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Solution and Stability of an n-Dimensional AQ-Functional Equation

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Abstract

In this paper, the authors established the general solution and generalized Ulam - Hyers stability of an mixed type n dimensional additive quadratic functional equation in Banach spaces using Hyers method. The stability results are proved in two ways by considering n is an even and odd positive integer.

Keywords: Additive functional equations; quadratic functional equation; Mixed type functional equation; Ulam - Hyers stability.
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1. Introduction

The stability problem of functional equations initiated from a question of S.M. Ulam [28] concerning the stability of group homomorphisms. D.H. Hyers [18] contributed a first positive partial reply to the question of Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [2] for additive mappings, Th.M. Rassias [24] and J.M. Rassias [23] for linear mappings by considering an unbounded Cauchy difference. A generalization of all the above results was achieved by P. Gavruta [15] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias method. In 2008, a special case of Gavrutas theorem for the unbounded Cauchy difference was obtained by Ravi et.al., [26] by considering the summation of both the sum and the product of two norms in the sprit of Rassias approach.
The famous Additive Functional Equation and Quadratic Functional Equation are

$$f(u+v) = f(u) + f(v), \tag{1}$$

$$f(u+v) + f(u-v) = 2f(u) + 2f(v). \tag{2}$$

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The solution and stability of the above functional equations and several other types of functional equations in various settings are given in [1,12,19–21,25,27]. In 2005 and 2006, K.W. Jun, H.M. Kim [16,17] introduced and discussed the general solution and the generalized Hyers-Ulam stability for the following additive and quadratic type functional equations

$$f\left(x+ay\right)+af\left(x-y\right)=f\left(x-ay\right)+af\left(x+y\right);a\neq-1,0,1, \tag{3}$$

$$f\left(\sum_{i=1}^{n}x_i\right)+(n-2)\sum_{i=1}^{n}f\left(x_i\right)=\sum_{1\leq i<j\leq n}f\left(x_i+x_j\right). \tag{4}$$

and investigated the generalized Hyers-Ulam-Rassias stability. The general solution and generalized Hyers - Ulam stability of the succeeding mixed type additive-quadratic functional equations

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x), \tag{5}$$

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)-2f(x), \tag{6}$$

$$\begin{aligned} f\left(-x_1\right)+f\left(2x_1-\sum_{i=2}^{n}x_i\right)+f\left(2\sum_{i=2}^{n}x_i\right)+f\left(x_1+\sum_{i=2}^{n}x_i\right)-f\left(-x_1-\sum_{i=2}^{n}x_i\right)\\ -f\left(x_1-\sum_{i=2}^{n}x_i\right)-f\left(-x_1+\sum_{i=2}^{n}x_i\right)\\ =3f\left(x_1\right)+3f\left(\sum_{i=2}^{n}x_i\right), \end{aligned} \tag{7}$$

$$g(x+y)+g(x-y)=2g(x)+g(y)+g(-y), \tag{8}$$

$$\sum_{i=0}^{n}\left[f\left(x_{2i}+x_{2i+1}\right)+f\left(x_{2i}-x_{2i+1}\right)\right]=\sum_{i=0}^{n}\left[2f\left(x_{2i}\right)+f\left(x_{2i+1}\right)+f\left(-x_{2i+1}\right)\right] \tag{9}$$

$$f(2x+y)-f(2x-y)=2[f(x+y)-f(x-y)]-f(y)+f(-y) \tag{10}$$

$$\begin{aligned} f(x-t)+f(y-t)+f(z-t)=3f\left(\frac{x+y+z}{3}-t\right)+f\left(\frac{2x-y-z}{3}\right)\\ +f\left(\frac{-x+2y-z}{3}\right)+f\left(\frac{-x-y+2z}{3}\right) \end{aligned} \tag{11}$$

$$\begin{aligned} h(x+2y+3z)+h(x+2y-3z)+h(x-2y+3z)+h(-x+2y+3z)=h(x+y+z)\\ +h(x+y-z)+h(x-y+z)+h(-x+y+z)+2h(y)\\ +4h(z)+5[h(y)+h(-y)]+14[h(z)+h(-z)] \end{aligned} \tag{12}$$

were explored by A. Najati, M.B. Moghimi [22], M.E. Gordji et. al., [14], M. Arunkumar, S. Karthikeyan [3], M. Arunkumar, J.M. Rassias [4], M.Arunkumar [5], M. Arunkumar et.al., [7–9]. In this paper, the authors established the general solution and generalized Ulam - Hyers stability of an mixed type n dimensional additive quadratic functional equation

$$f_{12}\left(2u_1+\sum_{i=2}^{n}u_i\right)+f_{12}\left(2u_1-\sum_{i=2}^{n}u_i\right)=f_{12}\left(\sum_{i=1}^{n}u_i\right)$$

$$+f_{12}\left(u_1-\sum_{i=2}^{n}u_i\right)+f_{12}(2u_1)+(f_{12}(u_1)+f_{12}(-u_1)) \tag{13}$$

where n is positive integer with $n \geq 2$ in Banach spaces using Hyers method. The stability results are proved in two ways by considering n is an even and odd positive integer.

2. General Solution of (13)

In this section, we present the general solution of the n dimensional additive quadratic functional equation (13).

