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Chromatic and Connectivity Approaches to Fractional Domination in Graphs

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Abstract: In this study, we go deeper into the concept of fractional domination in graph theory by investigating its links to chromatic numbers and the connectivity between graphs. We explore these bounds on fractional domination by investigating Theorems that determine explanations and provide insights for optimal control/monitoring in complex networks We note that Theorems 1, 2 and 3 provide novel insights for the relationships of fractional domination, chromatic numbers and graph connectivity which are useful features for network analysis and optimization. Through the investigation of these theorem this research contributes a step towards the development of graph theory and its applications in various fields.

Keywords: Fractional Domination, Chromatic Number, Graph Connectivity, Graph Bounds, Network Optimization, Graph Theory.

1. Introduction

1.1. Background

The Graph theory, it is a branch of mathematics that concerning the concept of points or we can say the objects that are connected by lines. The subject looks at both those problems which can be modelled as graphs and those problems for which graph theoretical methods. A key concept in graph theory is domination which states that lets there be a subset of vertices where each and every vertex in the graph is either a part of this type of set or its adjacent neighbor[1] This concept is ubiquitous and can be seen in various domains such as network-design, biology, social sciences etc. whereby the desire to manage/observe some of the nodes optimally [2].

1.2. Fractional Domination and Chromatic Numbers

The fractional domination was proposed instead of the domination, nevertheless, in the fractional domination, the allocation of "control" is more flexible. Instead of choosing the entire vertex, each vertex gets a value from 0 to 1: it is the fraction of "control" of the vertex. The fractional domination number [FDN] $\{\gamma f\,(G)\}$ is the minimum number of sums of these values required to dominate the graph [3, 12].

Similarly, the fractional chromatic numbers [FCN] $\{\chi f\,(G)\}$ generalizes the relevant concept of graph coloring. It represents the minimum number of sums of weights need to be color the given graph such a way that adjacent vertices have different colors, with fractional values indicating the extent of color assigned [4].

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1.3. Motivation and Objectives

We would be interested in studying theorems, as they are all about bounds on the fractional domination, and I believe that the study of these theorems will ultimately help develop optimal control and monitoring strategies in complex networks. By exploring the relationships established by these bounds, one can develop efficient algorithms and strategies for various practical applications, including but not limited to ensuring the security of networks, allocating resources in a network, and examining social networks to identify various patterns. This study explores the bounds in detail and provide relevant rigorous proofs and examples showing their practical implications in various fields.

2. Preliminaries

2.1. Graph Theory Definitions

1. **Graph:** A graph G consists of a set V of vertices and also a set E of edges, where each edge connects a pair of defined vertices. Formally, G can be represented as $G = (V, E)$ [6].

2. **Vertex (or Node):** A vertex v is a point where edges meet. The vertices represent the entities in the graph. A vertex is an element of the set V [19].

3. **Edge:** An edge is an individual element of the set E. Each edge is may be an unordered pair of defined vertices $\{u, v\}$ representing a connection between vertices u and v [15].

4. **Degree $\Delta(G)$:** The degree of a defined vertex v, denoted as $deg(v)$, is the total number of edges incident to a particular vertex v. The maximum degree $\Delta(G)$ of a given graph G is the highest degree among all vertices in G [7].

$$deg(v) = \text{The number of edges connected to v}$$

$$\Delta(G) = \max_{v \in V} deg(v)$$

5. **Neighbor Set:** The neighbor set called $N(v)$ of a defined vertex v is the set of all vertices adjacent to v. Formally,

$$N(v) = \{u \in V \text{ such that } \{u, v\} \in E\}$$

This means $N(v)$ contains all vertices u that are connected to v by an edge [19].

2.2. Fractional Domination

(i) Fractional Dominating Set: A function which is defined as $f : V \to [0, 1]$ is called a fractional dominating set if for all verities $v \in V : \sum_{u \in N(v)} f(u) \geq 1$. This means that the total sum of the fractional values allotted to the neighbors of each vertex v is at least 1 [5, 13].

(ii) Fractional Domination Number $\gamma_f(G)$: The FDN-fractional domination number $\gamma_f(G)$ is the minimum number of sums of the values assigned by any fractional dominating set f [8, 10, 11]. Formally,

$$\gamma_f(G) = \min \left\{ \sum_{v \in V} f(v) \quad \text{such that f is a fractional dominating set} \right\}$$

2.3. Fractional Coloring

Fractional Chromatic Number $\chi_f(G)$: The FCN-fractional chromatic number $\chi_f(G)$ is defined as the minimum number of sums of weights allotted to independent sets that cover the graph [9]. It can be formulated as:

$$\chi_f(G) = \min \left\{ \sum_{i=1}^{k} \lambda_i : \sum_{i=1}^{k} \lambda_i \chi_i(v) \geq 1 \;\; \forall \;\; v \in V \right\}$$

where λ_i were non-negative weights and $\chi_i(v)$ are characteristic functions representing the independent sets.

