International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **12(2)**(2024), 117–131
Available Online: http://ijmaa.in

Hyers Ulam Stability and Solutions for a Class of Nonlinear Integral Equations by Fixed Point Technique

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Abstract

Our intention of this paper is to prove the existence, uniqueness and stability of solution for some nonlinear functional-integral equations by using generalized tripled Lipschitz condition. We also prove a fixed point theorem to obtain the mentioned aim in Banach space $X = C([a,b],\mathbb{R})$. As application we study some Volterra integral equations with linear, non-linear and single kernel.

Keywords: Nonlinear functional-integral equation; Hyers-Ulam stability; iterative method; fixed point theorem.

2020 Mathematics Subject Classification: 34A34, 34A12, 45G10.

1. Introduction and preliminaries

Let us consider the following non homogeneous nonlinear Volterra integral equation,

$$\left.\begin{aligned} u(x) &= f_1(x) + \varphi\left(\int_a^x F(x,t,u(t),v(t),w(t))dt\right) \equiv T(u,v,w),\\ v(x) &= f_2(x) + \varphi\left(\int_a^x F(x,t,v(t),u(t),v(t))dt\right) \equiv T(v,u,v)\\ w(x) &= f_3(x) + \varphi\left(\int_a^x F(x,t,w(t),v(t),u(t))dt\right) \equiv T(w,v,u) \end{aligned}\right\} \tag{1}$$

where, $x, t \in I = [a,b]$, $-\infty < a < b < \infty$ and φ is a bounded linear mapping on X. this functional integral equation produces many integral equations which have arisen in different science fields such as theory of optimal control, economics and etc. Investigation on existence theorems for diverse nonlinear functional-integral equations has been presented in other references.

In this study, we will use the iterative method to prove that equation 1 has the mentioned cases under some appropriate conditions. On the other hand, in this paper, we prove the Hyers-Ulam stability (HUs) theorem of 1 under generalized tripled Lipschitz condition on F.

We say a functional equation is stable if for every approximate solution there exists an exact solution

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near it. In 1940, answering a problem of Ulam [23] affirmatively, Hyers [12] proof the following result (which is nowadays called the HUs stability theorem):
Let $S = (S, +)$ be a Abelian semigroup and assume that a function $f : S \to \mathbb{R}$ satisfies the inequality

$$|f(x+y) - f(x) - f(y)| \leq \epsilon,$$

$x, y \in S$ for some nonegative ϵ. Then there exists an additive function $h : S \to \mathbb{R}$ that

$$|h(x) - f(x)| \leq \epsilon,$$

holds. Ever since, the stability problems of functional equations have been extensively investigated by several mathematicians. Example of some recent developments, discussions and critiques of that idea if stability can be found in [5,6,10] and [11].
In this section, we introduce and recall some basic definitions and use them to obtain our aims in Section 2 and 3. Finally in Section 4 we offer some examples that verify the application of this kind of nonlinear functional-integral equations. In this section, we recall basic result which we will need in this paper.
Consider the non-homogeneous non-linear Volterra integral equation 1. We assume that $f : [a, b] \to \mathbb{R}$ is continuous and F is a mapping in the domain

$$D = \{(x, t, u) : x, t \in [a, b], u \in X\}.$$

Throughout this article, we consider the complete metric space (X, d), which is define as

$$d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|,$$

for all $f, g \in X$ and as we mentioned, we assume that φ is a bounded linear mapping on X.
Note that, the linear mapping $\varphi : X \to X$ is called bounded, if there exists $M > 0$ such that $\|\varphi(x)\| \leq M\|x\|$; for all $x \in X$. In this case, we define $\|\varphi\| = \sup\left\{\frac{\|\varphi x\|}{\|x\|}; x \neq 0, x \in X\right\}$. This φ is bounded iff $\|\varphi\| < \infty$.
Note: As φ is bounded linear mapping on X, then $\varphi(x) = \lambda x$ where λ does not depend on $x \in X$.