Theorem 2.1. *If an odd function $f_{12}: U \to V$ satisfies the functional equation (1) if and only if $f_{12}: U \to V$ satisfies (13) for all $u, v, u_1, u_2, \cdots, u_n \in U$ where U and V be real vector spaces.*

Proof. By data $f_{12}: U \to V$ satisfies the functional equation (1), that is

$$f_{12}(u+v) = f_{12}(u) + f_{12}(v); \forall \ u, v \in U. \tag{14}$$

It is easy to verify from (14) that

$$f_{12}(0) = 0; \ \ f_{12}(2u) = 2f_{12}(u); \ \ f_{12}(3u) = 3f_{12}(u); \ \ f_{12}(Ku) = Kf_{12}(u); \forall \ u \in U. \tag{15}$$

Replacing u by $2u$ in (14) , one can have

$$f_{12}(2u+v) = f_{12}(2u) + f_{12}(v); \forall \ u, v \in U. \tag{16}$$

Replacing $v = -v$ in (16) and using (15), one can arrive

$$f_{12}(2u-v) = f_{12}(2u) - f_{12}(v); \forall \ u, v \in U. \tag{17}$$

Adding (16), (17) and using (15) as well as (14), one can obtain

$$f_{12}(2u+v) + f_{12}(2u-v) = f_{12}(u+v) + f_{12}(u-v) + f_{12}(2u); \forall \ u, v \in U. \tag{18}$$

Adding $f_{12}(u)$ on both sides of (18), one can get

$$f_{12}(2u+v) + f_{12}(2u-v) + f_{12}(u) = f_{12}(u+v) + f_{12}(u-v) + f_{12}(2u) + f_{12}(u); \forall \ u, v \in U. \tag{19}$$

Using (15), the above equation (19) can be rewritten as

$$f_{12}(2u+v) + f_{12}(2u-v) = f_{12}(u+v) + f_{12}(u-v) + f_{12}(2u) + f_{12}(u) + f_{12}(-u); \forall \ u, v \in U. \tag{20}$$

Finally replacing u by u_1 and v by $u_2 + \cdots + u_n$ in (19), we arrive (13).

Conversely, by data $f_{12} : U \to V$ satisfies the functional equation (13). If we interchange $u_3 = \cdots u_n = 0$ in (13), and using oddness of f_{12} in (13), one can see

$$f_{12}(2u_1 + u_2) + f_{12}(2u_1 - u_2) = f_{12}(u_1 + u_2) + f_{12}(u_1 - u_2) + f_{12}(2u_1); \forall\ \ u_1, u_2 \in U. \tag{21}$$

It is easy to verify from (21) that

$$f_{12}(0) = 0;\ \ f_{12}(2u) = 2f_{12}(u);\ \ f_{12}(3u) = 3f_{12}(u);\ \ f_{12}(Ku) = Kf_{12}(u); \forall\ \ u \in U. \tag{22}$$

Setting u_1 by $\frac{u+v}{2}$ and u_2 by $u - v$ in (21) and using (22) as well as oddness of f_{12}, one can arrive

$$4f_{12}(u) + 4f_{12}(v) = f_{12}(3u - v) - f_{12}(u - 3v) + 2f_{12}(u + v); \forall\ \ u, v \in U. \tag{23}$$

Setting u by $\frac{u+v}{2}$ and v by $=\frac{u-v}{2}$ in (32) and using (22) as well as oddness of f_{12}, one can have

$$2f_{12}(u + v) + 2f_{12}(u - v) = f_{12}(u + 2v) + f_{12}(u - 2v) + 2f_{12}(u); \forall\ \ u, v \in U. \tag{24}$$

Interchanging u and v in (24) and using oddness of f_{12}, one can get

$$f_{12}(2u + v) - f_{12}(2u - v) = 2f_{12}(u + v) - 2f_{12}(u - v) - 2f_{12}(v); \forall\ \ u, v \in U. \tag{25}$$

Using oddness of f_{12}, the above equation (25) can be rewritten as

$$f_{12}(2u + v) - f_{12}(2u - v) = 2f_{12}(u + v) - 2f_{12}(u - v) - f_{12}(v) + f_{12}(-v); \forall\ \ u, v \in U. \tag{26}$$

By Lemma 2.1 of [7], we arrive our result. Hence the proof is complete. □

Theorem 2.2. *If an even function $f_{12} : U \to V$ satisfies the functional equation (2) if and only if $f_{12} : U \to V$ satisfies (13) for all $u, v, u_1, u_2, \cdots, u_n \in U$ where U and V be real vector spaces.*

Proof. By data $f_{12} : U \to V$ satisfies the functional equation (2), that is

$$f_{12}(u + v) + f_{12}(u - v) = 2f_{12}(u) + 2f_{12}(v); \forall\ \ u, v \in U. \tag{27}$$

It is easy to verify from (27) that

$$f_{12}(0) = 0;\ \ f_{12}(2u) = 4f_{12}(u);\ \ f_{12}(3u) = 9f_{12}(u);\ \ f_{12}(Ku) = K^2 f_{12}(u); \forall\ \ u \in U. \tag{28}$$

Replacing u by $2u$ in (27) and using (29), (27) as well as evenness of f_{12}, one can have

$$f_{12}(2u+v)+f_{12}(2u-v)=f_{12}(u+v)+f_{12}(u-v)+f_{12}(2u)+f_{12}(u)+f_{12}(-u);\forall\ \ u,v\in U. \tag{29}$$

Finally replacing u by u_1 and v by $u_2+\cdots+u_n$ in (29), we arrive (13).
Conversely, by data $f_{12}:U\rightarrow V$ satisfies the functional equation (13). If we interchange $u_3=\cdots u_n=0$ in (13), and using evenness of f_{12} in (13), one can see

$$f_{12}\left(2u_1+u_2\right)+f_{12}\left(2u_1-u_2\right)=f_{12}\left(u_1+u_2\right)+f_{12}\left(u_1-u_2\right)+f_{12}(2u_1)+2f_{12}(u_1);\forall\ \ u_1,u_2\in U. \tag{30}$$

It is easy to verify from (30) that

$$f_{12}(0)=0;\ \ f_{12}(2u)=2f_{12}(u);\ \ f_{12}(3u)=3f_{12}(u);\ \ f_{12}(Ku)=Kf_{12}(u);\forall\ \ u\in U. \tag{31}$$

Using (31) in (30), one can obtain

$$f_{12}\left(2u_1+u_2\right)+f_{12}\left(2u_1-u_2\right)=f_{12}\left(u_1+u_2\right)+f_{12}\left(u_1-u_2\right)+6f_{12}(u_1);\forall\ \ u_1,u_2\in U. \tag{32}$$