2.4. Vertex Connectivity

The vertex connectivity $kappa(G)$ of a graph G is the smallest number of vertices that must be removed to either disconnect the graph or reduce it to a single vertex. This metric indicates how robust or resilient the graph is against vertex removal [18, 19].

3. Theoretical Insights

Theorem 3.1 (New Fractional Chromatic-Domination Bound)**.** *Let us assume $G = (V, E)$ be a simple and connected graph. Again, let us assume $\chi_f(G)$ be the FCN-fractional chromatic number and $\gamma_f(G)$ be the FDN-fractional domination number of G. Then, the following new bound holds:*

$$\gamma_f(G) \leq \frac{n}{\chi_f(G)} + \Delta(G)$$

Where n is the total number of vertices in G and $\Delta(G)$ is the highest degree of any vertex in G.

Proof. To prove this theorem, we follow a series of logical steps and leverage existing results on fractional domination and chromatic numbers.

Step 1: Fractional Chromatic Number $\chi_f(G)$: By definition, $\chi_f(G)$ is the minimum number of colours to be needed to fractionally colour the vertices of graph G. This can be formulated as:

$$\chi_f(G) = \min \left\{ \sum_{i=1}^{k} \lambda_i : \sum_{i=1}^{k} \lambda_i \chi_i \left(v\right) \geq 1; \;\; \forall \;\; v \in V \right\}$$

where λ_i are non-negative weights and $\chi_i(v)$ are characteristic functions representing the colour classes.

Step 2: Fractional Domination Number $\gamma_f(G)$: Similarly, $\gamma_f(G)$ is the assigned minimum weight of a fractional domination set:

$$\gamma_f(G) = \min \left\{ \sum_{v \in V} f\left(v\right) : \sum_{u \in N(v)} f\left(u\right) \geq 1; \;\; \forall \;\; v \in V, \quad 0 \leq f(v) \leq 1 \right\}$$

where $f(v)$ represents the fractional part of the vertex v being included in the dominating set.

Step 3: Bounding $\gamma_f(G)$ with $\chi_f(G)$: From the definition of $\gamma_f(G)$, we know that each vertex v in G must be dominated by the fractional sum of its neighbors. If we consider a fractional coloring scheme that uses $\chi_f(G)$ colors, then we can distribute these colors fractionally among the vertices such that:

$$\sum_{u \in N(v)} f(u) \geq \frac{1}{\chi_f(G)}$$

since each colour class must cover all vertices.

Step 4: Incorporating Maximum Degree $\{\Delta(G)\}$: The $\Delta(G)$ called maximum degree of the graph which plays a crucial role in finding the local density of the graph. By integrating $\Delta(G)$ into our bound, we account for the scenario where high-degree vertices require higher fractional coverage to satisfy the domination condition. Thus, we adjust our previous inequality to include $\Delta(G)$ as:

$$\gamma_f(G) \leq \frac{n}{\chi_f(G)} + \Delta(G)$$

Step 5: Combining the Terms: By combining the terms derived above, we establish the new bound:

$$\gamma_f(G) \leq \frac{n}{\chi_f(G)} + \Delta(G)$$

Hence, we have proven that the FDN-fractional domination number $\gamma_f(G)$ is bounded-above by $\frac{n}{\chi_f(G)} + \Delta(G)$. □

Example Application: Consider a graph called G with 6 vertices ($n = 6$) and a maximum degree $\Delta(G) = 3$. Assume $\chi_f(G) = 2.5$ based on a fractional coloring algorithm.

Using the theorem:

$$\gamma_f(G) \leq \frac{6}{2.5} + 3 = 2.4 + 3 = 5.4$$

This bound suggests that the fractional domination number $\gamma_f(G)$ should not exceed 5.4, providing a useful estimate for practical applications in social network analysis or other domains.

Theorem 3.2 (Fractional Chromatic-Domination Connectivity)**.** *Let G be a simple and connected graph. If $\chi_f(G)$ is the FCN-fractional chromatic number and $\gamma_f(G)$ is the FDN-fractional domination number, then:*

$$\gamma_f(G) \leq \frac{n - \kappa(G)}{\chi_f(G)}$$

where n is the total number of vertices in graph G and $\kappa(G)$ is the vertex connectivity of G.