Definition 1.1. *We say that equation* (1) *has the HUs if there exists a constant* $K \geq 0$ *with the following property; for every* $\epsilon > 0, (y_1, y_2, y_3) \in X^3$, *if*

$$\left|y_1(x) - f_1(x) - \varphi\left(\int_a^x F(x, t, u(t), v(t), w(t))dt\right)\right| \leq \epsilon,$$
$$\left|y_2(x) - f_2(x) - \varphi\left(\int_a^x F(x, t, v(t), u(t), v(t))dt\right)\right| \leq \epsilon,$$
$$\left|y_3(x) - f_3(x) - \varphi\left(\int_a^x F(x, t, w(t), v(t), u(t))dt\right)\right| \leq \epsilon$$

then there exists some $(u, v, w) \in X^3$ *satisfying*

$$u(x) = f_1(x) + \varphi\left(\int_a^x F(x,t,u(t),v(t),w(t))dt\right),$$
$$v(x) = f_2(x) + \varphi\left(\int_a^x F(x,t,v(t),u(t),v(t))dt\right),$$
$$w(x) = f_3(x) + \varphi\left(\int_a^x F(x,t,w(t),v(t),u(t))dt\right)$$

such that

$$|u(x) - y_1(x)| \leq K\epsilon,$$

$$|v(x) - y_2(x)| \leq K\epsilon,$$

$$|w(x) - y_3(x)| \leq K\epsilon.$$

We call such K a HUs constant for equation (1).

Definition 1.2 ([6])**.** *Let* δ *denote the class of those functions* $\beta : [0, \infty) \to [0, 1)$ *which satisfies the condition*

$$\beta(t_n) \to 1 \quad implies \quad t_n \to 0.$$

Definition 1.3 ([6])**.** *Let* $\mathbb{B}$ *denote the class of those functions* $\phi : [0, \infty) \to [0, \infty)$ *which satisfies the following condition :*

(i). ϕ *is increasing,*

(ii). for each $x > 0$, $\phi(x) < x$,

(iii). $\beta(x) = \frac{\phi(x)}{x} \in \delta, \quad x \neq 0.$

For example, $\phi(t) = \mu t$, where $0 \leq \mu < 1$, $\phi(t) = \frac{t}{t+1}$ and $\phi(t) = \log(1 + t)$ are in $\mathbb{B}$.

2. Existence and Uniqueness of the Solution of Nonlinear Integral Equations

In this section we will study the existence and uniqueness of the nonlinear functional integral equation 1 in X

Theorem 2.1. *Consider the integral equation 1 such that*

(i). $F : D \times D \times D \to \mathbb{R}$ *and* $f : [a, b] \to \mathbb{R}$ *are continuous.*

(ii). $\varphi : X \to X$ *is bounded linear transformation.*

(iii). There exists an integrable function $p : [a, b] \times [a, b] \to \mathbb{R}$ *and* $\phi \in \mathbb{R}$ *such that*

$$|F(x,t,u_1,v_1,w_1) - |F(x,t,u_2,v_2,w_2)| \leq p(x,t)\phi(\max\{|u_1 - u_2|, |v_1 - v_2|, |w_1 - w_2|\}), \tag{2}$$

for each $x, t \in [a, b]$ *and* $(u_1, v_1, w_1), (u_2, v_2, w_2) \in X^3$.

(iv) $\sup_{x\in[a,b]} \int_a^b p^2(x,t)dt \leq \frac{1}{\|\varphi\|^2(b-a)}$.

Then the integral equation 1 has an unique tripled fixed point $(u, v, w) \in X^3$.

Note: We define 2 as a generalized Lipschitz condition.

Proof. Consider the iterative scheme

$$\left.\begin{aligned} u_{n+1}(x) &= f_1(x) + \varphi\left(\int_a^x F(x,t,u_n(t),v_n(t),w_n(t))dt\right) \equiv T(u_n,v_n,w_n), \\ v_{n+1}(x) &= f_2(x) + \varphi\left(\int_a^x F(x,t,v_n(t),u_n(t),v_n(t))dt\right) \equiv T(v_n,u_n,v_n), \\ w_{n+1}(x) &= f_3(x) + \varphi\left(\int_a^x F(x,t,w_n(t),v_n(t),u_n(t))dt\right) \equiv T(w_n,v_n,u_n) \end{aligned}\right\} \tag{3}$$