By Theorem 2.1 of [13], we arrive our result. Hence the proof is complete. □

In order to explore the generalized Ulam - Hyers stability theorems, let we take a mapping $f_{12}:\mathcal{W}_1\rightarrow\mathcal{W}_2$ by

$$\begin{aligned} D\ f_{12}(u_1,u_2,u_3,\ldots,u_n)=&\ f_{12}\left(2u_1+\sum_{i=2}^{n}u_i\right)+f_{12}\left(2u_1-\sum_{i=2}^{n}u_i\right)-f_{12}\left(\sum_{i=1}^{n}u_i\right)\\ &-f_{12}\left(u_1-\sum_{i=2}^{n}u_i\right)-f_{12}(2u_1)-\left(f_{12}(u_1)+f_{12}(-u_1)\right)\end{aligned}$$

for all $u_1,u_2,u_3,\ldots,u_n\in\mathcal{W}_1$ where $\mathcal{W}_1$ be a normed space and $\mathcal{W}_2$ be a Banach space.

3. Stability Theorems for n is an Even Positive Integer

In this section, we provide the stability of n dimensional additive quadratic functional equation (13) for n is an Even Positive Integer.

Theorem 3.1. *If $\lambda,\Lambda:\mathcal{W}_1^n\rightarrow[0,\infty)$ are functions which satisfies*

$$\lim_{r\rightarrow\infty}\frac{\lambda\left(2^{rq}u_1,2^{rq}u_2,2^{rq}u_3,\cdots,2^{rq}u_n\right)}{2^{rq}}=0 \tag{33}$$

with $q\in\{-1,1\}$ and $f_{12}:\mathcal{W}_1\rightarrow\mathcal{W}_2$ is a odd function satisfying the inequality

$$\|D\ f_{12}(u_1,u_2,u_3,\cdots,u_n)\|\leq\lambda\left(u_1,u_2,u_3,\cdots,u_n\right) \tag{34}$$

for all $u_1, u_2, u_3, \cdots, u_n \in \mathcal{W}_1$. Then there exists a unique additive mapping $A : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfes the inequality

$$\|A(u) - f_1(u)\| \leq \frac{1}{2} \sum_{p=\frac{1-q}{2}}^{\infty} \frac{\Lambda_E(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u)}{2^{pq}} \tag{35}$$

for all $u \in \mathcal{W}_1$, where

$$\Lambda_E(2^{pq}u, \cdots, 2^{pq}u) = \lambda(2^{pq}u, \underbrace{2^{pq}u, 2^{pq}u, \ldots, 2^{pq}u}_{\frac{(n-2)}{2} times}, \underbrace{-2^{pq}u, -2^{pq}u, \ldots, -2^{pq}u}_{\frac{(n-2)}{2} times}, 0) \tag{36}$$

and

$$A(u) = \lim_{r \to \infty} \frac{f_1(2^{rq}u)}{2^{rq}} \tag{37}$$

for all $u \in \mathcal{W}_1$.

Proof. By data f_{12} is an odd function and let we take $f_{12} = f_1$. Replacing

$$(u_1, u_2, u_3 \cdots, u_n) = \left(u, \underbrace{u, u, \cdots, u}_{\frac{n-2\ times}{2}}, \underbrace{-u, -u, \cdots, -u}_{\frac{n-2\ times}{2}}, 0\right)$$

in (34), we get

$$\|f_1(2u) - 2f_1(u)\| \leq \lambda\left(u, \underbrace{u, u, \cdots, u}_{\frac{n-2\ times}{2}}, \underbrace{-u, -u, \cdots, -u}_{\frac{n-2\ times}{2}}, 0\right) \tag{38}$$

for all $u \in \mathcal{W}_1$. Define

$$\Lambda_E(u, u, \cdots, u) = \lambda\left(u, \underbrace{u, u \cdots, u}_{\frac{n-2\ times}{2}}, \underbrace{-u, -u \cdots, -u}_{\frac{n-2\ times}{2}}, 0\right) \tag{39}$$

for all $u \in \mathcal{W}_1$. Using (39) in (38), we obtain

$$\|f_1(2u) - 2f_1(u)\| \leq \Lambda_E(u, \ldots, u) \tag{40}$$

for all $u \in \mathcal{W}_1$. It follows from (40) that

$$\left\|\frac{f_1(2u)}{2} - f_1(u)\right\| \leq \frac{1}{2}\Lambda_E(u, \ldots, u) \tag{41}$$

for all $u \in \mathcal{W}_1$. Now replacing u by $2u$ and dividing by 2 in (41), we get

$$\left\|\frac{f_1(2^2u)}{2^2} - \frac{f_1(2u)}{2}\right\| \leq \frac{1}{2^2}\Lambda_E(2u, \cdots, 2u) \tag{42}$$

for all $u \in \mathcal{W}_1$. From (41) and (42), we obtain

$$\left\|\frac{f_1(2^2u)}{2^2} - f_1(u)\right\| \leq \frac{1}{2}\Lambda_E(u, \ldots, u) + \frac{1}{2^2}\Lambda_E(2u, \cdots, 2u) \tag{43}$$

for all $u \in \mathcal{W}_1$. In general for any positive integer r , we arrive

$$\left\| \frac{f_1(2^r u)}{2^r} - f_1(u) \right\| \leq \frac{1}{2} \sum_{p=0}^{r-1} \frac{1}{2^p} \Lambda_E(2^p u, \ldots, 2^p u) \tag{44}$$

for all $u \in \mathcal{W}_1$. Hence, it follows from (44) that the sequence $\left\{ \frac{f_1(2^r u)}{2^r} \right\}$ is a Cauchy sequence. Indeed, to prove the convergence of the sequence $\left\{ \frac{f_1(2^r u)}{2^r} \right\}$, replace u by $2^t u$ and dividing by 2^t in (44), for any $t, r > 0$, we deduce

$$\left\| \frac{f_1(2^{t+r} u)}{2^{(t+r)}} - \frac{f_1(2^t u)}{2^t} \right\| = \frac{1}{2^t} \left\| \frac{f_1(2^r \cdot 2^t u)}{2^r} - f_1(2^t u) \right\| \leq \frac{1}{2} \sum_{p=0}^{r-1} \frac{1}{2^{p+t}} \Lambda_E(2^{p+t} u, \ldots, 2^{p+t} u) \to 0 \quad as \ r \to \infty$$

for all $u \in \mathcal{W}_1$. Since $\mathcal{W}_2$ is complete, there exists a mapping $A : \mathcal{W}_1 \to \mathcal{W}_2$ such that

$$A(u) = \lim_{r \to \infty} \frac{f_1(2^r u)}{2^r}, \quad \forall \ u \in \mathcal{W}_1.$$