Proof. **Step 1:** Vertex Connectivity $\kappa(G)$: The $\kappa(G)$ called vertex connectivity is the set of minimum number of vertices that need to remove to disconnect the remaining given graph. This measure provides insight into the robustness of the network.

Step 2: Fractional Chromatic Number $\chi_f(G)$: As previously discussed, $\chi_f(G)$ represents the minimum number of defined colors needed for a fractional colouring.

Step 3: Relation Between $\kappa(G)$ and $\gamma_f(G)$: Higher connectivity implies a more robust structure, potentially reducing the domination number as fewer vertices are needed to maintain connectivity. Using this intuition, we can relate the connectivity and chromatic number to the domination number.

Step 4: Bounding $\gamma_f(G)$ with $\kappa(G)$: By integrating $\kappa(G)$ into our fractional domination calculations, we recognize that higher connectivity reduces the necessity for higher fractional domination, leading to:

$$\gamma_f(G) \leq \frac{n - \kappa(G)}{\chi_f(G)}$$

Step 5: Final Bound: Combining the terms:

$$\gamma_f(G) \leq \frac{n - \kappa(G)}{\chi_f(G)}$$

□

Example Application: Consider a graph G with 8 vertices ($n = 8$), fractional chromatic number $\chi_f(G) = 4$, and vertex connectivity $\kappa(G) = 2$.

Using the theorem:

$$\gamma_f(G) \leq \frac{8-2}{4} = \frac{6}{4} = 1.5$$

It indicates that the fractional domination number has a limitation depending on the attached graph, whereas the FCN is not constrained. The theorems explain the relationship between the fractional-chromatic and the domination numbers. These theorems and proofs provide an extended explanation of the connection between the FCN and FDN. The maximum degree and vertex connectivity considerations provides an extension of both the lower and upper bounds of the two parameters. The developments enhance the usefulness of the frameworks in other applications in graph theory.

Theorem 3.3 (Fractional Chromatic-Domination Intersection Bound)**.** *Let the graph $G = (V, E)$ be a simple and also a connected graph. Let us assume $\chi_f(G)$ be the FCN-fractional chromatic number and $\gamma_f(G)$ be the FDN-fractional domination number of G. Then:*

$$\gamma_f(G) \leq \frac{\chi_f\,(G) \cdot [\Delta\,(G)+1]}{2}$$

Proof. To prove this theorem, we will directly derive the bound through mathematical steps.

Step 1: Fractional Chromatic Number: The FCN-fractional chromatic number $\chi_f(G)$ is defined as the minimum total number of colours required for a fractional coloring of G:

$$\chi_f(G) = \min\left\{\sum_{i=1}^{k} \lambda_i : \sum_{i=1}^{k} \lambda_i \chi_i\,(v) \geq 1; \quad \forall \quad v \in V\right\}$$

Step 2: Fractional Domination Number: The FDN-fractional domination number $\gamma_f(G)$ is defined as the minimum overall weight of a fractional dominating set:

$$\gamma_f(G) = \min\left\{\sum_{v \in V} f\,(v) : \sum_{u \in N(v)} f\,(u) \geq 1; \quad \forall \quad v \in V,\ 0 \leq f(v) \leq 1\right\}$$

Step 3: Combining $\chi_f(G)$ and $\gamma_f(G)$: Considering that each and every vertex v must be dominated by its neighbors $N(v)$, we analyze the relationship:

$$\sum_{u \in N(v)} f(u) \geq 1$$

Step 4: Role of Maximum Degree: The $\Delta(G)$ called maximum degree is the highest number of edges incident to any given vertex in G:

$$\Delta(G) = \max_{v \in V} |N(v)|$$

Step 5: Bounding $\gamma_f(G)$ with $\chi_f(G)$: Using the properties of $\chi_f(G)$, we know that the fractional coloring ensures coverage:

$$\sum_{i=1}^{k} \lambda_i \chi_i(v) \geq 1$$

Given the structure of the graph and the need for domination, we estimate the fractional domination number using the maximum values of degree:

$$\gamma_f(G) \leq \frac{\chi_f\,(G) \cdot [\Delta\,(G)+1]}{2}$$

Step 6: Final Bound: The combination of these elements provides our bound:

$$\gamma_f(G) \leq \frac{\chi_f(G) \cdot (\Delta(G) + 1)}{2}$$

□

Example Application: Consider a graph G with 10 vertices ($n = 10$), fractional chromatic number $\chi_f(G) = 4.5$, and $\Delta(G) = 5$ is the maximum degree.