where, $(u_0, v_0, w_0) \in X^3$ is an arbitrary initial guess. So

$$\begin{aligned} |u_{n+1}(x) - u_n(x)| &= |T(u_n,v_n,w_n) - T(u_{n-1},v_{n-1},w_{n-1})| \\ &\leq \left|\varphi\left(\int_a^x F(x,t,u_n(t),v_n(t),w_n(t))dt\right) - \varphi\left(\int_a^x F(x,t,u_{n-1}(t),v_{n-1}(t),w_{n-1}(t))dt\right)\right| \\ &\leq \left|\varphi\left(\int_a^x (F(x,t,u_n(t),v_n(t),w_n(t)) - F(x,t,u_{n-1}(t),v_{n-1}(t),w_{n-1}(t)))dt\right)\right| \\ &\leq \|\varphi\|| \int_a^x (F(x,t,u_n(t),v_n(t),w_n(t)) - F(x,t,u_{n-1}(t),v_{n-1}(t),w_{n-1}(t))dt| \\ &\leq \|\varphi\| \int_a^x |(F(x,t,u_n(t),v_n(t),w_n(t)) - F(x,t,u_{n-1}(t),v_{n-1}(t),w_{n-1}(t)))|dt \\ &\leq \|\varphi\|| \int_a^x (F(x,t,u_n(t),v_n(t),w_n(t)) - F(x,t,u_{n-1}(t),v_{n-1}(t),w_{n-1}(t)))dt| \\ &\leq \|\varphi\| \int_a^b p(x,t)\phi(\max\{|u_n(x) - u_{n-1}(x)|, |v_n(x) - v_{n-1}(x)|, |w_n(x) - w_{n-1}(x)|\})dt \\ &\leq \|\varphi\| \int_a^b p(x,t)\phi(|u_n(x) - u_{n-1}(x)|)dt \\ &\leq \|\varphi\| \left(\int_a^b p^2(x,t)dt\right)^2 \left(\int_a^b |\phi(|u_n(x) - u_{n-1}(x)|)dt\right)^2. \end{aligned} \tag{4}$$

Similarly we have

$$|v_{n+1}(x) - v_n(x)| \leq \|\varphi\| \left(\int_a^b p^2(x,t)dt\right)^2 \left(\int_a^b |\phi(|v_n(x) - v_{n-1}(x)|)dtdt\right)^2 \tag{5}$$

and

$$|w_{n+1}(x) - w_n(x)| \leq \|\varphi\| \left(\int_a^b p^2(x,t)dt\right)^2 \left(\int_a^b |\phi(|w_n(x) - w_{n-1}(x)|)dtdt\right)^2 \tag{6}$$

As the function ϕ is increasing then

$$\phi(|u_n(x) - u_{n-1}(x)|) \leq \phi(d(u_n, u_{n-1})),$$

$$\phi(|v_n(x) - v_{n-1}(x)|) \leq \phi(d(v_n, v_{n-1})),$$
$$\phi(|w_n(x) - w_{n-1}(x)|) \leq \phi(d(w_n, w_{n-1})),$$

so, we obtain

$$\begin{aligned} d^2(u_{n+1}, u_n) &\leq \|\varphi\|^2 \left(\sup_{x\in[a,b]} \int_a^b p^2(x,t)dt \right) \left(\int_a^b \phi^2(d(u_n, u_{n-1}))dt \right) \\ &\leq \phi^2(d(u_n, u_{n-1})), \quad (7) \\ d^2(v_{n+1}, v_n) &\leq \|\varphi\|^2 \left(\sup_{x\in[a,b]} \int_a^b p^2(x,t)dt \right) \left(\int_a^b \phi^2(d(v_n, v_{n-1}))dt \right) \\ &\leq \phi^2(d(v_n, v_{n-1}) \quad (8) \end{aligned}$$

and

$$\begin{aligned} d^2(w_{n+1}, w_n) &\leq \|\varphi\|^2 \left(\sup_{x\in[a,b]} \int_a^b p^2(x,t)dt \right) \left(\int_a^b \phi^2(d(w_n, w_{n-1}))dt \right) \\ &\leq \phi^2(d(w_n, w_{n-1})). \quad (9) \end{aligned}$$

By (7), (8) and (9) we have

$$\begin{aligned} d(u_{n+1}, u_n) + d(v_{n+1}, v_n) + d(w_{n+1}, w_n) &\leq \phi(d(u_n, u_{n-1})) + \phi(d(v_n, v_{n-1})) + \phi(d(w_n, w_{n-1})) \\ &\leq \phi(d(u_n, u_{n-1}) + d(v_n, v_{n-1}) + d(w_n, w_{n-1})) \\ &= \frac{\phi(d(u_n, u_{n-1}) + d(v_n, v_{n-1}) + d(w_n, w_{n-1}))}{d(u_n, u_{n-1}) + d(v_n, v_{n-1}) + d(w_n, w_{n-1})} \\ &\quad (d(u_n, u_{n-1}) + d(v_n, v_{n-1}) + d(w_n, w_{n-1})) \\ &= \beta(d(u_n, u_{n-1}) + d(v_n, v_{n-1}) + d(w_n, w_{n-1}))(d(u_n, u_{n-1}) \\ &+ d(v_n, v_{n-1}) + d(w_n, w_{n-1})) \quad (10) \end{aligned}$$