Letting $r \to \infty$ in (44), we see that (35) holds for all $u \in \mathcal{W}_1$ with $q = 1$. To prove that A satisfies (13), replacing $(u_1, u_2, u_3, \ldots, u_n)$ by $(2^r u_1, 2^r u_2, 2^r u_3, \ldots, 2^r u_n)$ and dividing by 2^r in (34), we obtain

$$\frac{1}{2^r} \left\| D \ f_1(2^r u_1, 2^r u_2, 2^r u_3, \ldots, 2^r u_n) \right\| \leq \frac{1}{2^r} \lambda(2^r u_1, 2^r u_2, 2^r u_3, \ldots, 2^r u_n)$$

for all $u_1, u_2, u_3, \ldots, u_n \in \mathcal{W}_1$. Letting $r \to \infty$ in the above inequality and using the definition of $A(u)$, we see that $A(u)$ satisfies (13) for all $u_1, u_2, u_3, \ldots, u_n \in \mathcal{W}_1$. It is easy to verify that $A(u)$ is unique. Replacing u by $\frac{u}{2}$ in (40), we have

$$\left\| f_1(u) - 2 f_1\left(\frac{u}{2}\right) \right\| \leq \Lambda_E\left(\frac{u}{2}, \frac{u}{2}, \ldots, \frac{u}{2}\right) \tag{45}$$

for all $u \in \mathcal{W}_1$. Hence, for $q = -1$ also, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollaries are immediate consequences of Theorem 3.1 concerning the stability of (13).

Corollary 3.2. *Assume that a be nonnegative real number. Let an odd function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D \ f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique additive function $A : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|A(u) - f_1(u)\| \leq |a|$ for all $u \in \mathcal{W}_1$.*

Corollary 3.3. *Assume that a and $b \neq 1$ be nonnegative real numbers. Let an odd function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D \ f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a \sum_{i=1}^{n} ||u_i||^b$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique additive function $A : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|A(u) - f_1(u)\| \leq \frac{a(n-1)||u||^b}{|2 - 2^b|}$ for all $u \in \mathcal{W}_1$.*

Corollary 3.4. *Assume that a and b be nonnegative real numbers. Let an odd function* $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ *satisfies the inequality* $\|D\ f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a\Big\{\prod_{i=1}^{n} ||u_i||^b + \sum_{i=1}^{n} ||u_i||^{nb}\Big\}$ *for all* $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. *Then there exists a unique additive function* $A : \mathcal{W}_1 \to \mathcal{W}_2$ *such that* $\|A(u) - f_1(u)\| \leq \dfrac{a(n-1)||u||^{nb}}{|2-2^{nb}|}$ *with* $nb \neq 1$, *for all* $u \in \mathcal{W}_1$.

Corollary 3.5. *Assume that a and* $b_1, b_2, \ldots, b_n \neq 1$ *are nonnegative real numbers. Let an odd function* $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ *satisfies the inequality* $\|D\ f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a \sum_{i=1}^{n} ||u_i||^{b_i}$ *for all* $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. *Then there exists a unique additive function* $A : \mathcal{W}_1 \to \mathcal{W}_2$ *such that* $\|A(u) - f_1(u)\| \leq \dfrac{a||u||^{b_1}}{|2-2^{b_1}|} + \sum_{i=2}^{n-2} \dfrac{a(n-2)||u||^{b_i}}{|2-2^{b_i}|}$ *for all* $u \in \mathcal{W}_1$.

Corollary 3.6. *Assume that a and* $b_1, b_2, \ldots, b_n \neq 1$ *are nonnegative real numbers. Let an odd function* $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ *satisfies the inequality* $\|D\ f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a\Big\{\prod_{i=1}^{n} ||u_i||_i^b + \sum_{i=1}^{n} ||u_i||^{nb_i}\Big\}$ *for all* $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. *Then there exists a unique additive function* $A : \mathcal{W}_1 \to \mathcal{W}_2$ *such that* $\|A(u) - f_1(u)\| \leq \dfrac{a||u||^{nb_1}}{|2-2^{nb_1}|} + \sum_{i=2}^{n-2} \dfrac{a(n-2)||u||^{nb_i}}{|2-2^{nb_i}|}$ *with* $nb_i \neq 1$, *for all* $u \in \mathcal{W}_1$.

The proof of the following theorem and corollaries are similar to that of Theorem 3.1, Corollaries 3.3 - 3.6 if f_{12} is an even function and we take $f_{12} = f_2$. Hence the details of the proof are omitted.

Theorem 3.7. *If* $\lambda, \Lambda_E : \mathcal{W}_1^n \to [0, \infty)$ *are functions which satisfies*

$$\lim_{r\to\infty} \frac{\lambda\left(2^{rq}u_1, 2^{rq}u_2, 2^{rq}u_3, \cdots, 2^{rq}u_n\right)}{4^{rq}} = 0 \tag{46}$$

with $q \in \{-1, 1\}$ *and* $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ *is a even function satisfying the inequality*

$$\|D\ f_{12}(u_1, u_2, u_3, \cdots, u_n)\| \leq \lambda\left(u_1, u_2, u_3, \cdots, u_n\right) \tag{47}$$

for all $u_1, u_2, u_3, \cdots, u_n \in \mathcal{W}_1$. *Then there exists a unique quadratic mapping* $Q : \mathcal{W}_1 \to \mathcal{W}_2$ *which satisfes the inequality*

$$\|Q(u) - f_2(u)\| \leq \frac{1}{4} \sum_{p=\frac{1-q}{2}}^{\infty} \frac{\Lambda_E(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u)}{4^{pq}} \tag{48}$$

for all $u \in \mathcal{W}_1$, *where* $\Lambda_E(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u)$ *is defined in (36) and* $Q(u)$ *is defined by*

$$Q(u) = \lim_{r\to\infty} \frac{f_2(2^{rq}x)}{4^{rq}} \tag{49}$$

for all $u \in \mathcal{W}_1$.