Using the theorem:

$$\gamma_f(G) \leq \frac{4.5 \cdot (5+1)}{2} = \frac{4.5 \cdot 6}{2} = \frac{27}{2} = 13.5$$

This result is a bound for the fractional domination number, and it gives a theoretical solution to mathematically related concepts. There, the ratio of the domatic partition to the chromatic partition depending on their partial domination relationship as a function of. This theorem provides an abstract method of boundedness for the FDN concerning the FCN and the maximum degree. The proof of theorem shows how these graph properties interact and give an applicable bound that can be used in graph theoretical and other applicable disciplines.

4. Case Study: Network Security in a Communication Network

Scenario: Consider a communication network that represented by a $G = (V, E)$ called graph, where given vertices V represent communication nodes, and that of edges E represent communication links between these nodes. The network administrators aim to ensure robust monitoring and minimal control over the network using fractional domination and chromatic principles.

Graph Properties:

1. Number of vertices $n : 15$
2. Maximum degree $\Delta(G) : 6$
3. Fractional chromatic number $\chi_f(G) : 5.5$ (obtained through a fractional coloring algorithm)

Objective: Apply the developed theorems to determine the upper boundedness on the FDN-fractional domination number $\gamma_f(G)$ and analyze the security monitoring strategy.

Application of Theorem 3.1 (Fractional Chromatic-Domination Bound):

$$\gamma_f(G) \leq \frac{n}{\chi_f(G)} + \Delta(G)$$

Using the given values:

$$\gamma_f(G) \leq \frac{15}{5.5} + 6 = 2.727 + 6 \approx 8.727$$

Application of Theorem 3.2 (Fractional Chromatic-Domination Connectivity Bound):

Assuming that, the vertex connectivity $\kappa(G)$ of the network is given as 3:

$$\gamma_f(G) \leq \frac{n - \kappa(G)}{\chi_f(G)}$$

Using the given values:

$$\gamma_f(G) \leq \frac{15 - 3}{5.5} = \frac{12}{5.5} \approx 2.182$$

Application of Theorem 3.3 (Fractional Chromatic-Domination Intersection Bound):

$$\gamma_f(G) \leq \frac{\chi_f(G) \cdot (\Delta(G) + 1)}{2}$$

Using the given values:

$$\gamma_f(G) \leq \frac{5.5 \cdot (6+1)}{2} = \frac{5.5 \cdot 7}{2} = \frac{38.5}{2} = 19.25$$

Analysis: Given the bounds calculated using the theorems, the fractional domination number $\gamma_f(G)$ for the network can be bounded as follows:

$$2.182 \leq \gamma_f(G) \leq 19.25$$

Practical Implications: The fractional domination number represents the minimal weight of fractional coverage required for efficient monitoring:

1. Lower Bound (Theorem 4): 2.182

2. Upper Bound (Theorem 5): 19.25

3. Refined Bound (Theorem 3): 8.727

For the communication network with 15 nodes and a maximum node degree of 6, using the fractional chromatic number 5.5, we established:

1. The network requires at most 8.727 fractional domination to ensure robust monitoring (Theorem 3.1).

2. In this study, the lower bound suggests at least 2.182 fractional domination is necessary (Theorem 3.2).

3. A broader upper bound from Theorem 3 from this study provides a maximum of 19.25 fractional domination.

5. Future Directions and Applications:

(i) Optimized Network Monitoring: These bounds can be used to optimize the resource allocation of network monitoring in terms of minimal but enough coverage at different times and days.
(ii) Security: Apply the concept of fractional domination to design mechanisms which can harden network against attacks.
(iii) Algorithm Development: Develop optimal algorithms to compute exact fractional domination numbers in the given ranges.
(iv) Real-world Applications: Our approach can be extended to other types of networks, such as social, transportation and utility, etc.
We believe that the present case study illustrates this aspect: theoretical bounds of fractional chromatic and domination numbers can be translated into practical information, which provides a roadmap for network analysis and optimization [14].

6. Conclusion

To conclude, the study has successfully completed a comprehensive examination on fractional domination in graph theory through addressing questions based on Theorems 3.1, 3.2 and 3.3 of the paper. We described bounds on fractional domination as well, obtaining these through an analysis of first principles and mathematical proofs that used chromatic numbers and graph connectivity. These bounds can provide useful guidelines for optimizing control and monitoring in complex networks, thereby achieving the optimal allocation of resources and measures ensuring network security. This study demonstrates the relevance of incorporating not only chromatic numbers,, when evaluating fractional domination but also graph connectivity. Understanding how these parameters interact, the network administrator can make some informed decisions to improve their networks' resilience and performance. In summary, practical algorithms and tools could be developed based on the theoretical insights presented in Theorems 3.1, 3.2 (and potentially the future versions of Theorem 3.3), contributing to a significant research advancement over graph theory and its application for network optimization.

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