim_{n\to\infty} dist(x_n, F) = 0$. Now all the conditions of Theorem 3.5 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a point of F. □

If $S_1 = S_2 = S_3 = I$ is a identity mapping, the following corollaries are direct consequences of Theorems 3.4, 3.5 and 3.6.

Corollary 3.7. *Let K be a nonempty closed convex subset of a complete CAT(0) space X. Let $T_i : K \to CB(K)$ be multi-valued non-expansive mappings with $F = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ with $T_i p = \{p\}$ for each $p \in \bigcap_{i=1}^{3} F(T_i)$ for all $i = 1, 2, 3$. Let $\{x_n\}$ be SP-iterates is defined by (6). Assume that there exist constants $b, c \in (0, 1)$ and $0 < b(1 - c) \le \frac{1}{2}$ such that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [b, c]$, then $\{x_n\}$ $\triangle$-converges to a common fixed point of T_1, T_2 and T_3.*

Corollary 3.8. *Let K be a nonempty closed convex subset of a complete CAT(0) space X. Let $T_i : K \to CB(K)$ be multi-valued non-expansive mappings with $F = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ with $T_i p = \{p\}$ for each $p \in \bigcap_{i=1}^{3} F(T_i)$ for all $i = 1, 2, 3$. Let $\{x_n\}$ be SP-iterates is defined by (6). Assume that there exist constants $b, c \in (0, 1)$ and $0 < b(1 - c) \le \frac{1}{2}$ such that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [b, c]$, then $\{x_n\}$ strong converges to a common fixed point of T_1, T_2 and T_3 if and only if $\liminf_{n\to\infty} dist(x, F) = 0$.*

Corollary 3.9. *Let K be a nonempty closed convex subset of a complete CAT(0) space X. Let $T_1, T_2, T_3 : K \to C(K)$ be three multi-valued non-expansive mappings satisfying condition (A'). Assume that $F = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and $T_i(p) = p, (i = 1, 2, 3)$ for each $p \in F$. Let $\{x_n\}$ be the mix type SP-iterates is defined by (6), then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.*

If $T_1 = T_2 = T_3 = T$ is a multi-valued non-expansive mapping, the following corollaries are also direct consequences of Theorems 3.4, 3.5 and 3.6.

Corollary 3.10. *Let K be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : K \to CB(K)$ be a multi-valued non-expansive mapping with $F(T) \neq \emptyset$ with $Tp = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be SP-iterates is defined by (3). Assume that there exist constants $b, c \in (0, 1)$ and $0 < b(1 - c) \le \frac{1}{2}$ such that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [b, c]$, then $\{x_n\}$ $\triangle$-converges to a common fixed point of T.*

Corollary 3.11. *Let K be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : K \to CB(K)$ be a multi-valued non-expansive mapping with $F(T) \neq \emptyset$ with $Tp = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be SP-iterates is defined by* (3)*. Assume that there exist constants $b, c \in (0, 1)$ and $0 < b(1 - c) \leq \frac{1}{2}$ such that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [b, c]$, then $\{x_n\}$ strong converges to a fixed point of T if and only if* $\liminf_{n\to\infty} dist(x, F(T) = 0$.

Corollary 3.12. *Let K be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : K \to C(K)$ be a multi-valued non-expansive mapping satisfying condition (A'). Assume that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be the mix type SP-iterates is defined by* (3)*, then $\{x_n\}$ converges strongly to a fixed point of T.*

4. Conclusions

In this article, we extend known results on convergence of SP-iterations to fixed points of single-valued non-expansive mappings or multi-valued non-expansive mappings to single-valued non-expansive mappings and multi-valued non-expansive mappings mixed type version. In order to do so, we prove strong and $\triangle-$convergence theorems for the mixed type SP-iteration schemes involving three single-valued non-expansive mappings and three multi-valued non-expansive mappings in the framework of CAT(0) spaces.

Author contributions

YL and SW wrote the main manuscript text, XH prepared the research ideas of this paper. All authors reviewed the manuscript.

Funding

This work was supported by the Research Foundation of Science and Technology Research Project of Jiangxi Provincial Department of Education (grant no. GJJ219409), the Foundation of Teaching Construction of Changzhou Institute of Technology(grant no. JGKT2023-11).

References

[1] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Berlin: Springer, (1999).

[2] D. Burago, Y. Burago and S. Ivanov, *A course in metric geometry. In: Graduate Studies in Mathematics*, Rhode Island: American Mathematical Society Providence, (2001).

[3] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, In: Progress in Mathematics, vol. 152, Boston: Birkhäuse, (1999).

[4] I. Bartolini, P. Ciaccia and M. Patella, *String matching with metric trees using an approximate distance*, SPIRE Lecture Notes in Computer Science, 2476(2002), 271-283.

[5] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, New York: Dekker, (1984).