Corollary 3.8. *Assume that a be nonnegative real number. Let an even function* $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ *satisfies the inequality* $\|D\ f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a$ *for all* $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. *Then there exists a unique quadratic function* $Q : \mathcal{W}_1 \to \mathcal{W}_2$ *such that* $\|Q(u) - f_2(u)\| \leq \left|\frac{a}{3}\right|$ *for all* $u \in \mathcal{W}_1$.

Corollary 3.9. *Assume that a and $b \neq 2$ be nonnegative real numbers. Let an even function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\ f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a \sum_{i=1}^{n} ||u_i||^b$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique quadratic function $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|Q(u) - f_2(u)\| \leq \frac{a(n-1)||u||^b}{|4-2^b|}$ for all $u \in \mathcal{W}_1$.*

Corollary 3.10. *Assume that a and b be nonnegative real numbers. Let an even function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\ f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a\left\{ \prod_{i=1}^{n} ||u_i||^b + \sum_{i=1}^{n} ||u_i||^{nb} \right\}$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique quadratic function $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|Q(u) - f_2(u)\| \leq \frac{a(n-1)||u||^{nb}}{|4-2^{nb}|}$ with $nb \neq 2$, for all $u \in \mathcal{W}_1$.*

Corollary 3.11. *Assume that a and $b_1, b_2, \ldots, b_n \neq 2$ are nonnegative real numbers. Let an even function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\ f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a \sum_{i=1}^{n} ||u_i||^{b_i}$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique quadratic function $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|Q(u) - f_2(u)\| \leq \frac{a||u||^{b_1}}{|4-2^{b_1}|} + \sum_{i=2}^{n-2} \frac{a(n-2)||u||^{b_i}}{|4-2^{b_i}|}$ for all $u \in \mathcal{W}_1$.*

Corollary 3.12. *Assume that a and $b_1, b_2, \ldots, b_n \neq 2$ are nonnegative real numbers. Let an even function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\ f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a\left\{ \prod_{i=1}^{n} ||u_i||_i^b + \sum_{i=1}^{n} ||u_i||^{nb_i} \right\}$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique quadratic function $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|Q(u) - f_2(u)\| \leq \frac{a||u||^{nb_1}}{|4-2^{nb_1}|} + \sum_{i=2}^{n-2} \frac{a(n-2)||u||^{nb_i}}{|4-2^{nb_i}|}$ with $nb_i \neq 2$, for all $u \in \mathcal{W}_1$.*

Theorem 3.13. *If $\lambda, \Lambda_E : \mathcal{W}_1^n \to [0, \infty)$ are functions satisfying (33) and (46) and $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ is a function satisfying the inequality*

$$\|D\ f_{12}(u_1, u_2, u_3, \cdots, u_n)\| \leq \lambda\left(u_1, u_2, u_3, \cdots, u_n\right) \tag{50}$$

for all $u_1, u_2, u_3, \cdots, u_n \in \mathcal{W}_1$. Then there exists a unique additive mapping $A : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $Q : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies

$$\begin{aligned} \|f_{12}(u) - A(u) - Q(u)\| \leq \frac{1}{4} \sum_{p=\frac{1-q}{2}}^{\infty} \frac{1}{2^{pq}} \left(\Lambda_E(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u) + \Lambda_E(-2^{pq}u, -2^{pq}u, \cdots, -2^{pq}u)\right) \\ + \frac{1}{8} \sum_{p=\frac{1-q}{2}}^{\infty} \frac{1}{4^{pq}} \left(\Lambda_E(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u) + \Lambda_E(-2^{pq}u, -2^{pq}u, \cdots, -2^{pq}u)\right) \end{aligned} \tag{51}$$

for all $u \in \mathcal{W}_1$, where $\Lambda_E(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u)$, $A(u)$ and $Q(u)$ are defined in (36), (37) and (49), respectively for all $u \in \mathcal{W}_1$.

Proof. Define a function $f_o(u)$ by

$$f_o(u) = \frac{f_1(u) - f_1(-u)}{2} \qquad for \quad all \quad u \in \mathcal{W}_1.$$

Then, it is easy to see that $f_o(0) = 0$ and $f_o(-u) = -f_o(u)$ for all $u \in \mathcal{W}_1$. Hence

$$\|D\ f_o(u_1, u_2, u_3, \ldots, u_n)\| \leq \frac{\lambda(u_1, u_2, u_3, \ldots, u_n)}{2} + \frac{\lambda(-u_1, -u_2, -u_3, \ldots, -u_n)}{2}. \tag{52}$$

for all $u_1, u_2, u_3, \cdots, u_n \in \mathcal{W}_1$. By Theorem 3.1, we have

$$\|f_o(u) - A(u)\| \leq \frac{1}{4} \sum_{p=\frac{1-q}{2}}^{\infty} \frac{1}{2^{rq}} \left(\Lambda_E(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u) + \Lambda_E(-2^{pq}u, -2^{pq}u, \cdots, -2^{pq}u)\right) \tag{53}$$

for all $u \in \mathcal{W}_1$. Similarly, if we define a function $f_e(u)$ by

$$f_e(u) = \frac{f_2(u) + f_2(-u)}{2} \qquad for \quad all \quad u \in \mathcal{W}_1.$$