and so the sequence $\{d(u_{n+1}, u_n) + d(v_{n+1}, v_n) + d(w_{n+1}, w_n)\}$ is decreasing and bounded. Thus there exists $r \geq 0$ such that

$$\lim_{n\to\infty} [d(u_{n+1}, u_n) + d(v_{n+1}, v_n) + d(w_{n+1}, w_n)] = r.$$

Assume $r > 0$. Then from (10) we have

$$\frac{d(u_{n+1}, u_n) + d(v_{n+1}, v_n) + d(w_{n+1}, w_n)}{d(u_n, u_{n-1}) + d(v_n, v_{n-1}) + d(w_n, w_{n-1})} \leq \beta(d(u_n, u_{n-1}) + d(v_n, v_{n-1}) + d(w_n, w_{n-1})) \quad (11)$$

where $n = 1, 2, 3, \ldots$. Then (11) yields

$$\lim_{n\to\infty} \beta(d(u_{n+1}, u_n) + d(v_{n+1}, v_n) + d(w_{n+1}, w_n)) = 1.$$

Then $\beta \notin \delta$ and this contradiction. So $r = 0$ and then

$$\lim_{n\to\infty}(d(u_{n+1}, u_n) + d(v_{n+1}, v_n) + d(w_{n+1}, w_n)) = 0.$$

Next we show that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are Cauchy sequence. On the contrary, assume that

$$\lim_{n,m\to\infty}(\sup(d(u_n, u_m) + d(v_n, v_m) + d(w_n, w_m))) > 0. \tag{12}$$

By the triangle inequality and relation (10) we have

$$\begin{aligned} d(u_n, u_m) + d(v_n, v_m) + d(w_n, w_m) &\leq d(u_n, u_{n+1}) + d(v_n, v_{n+1}) + d(w_n, w_{n+1}) + d(u_{n+1}, u_{m+1}) \\ &\quad + d(v_{n+1}, v_{m+1}) + d(w_{n+1}, w_{m+1}) + d(u_{m+1}, u_m) \\ &\quad + d(v_{m+1}, v_m) + d(w_{m+1}, w_m) \\ &\leq d(u_n, u_{n+1}) + d(v_n, v_{n+1}) + d(w_n, w_{n+1}) + d(u_{m+1}, u_m) \\ &\quad + d(v_{m+1}, v_m) + d(w_{m+1}, w_m) + (\beta(d(u_n, u_m) + d(v_n, v_m) \\ &\quad + d(w_n, w_m)))[d(u_n, u_m) + d(v_n, v_m) + d(w_n, w_m)] \end{aligned}$$

hence

$$\begin{aligned} [d(u_n, u_m) + d(v_n, v_m) + d(w_n, w_m)] - (\beta(d(u_n, u_m) + d(v_n, v_m) + d(w_n, w_m))) \\ [d(u_n, u_m) + d(v_n, v_m) + d(w_n, w_m)] \\ \leq d(u_n, u_{n+1}) + d(v_n, v_{n+1}) + d(w_n, w_{n+1}) \\ + d(u_{m+1}, u_m) + d(v_{m+1}, v_m) + d(w_{m+1}, w_m) \\ d(u_n, u_m) + d(v_n, v_m) + d(w_n, w_m) - (1 - \beta(d(u_n, u_m) + d(v_n, v_m) + d(w_n, w_m))) \\ \leq d(u_n, u_{n+1}) + d(v_n, v_{n+1}) + d(w_n, w_{n+1}) + d(u_{m+1}, u_m) \\ + d(v_{m+1}, v_m) + d(w_{m+1}, w_m) \\ d(u_n, u_m) + d(v_n, v_m) + d(w_n, w_m) \leq (1 - \beta(d(u_n, u_m) + d(v_n, v_m) + d(w_n, w_m)))^{-1} \\ [d(u_n, u_{n+1}) + d(v_n, v_{n+1}) + d(w_n, w_{n+1}) \\ + d(u_{m+1}, u_m) + d(v_{m+1}, v_m) + d(w_{m+1}, w_m)] \end{aligned}$$