[6] S. Dhompongsa, A. Kaewkhao and B. Panyanak, *Lim's theorems for multi-valued mappings in CAT(0) spaces*, J. Math. Anal. Appl., 312(2005), 478-487.

[7] W. Laowang and B. Panyanak, *Strong and $\triangle$-convergence theorems for multi-valued mappings in CAT(0) spaces*, J. Inequal. Appl., 2009(2009), Article ID 730132.

[8] N. Shahzad, *Invariant approximations in CAT(0) spaces*, Nonlinear Anal., 70(2009), 4338-4340.

[9] N. Shahzad and J. Markin, *Invariant approximations for commuting mappings in hyperconvex and CAT(0) spaces*, J. Math. Anal. Appl., 337(2008), 1457-1464.

[10] N. Shahzad, *Fixed point result for multimaps in CAT(0) space*, Topol. Appl., 156(2009), 997-1001.

[11] A. Abkar and M. Eslamian, *Convergence theorems for a finite family of generalized non-expansive multi-valued mappings in CAT(0) spaces*, Nonlinear Anal., 75(2012), 1895-1903.

[12] K.K. Tan and H.K. Xu, *Approximating Fixed Points of Nonexpansive Mappings by the Ishikawa Iteration Process*, J. Math. Anal. Appl., 178(1993), 301-308.

[13] A. Abkar and M. Eslamian, *New iteration scheme for numerical reckoning fixed points of non-expansive mappings*, J. Inequal. Appl., 2014(2014), Article ID 328.

[14] K. P. R. Sastry and G. V. R. Babu, *Convergence of Ishikawa Iterates for a Multi-Valued Mapping with a Fixed Point*, Czech. Math. J., 55(2005), 817-826.

[15] B. Panyanak, *Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces*, Comput. Math. Appl., 54(2007), 872-877.

[16] Y. Song and H. Wang, *Erratum to "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces*, Comput. Math. Appl., 55(2008), 2999-3002.

[17] Y. Song and H. Wang, *Convergence of iterative algorithms for multivalued mappings in Banach spaces*, Comput. Math. Appl., 70(2009), 1547-1556.

[18] N. Shahzad and H. Zegeye, *On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces*, Nonlinear Anal., 71(2009), 838-844.

[19] S. Dhompongsa and B. Panyanak, *On $\triangle$-convergence theorems in CAT(0) spaces*, Comput. Math. Appl., 56(2008), 2572-2579.

[20] W. Phuengrattana and S. Suantai, *On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval*, J. Comput. Appl. Math., 235(2011), 3006-3014.

[21] R. P. Pathak, *On Noor-type iteration schemes for multivalued mappings in CAT(0) spaces*, Fixed Point Theory Appl., 2015(2015), 133.

[22] K. Sokhuma, *On SP-iteration schemes for multi-valued mappings in CAT(0) spaces*, Fixed Point Theory, 19(2018), 775-784.

[23] Y. Liu, *Convergence theorems for mixed type iterative process of single-valued and multi-valued non-expansive mappings and applications*, Asian-European J. Math., 14(2021), 775-784.

[24] A. Sahin and M. Basarır, *On the strong and △-convergence of SP-iteration on CAT(0) space*, J. Inequal. Appl., 2013(2013), 311.

[25] I. Uddin, J. J. Nieto and J. Ali, *One-step iteration scheme for multivalued non-expansive mappings in CAT(0) spaces*, J. Inequal. Appl., 13(2016), 1211-1225.

[26] I. Uddin, J. J. Nieto and J. Ali, *On SP-type iteration Schemes for multivalued mappings in CAT(0) spaces*, Far East J. Math. Sci., 100(2016), 681-696.

[27] S. S. Chang, *Demiclosed principle and △-convergence theorems for total asymptotically non-expansive mappings in CAT(0) spaces*, Appl. Math. Comput., 219(2012), 2611-2617.

[28] S. Dhompongsa, W. A. Kirk and B. Sims, *Fixed points of uniformly lipschitzian mappings*, Appl. Math. Comput., 4(2006), 762-772.

[29] S. H. Khan and H. Fukhar-ud-din, *Weak and strong convergence of a scheme with errors for two non-expansive mappings*, Nonlinear Analysis: Theory, Methods and Applications, 6(2005), 1295-1301.

[30] H. Fukhar-ud-din and S. H. Khan, *Convergence of iterates with errors of asymptotically quasi-non-expansive mappings and applications*, J. Math. Anal. Appl., 328(2007), 821-829.

[31] M. Abbas, *Common fixed points of two multivalued non-expansive mappings by one-step iterative scheme*, Appl. Math. Letters., 24(2011), 97-102.

[32] W.A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Analysis: Theory, Methods and Applications, 68(2008), 3689-3696.

[33] T. C. Lim, *Remarks on some fixed point theorems*, Proc. Am. Math. Soc., 60(1976), 179-182.

[34] S. Dhompongsa, W. A. Kirk and B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal., 8(2007), 35-45.