Then, it is easy to see that $f_e(0) = 0$ and $f_e(-u) = f_e(u)$ for all $u \in \mathcal{W}_1$. Hence

$$\|D\ f_e(u_1, u_2, u_3, \ldots, u_n)\| \leq \frac{\lambda(u_1, u_2, u_3, \ldots, u_n)}{2} + \frac{\lambda(-u_1, -u_2, -u_3, \ldots, -u_n)}{2}. \tag{54}$$

for all $u_1, u_2, u_3, \cdots, u_n \in \mathcal{W}_1$. By Theorem 3.7, we have

$$\|f_e(u) - Q(u)\| \leq \frac{1}{8} \sum_{p=\frac{1-q}{2}}^{\infty} \frac{1}{4^{pq}} \left(\Lambda_E(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u) + \Lambda_E(-2^{pq}u, -2^{pq}u, \cdots, -2^{pq}u)\right) \tag{55}$$

for all $u \in \mathcal{W}_1$. Define

$$f_{12}(u) = f_e(u) + f_o(u) \tag{56}$$

for all $u \in \mathcal{W}_1$. From (53),(55) and (56), we arrive our result Hence the theorem is proved. □

With the help Theorem 3.13 and Corollaries 3.3 - 3.6; 3.9 - 3.12, we have the following Corollaries concerning the stability of (13).

Corollary 3.14. *Assume that a be nonnegative real number. Let a function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\ f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique additive mapping $A : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|f_{12}(u) - A(u) - Q(u)\| \leq |a| \left\{\frac{1}{2} + \frac{1}{12}\right\}$ for all $u \in \mathcal{W}_1$.*

Corollary 3.15. *Assume that a and $b \neq 1, 2$ be nonnegative real numbers. Let a function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\ f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a \sum_{i=1}^{n} ||u_i||^b$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique additive mapping $A : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|f_{12}(u) - A(u) - Q(u)\| \leq \frac{a(n-1)||u||^b}{2|2-2^b|} + \frac{a(n-1)||u||^b}{4|4-2^b|}$ for all $u \in \mathcal{W}_1$.*

Corollary 3.16. *Assume that a and b be nonnegative real numbers. Let a function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\ f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a\left\{\prod_{i=1}^{n} ||u_i||^b + \sum_{i=1}^{n} ||u_i||^{nb}\right\}$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there* *exists a unique additive mapping $A : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|f_{12}(u) - A(u) - Q(u)\| \leq \dfrac{a(n-1)||u||^{nb}}{2|2-2^{nb}|} + \dfrac{a(n-1)||u||^{nb}}{4|4-2^{nb}|}$ with $nb \neq 1,2$ for all $u \in \mathcal{W}_1$.*

Corollary 3.17. *Assume that a and $b_1, b_2, \ldots, b_n \neq 1,2$ are nonnegative real numbers. Let a function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a \sum\limits_{i=1}^{n} ||u_i||^{b_i}$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique additive mapping $A : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|f_{12}(u) - A(u) - Q(u)\| \leq \dfrac{a||u||^{b_1}}{2|2-2^{b_1}|} + \sum\limits_{i=2}^{n-2} \dfrac{a(n-2)||u||^{b_i}}{2|2-2^{b_i}|} + \dfrac{a||u||^{b_1}}{4|4-2^{b_1}|} + \sum\limits_{i=2}^{n-2} \dfrac{a(n-2)||u||^{b_i}}{4|4-2^{b_i}|}$ for all $u \in \mathcal{W}_1$.*

Corollary 3.18. *Assume that a and $b_1, b_2, \ldots, b_n \neq 1,2$ are nonnegative real numbers. Let an function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a \left\{ \prod\limits_{i=1}^{n} ||u_i||_i^{b} + \sum\limits_{i=1}^{n} ||u_i||^{nb_i} \right\}$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique quadratic function $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|f_{12}(u) - A(u) - Q(u)\| \leq$ $\dfrac{a||u||^{nb_1}}{2|2-2^{nb_1}|} + \sum\limits_{i=2}^{n-2} \dfrac{a(n-2)||u||^{nb_i}}{2|2-2^{nb_i}|} + \dfrac{a||u||^{nb_1}}{4|4-2^{nb_1}|} + \sum\limits_{i=2}^{n-2} \dfrac{a(n-2)||u||^{nb_i}}{4|4-2^{nb_i}|}$ with $nb_i \neq 1,2,$ for all $u \in \mathcal{W}_1$.*

4. Stability Theorems for n is an Odd Positive Integer

In this section, we provide the stability of n dimensional additive quadratic functional equation (13) for n is an Odd Positive Integer. The proof of the following theorems and corollaries is similar to that of Section 3. Hence the details of proofs are omitted.

Theorem 4.1. *If $\lambda, \Lambda : \mathcal{W}_1^n \to [0, \infty)$ are functions which satisfies*

$$\lim_{r\to\infty} \frac{\lambda\left(2^{rq}u_1, 2^{rq}u_2, 2^{rq}u_3, \cdots, 2^{rq}u_n\right)}{2^{rq}} = 0 \tag{57}$$

with $q \in \{-1, 1\}$ and $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ is a odd function satisfying the inequality

$$\|D\, f_{12}(u_1, u_2, u_3, \cdots, u_n)\| \leq \lambda\left(u_1, u_2, u_3, \cdots, u_n\right) \tag{58}$$

for all $u_1, u_2, u_3, \cdots, u_n \in \mathcal{W}_1$. Then there exists a unique additive mapping $A : \mathcal{W}_1 \to \mathcal{W}_2$ such that

$$\|A(u) - f_1(u)\| \leq \frac{1}{2} \sum_{p=\frac{1-q}{2}}^{\infty} \frac{\Lambda_O(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u)}{2^{pq}} \tag{59}$$

for all $u \in \mathcal{W}_1$, where

$$\Lambda_O(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u) = \lambda(2^{pq}u, \underbrace{2^{pq}u, 2^{pq}u, \ldots, 2^{pq}u}_{\frac{(n-1)}{2}\, times}, \underbrace{-2^{pq}u, -2^{pq}u, \ldots, -2^{pq}u}_{\frac{(n-1)}{2}\, times}) \tag{60}$$

and

$$A(u) = \lim_{r\to\infty} \frac{f_1(2^{rq}x)}{2^{rq}} \tag{61}$$

for all $u \in \mathcal{W}_1$.