Since

$$\limsup_{n,m\to\infty}[d(u_n, u_m) + d(v_n, v_m) + d(w_n, w_m)] > 0$$

and

$$\lim_{n,m\to\infty}[d(u_n, u_m) + d(v_n, v_m) + d(w_n, w_m)] = 0$$

then

$$\lim_{n,m\to\infty} \sup((1-\beta(d(u_n,u_m+d(v_n,v_m)+d(w_n,w_m))^{-1})=+\infty,$$

from which we obtain

$$\lim_{n,m\to\infty} \sup(\beta(d(u_n,u_m)+d(v_n,v_m)+d(w_n,w_m)))=1.$$

But since $\beta \in \delta$, we get

$$\lim_{n,m\to\infty} \sup(d(u_n,u_m)+d(v_n,v_m)+d(w_n,w_m))=0.$$

This contradiction (12) and shows $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are Cauchy sequence in X. Since (X,d) is a complete metric space, then there exists $(u,v,w)\in X^3$ such that

$$\lim_{n\to\infty}(u_n,v_n,w_n)=(u,v,w.$$

Now by taking the limit of both sides of (3), we have

$$\begin{aligned}
u &= \lim_{n\to\infty} u_{n-1}(x)=\lim_{n\to\infty}\left(f_1(x)+\varphi\left(\int_a^x F(x,t,u_n(t),v_n(t),w_n(t))dt\right)\right)\\
&= f_1(x)+\varphi\left(\int_a^x F(x,t,\lim_{n\to\infty}u_n(t),\lim_{n\to\infty}v_n(t),\lim_{n\to\infty}w_n(t))dt\right)\\
&= f_1(x)+\varphi\left(\int_a^x F(x,t,u(t),v(t),w(t))dt\right)\\
v &= \lim_{n\to\infty} v_{n-1}(x)=\lim_{n\to\infty}\left(f_2(x)+\varphi\left(\int_a^x F(x,t,v_n(t),u_n(t),v_n(t))dt\right)\right)\\
&= f_2(x)+\varphi\left(\int_a^x F(x,t,\lim_{n\to\infty}v_n(t),\lim_{n\to\infty}u_n(t),\lim_{n\to\infty}v_n(t))dt\right)\\
&= f_2(x)+\varphi\left(\int_a^x F(x,t,v(t),u(t),v(t))dt\right).
\end{aligned}$$

and

$$\begin{aligned}
w &= \lim_{n\to\infty} w_{n-1}(x)=\lim_{n\to\infty}\left(f_3(x)+\varphi\left(\int_a^x F(x,t,w_n(t),v_n(t),u_n(t))dt\right)\right)\\
&= f_3(x)+\varphi\left(\int_a^x F(x,t,\lim_{n\to\infty}w_n(t),\lim_{n\to\infty}v_n(t),\lim_{n\to\infty}u_n(t))dt\right)\\
&= f_3(x)+\varphi\left(\int_a^x F(x,t,w(t),v(t),u(t))dt\right)
\end{aligned}$$

So, there exists an unique solution $(u,v,w) \in X^3$ such that $T(u,v,w)=u$, $T(v,u,v)=v \ and \ T(w,v,u)=w.$ □

3. Stability of Nonlinear Integral Equation

Theorem 3.1. *The equation $T(x,y,z) = x$, $T(y,x,y) = y$ and $T(z,y,x) = z$ where T is defined by 1 under the assumption of Theorem 2.1, has the Hyers-Ulam stability that is for every $(\alpha,\beta,\gamma) \in X^3$ and $\epsilon > 0$ with*

$$d(T(\alpha,\beta,\gamma),\alpha) \le \epsilon,$$
$$d(T(\beta,\alpha,\beta),\beta) \le \epsilon$$
$$d(T(\gamma,\beta,\alpha),\gamma) \le \epsilon$$

there exists an unique solution $(u,v,w) \in X^3$ such that $T(u,v,w) = u,\quad T(v,u,v) = v,\quad T(w,v,u) = w$ and

$$d(\alpha,u) \le K\epsilon$$
$$d(\beta,v) \le K\epsilon$$
$$d(\gamma,w) \le K\epsilon$$

for some $K \ge 0$.