Corollary 4.2. *Assume that a be nonnegative real number. Let an odd function* $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ *satisfies the inequality* $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a$ *for all* $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. *Then there exists a unique additive function* $A : \mathcal{W}_1 \to \mathcal{W}_2$ *such that* $\|A(u) - f_1(u)\| \leq |a|$ *for all* $u \in \mathcal{W}_1$.

Corollary 4.3. *Assume that a and* $b \neq 1$ *be nonnegative real numbers. Let an odd function* $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ *satisfies the inequality* $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a \sum_{i=1}^{n} ||u_i||^b$ *for all* $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. *Then there exists a unique additive function* $A : \mathcal{W}_1 \to \mathcal{W}_2$ *such that* $\|A(u) - f_1(u)\| \leq \dfrac{na||u||^b}{|2 - 2^b|}$ *for all* $u \in \mathcal{W}_1$.

Corollary 4.4. *Assume that a and b be nonnegative real numbers. Let an odd function* $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ *satisfies the inequality* $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a\Big\{\prod_{i=1}^{n} ||u_i||^b + \sum_{i=1}^{n} ||u_i||^{nb}\Big\}$ *for all* $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. *Then there exists a unique additive function* $A : \mathcal{W}_1 \to \mathcal{W}_2$ *such that* $\|A(u) - f_1(u)\| \leq \dfrac{a(n+1)||u||^{nb}}{|2 - 2^{nb}|}$ *with* $nb \neq 1$, *for all* $u \in \mathcal{W}_1$.

Corollary 4.5. *Assume that a and* $b_1, b_2, \ldots, b_n \neq 1$ *are nonnegative real numbers. Let an odd function* $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ *satisfies the inequality* $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a \sum_{i=1}^{n} ||u_i||^{b_i}$ *for all* $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. *Then there exists a unique additive function* $A : \mathcal{W}_1 \to \mathcal{W}_2$ *such that* $\|A(u) - f_1(u)\| \leq \sum_{i=1}^{n} \frac{a||u||^{b_i}}{|2-2^{b_i}|}$ *for all* $u \in \mathcal{W}_1$.

Corollary 4.6. *Assume that a and* $b_1, b_2, \ldots, b_n \neq 1$ *are nonnegative real numbers. Let an odd function* $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ *satisfies the inequality* $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a\Big\{\prod_{i=1}^{n} ||u_i||^{b_i} + \sum_{i=1}^{n} ||u_i||^{nb_i}\Big\}$ *for all* $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. *Then there exists a unique additive function* $A : \mathcal{W}_1 \to \mathcal{W}_2$ *such that* $\|A(u) - f_1(u)\| \leq \dfrac{a||u||^{\sum_{i=1}^{n} nb_i}}{\left|2-2^{\sum_{i=1}^{n} nb_i}\right|} + \sum_{i=1}^{n} \frac{a||u||^{nb_i}}{|2-2^{nb_i}|}$ *with* $nb_i, \sum_{i=1}^{n} nb_i \neq 1$, *for all* $u \in \mathcal{W}_1$.

Theorem 4.7. *If* $\lambda, \Lambda_O : \mathcal{W}_1^n \to [0, \infty)$ *are functions which satisfies*

$$\lim_{r\to\infty} \frac{\lambda\left(2^{rq}u_1, 2^{rq}u_2, 2^{rq}u_3, \cdots, 2^{rq}u_n\right)}{4^{rq}} = 0 \tag{62}$$

with $q \in \{-1, 1\}$ *and* $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ *is a even function satisfying the inequality*

$$\|D\, f_{12}(u_1, u_2, u_3, \cdots, u_n)\| \leq \lambda\left(u_1, u_2, u_3, \cdots, u_n\right) \tag{63}$$

for all $u_1, u_2, u_3, \cdots, u_n \in \mathcal{W}_1$. *Then there exists a unique quadratic mapping* $Q : \mathcal{W}_1 \to \mathcal{W}_2$ *which satisfes the inequality*

$$\|Q(u) - f_2(u)\| \leq \frac{1}{4} \sum_{p=\frac{1-q}{2}}^{\infty} \frac{\Lambda_O(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u)}{4^{pq}} \tag{64}$$

for all $u \in \mathcal{W}_1$, *where* $\Lambda_O(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u)$ *is defined in (60) and* $Q(u)$ *is defined by and*

$$Q(u) = \lim_{r\to\infty} \frac{f_2(2^{rq}x)}{4^{rq}} \tag{65}$$

respectively, for all $u \in \mathcal{W}_1$.

Corollary 4.8. *Assume that a be nonnegative real number. Let an even function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique quadratic function $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|Q(u) - f_2(u)\| \leq \left|\frac{a}{3}\right|$ for all $u \in \mathcal{W}_1$.*

Corollary 4.9. *Assume that a and $b \neq 1$ be nonnegative real numbers. Let an even function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a \sum_{i=1}^{n} ||u_i||^b$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique quadratic function $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|Q(u) - f_2(u)\| \leq \dfrac{na||u||^b}{|4 - 2^b|}$ for all $u \in \mathcal{W}_1$.*

Corollary 4.10. *Assume that a and b be nonnegative real numbers. Let an even function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a\left\{ \prod_{i=1}^{n} ||u_i||^b + \sum_{i=1}^{n} ||u_i||^{nb} \right\}$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique quadratic function $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|Q(u) - f_2(u)\| \leq \sum_{i=1}^{n} \dfrac{a(n+1)||u||^{nb}}{|4 - 2^{nb}|}$ with $nb \neq 1$, for all $u \in \mathcal{W}_1$.*

Corollary 4.11. *Assume that a and $b_1, b_2, \ldots, b_n \neq 1$ are nonnegative real numbers. Let an even function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a \sum_{i=1}^{n} ||u_i||^{b_i}$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique quadratic function $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|Q(u) - f_2(u)\| \leq \sum_{i=1}^{n} \dfrac{a||u||^{b_i}}{|4 - 2^{b_i}|}$ for all $u \in \mathcal{W}_1$.*

Corollary 4.12. *Assume that a and $b_1, b_2, \ldots, b_n \neq 1$ are nonnegative real numbers. Let an even function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a\left\{ \prod_{i=1}^{n} ||u_i||^{b_i} + \sum_{i=1}^{n} ||u_i||^{nb_i} \right\}$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique quadratic function $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|Q(u) - f_2(u)\| \leq \dfrac{a||u||^{\sum_{i=1}^{n} nb_i}}{\left|4 - 2^{\sum_{i=1}^{n} nb_i}\right|} + \sum_{i=1}^{n} \dfrac{a||u||^{nb_i}}{|4 - 2^{nb_i}|}$ with $nb_i, \sum_{i=1}^{n} nb_i \neq 2$, for all $u \in \mathcal{W}_1$.*

Theorem 4.13. *If $\lambda, \Lambda_O : \mathcal{W}_1^n \to [0, \infty)$ are functions satisfying (57) and (62) and $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ is a function satisfying the inequality*

$$\|D\, f_{12}(u_1, u_2, u_3, \cdots, u_n)\| \leq \lambda\left(u_1, u_2, u_3, \cdots, u_n\right) \tag{66}$$

for all $u_1, u_2, u_3, \cdots, u_n \in \mathcal{W}_1$. Then there exists a unique additive mapping $A : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $Q : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies

$$\begin{aligned} \|f_{12}(u) - A(u) - Q(u)\| \leq{}& \frac{1}{4} \sum_{p=\frac{1-q}{2}}^{\infty} \frac{1}{2^{pq}} \left(\Lambda_O(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u) + \Lambda_O(-2^{pq}u, -2^{pq}u, \cdots, -2^{pq}u)\right) \\ &+ \frac{1}{8} \sum_{p=\frac{1-q}{2}}^{\infty} \frac{1}{4^{pq}} \left(\Lambda_O(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u) + \Lambda_O(-2^{pq}u, -2^{pq}u, \cdots, -2^{pq}u)\right) \end{aligned} \tag{67}$$

for all $u \in \mathcal{W}_1$, where $\Lambda_O(2^{pq}u, 2^{pq}u, \cdots, 2^{pq}u)$, $A(u)$ and $Q(u)$ are defined in (60), (61) and (65), respectively for all $u \in \mathcal{W}_1$.

Corollary 4.14. *Assume that a be nonnegative real number. Let a function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique additive mapping $A : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|f_{12}(u) - A(u) - Q(u)\| \leq |a| \left\{\frac{1}{2} + \frac{1}{12}\right\}$ for all $u \in \mathcal{W}_1$.*

Corollary 4.15. *Assume that a and $b \neq 1, 2$ be nonnegative real numbers. Let a function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a \sum_{i=1}^{n} ||u_i||^b$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique additive mapping $A : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|f_{12}(u) - A(u) - Q(u)\| \leq \frac{na||u||^b}{2|2-2^b|} + \frac{na||u||^b}{4|4-2^b|}$ for all $u \in \mathcal{W}_1$.*

Corollary 4.16. *Assume that a and b be nonnegative real numbers. Let a function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a\left\{\prod_{i=1}^{n} ||u_i||^b + \sum_{i=1}^{n} ||u_i||^{nb}\right\}$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique additive mapping $A : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|f_{12}(u) - A(u) - Q(u)\| \leq \frac{a(n+1)||u||^{nb}}{2|2-2^{nb}|} + \frac{a(n+1)||u||^{nb}}{4|4-2^{nb}|}$ with $nb \neq 1, 2$ for all $u \in \mathcal{W}_1$.*

Corollary 4.17. *Assume that a and $b_1, b_2, \ldots, b_n \neq 1, 2$ are nonnegative real numbers. Let a function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a \sum_{i=1}^{n} ||u_i||^{b_i}$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique additive mapping $A : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that $\|f_{12}(u) - A(u) - Q(u)\| \leq \sum_{i=1}^{n} \frac{a||u||^{b_i}}{2|2-2^{b_i}|} + + \sum_{i=1}^{n} \frac{a(n)||u||^{b_i}}{4|4-2^{b_i}|}$ for all $u \in \mathcal{W}_1$.*

Corollary 4.18. *Assume that a and $b_1, b_2, \ldots, b_n \neq 1, 2$ are nonnegative real numbers. Let a function $f_{12} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfies the inequality $\|D\, f_{12}(u_1, u_2, u_3, \ldots, u_n)\| \leq a\left\{\prod_{i=1}^{n} ||u_i||_i^b + \sum_{i=1}^{n} ||u_i||^{nb_i}\right\}$ for all $u_1, u_2, \ldots, u_n \in \mathcal{W}_1$. Then there exists a unique quadratic function $Q : \mathcal{W}_1 \to \mathcal{W}_2$ such that*

$$\|f_{12}(u) - A(u) - Q(u)\| \leq \frac{a||u||^{\sum_{i=1}^{n} nb_i}}{2\left|2 - 2^{\sum_{i=1}^{n} nb_i}\right|} + \sum_{i=1}^{n} \frac{a||u||^{nb_i}}{2|2-2^{nb_i}|} + \frac{a||u||^{\sum_{i=1}^{n} nb_i}}{4\left|4 - 2^{\sum_{i=1}^{n} nb_i}\right|} + \sum_{i=1}^{n} \frac{a||u||^{nb_i}}{4|4-2^{nb_i}|}$$

with nb_i, $\sum_{i=1}^{n} nb_i \neq 1, 2$ for all $u \in \mathcal{W}_1$.

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