Proof. By relation 4, 5 and 6 we can write

$$\begin{aligned}
|u_{n+1}(x) - u_n(x)| &= |T(u_n,v_n,w_n) - T(u_{n-1},v_{n-1},w_{n-1})| \\
&\le \left| |\varphi\left(\int_a^x F(x,t,u_n(t),v_n(t),w_n(t))dt\right) - \varphi\left(\int_a^x F(x,t,u_{n-1}(t),v_{n-1}(t),w_{n-1}(t))dt\right)\right| \\
|u_{n+1}(x) - u_n(x)| &\le \|\varphi\| \left(\frac{1}{\|\varphi\|^2(b-a)}\right)^{\frac{1}{2}} \left(\int_a^x |u_n(x) - u_{n-1}(x)|^2 dt\right)^{\frac{1}{2}}
\end{aligned}$$

Hence

$$\begin{aligned}
|u_{n+1}(x) - u_n(x)| &\le \left(\frac{1}{(b-a)}\int_a^x |u_n(t_1) - u_{n-1}(t_1)|^2 dt_1\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{(b-a)^2}\int_a^x\int_a^{t_1} |u_n(t_2) - u_{n-1}(t_2)|^2 dt_2 dt_1\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{(b-a)^3}\int_a^x\int_a^{t_1}\int_a^{t_2} |u_n(t_3) - u_{n-2}(t_3)|^2 dt_3 dt_2 dt_1\right)^{\frac{1}{2}} \\
&\vdots \\
&\le \left(\frac{1}{(b-a)^n}\int_a^x\int_a^{t_1}\int_a^{t_2}\cdots\int_a^{t_{n-1}} |u_n(t_3) - u_{n-2}(t_3)|^2 dt_n \ldots dt_3 dt_2 dt_1\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{(b-a)^n} d^2(T(u_0,v_0,w_0),u_0)\int_a^x\int_a^{t_1}\int_a^{t_2}\cdots\int_a^{t_{n-1}} dt_n \ldots dt_3 dt_2 dt_1\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{(b-a)^n}\frac{(x-a)^n}{(n)!} d^2(T(u_0,v_0,w_0),u_0)\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \frac{d(T(u_0, v_0, w_0), u_0)}{(n!)^{\frac{1}{2}}}.$$

Similarly we have

$$\begin{aligned}
|v_{n+1}(x) - v_n(x)| &= |T(v_n, u_n, v_n) - T(v_{n-1}, u_{n-1}, v_{n-1})| \\
&\leq \left|\varphi\left(\int_a^x F(x, t, v_n(t), u_n(t), v_n(t))dt\right) - \varphi\left(\int_a^x F(x, t, v_{n-1}(t), u_{n-1}(t), v_{n-1}(t))dt\right)\right| \\
&\quad |v_{n+1}(x) - v_n(x)| \\
&\leq \|\varphi\| \left(\frac{1}{\|\varphi\|^2(b-a)}\right)^{\frac{1}{2}} \left(\int_a^b |v_n(x) - v_{n-1}(x)|^2 dt\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
|v_{n+1}(x) - v_n(x)| &\leq \left(\frac{1}{(b-a)} \int_a^x |v_n(t_1) - v_{n-1}(t_1)|^2 dt_1\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{(b-a)^2} \int_a^x \int_a^{t_1} |v_n(t_2) - v_{n-1}(t_2)|^2 dt_2 dt_1\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{(b-a)^3} \int_a^x \int_a^{t_1} \int_a^{t_2} |v_n(t_3) - v_{n-2}(t_3)|^2 dt_3 dt_2 dt_1\right)^{\frac{1}{2}} \\
&\vdots \\
&\leq \left(\frac{1}{(b-a)^n} \int_a^x \int_a^{t_1} \int_a^{t_2} \cdots \int_a^{t_{n-1}} |v_n(t_3) - v_{n-2}(t_3)|^2 dt_n \ldots dt_3 dt_2 dt_1\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{(b-a)^n} d^2(T(v_0, u_0, v_0), v_0) \int_a^x \int_a^{t_1} \int_a^{t_2} \cdots \int_a^{t_{n-1}} dt_n \ldots dt_3 dt_2 dt_1\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{(b-a)^n} \frac{(x-a)^n}{(n)!} d^2(T(v_0, u_0, v_0), v_0)\right)^{\frac{1}{2}} \\
&\leq \frac{d(T(v_0, u_0, v_0), v_0)}{(n!)^{\frac{1}{2}}}.
\end{aligned}$$

and

$$\begin{aligned}
|w_{n+1}(x) - w_n(x)| &= |T(w_n, v_n, u_n) - T(w_{n-1}, v_{n-1}, u_{n-1})| \\
&\leq \left|\varphi\left(\int_a^x F(x, t, w_n(t), v_n(t), u_n(t))dt\right) - \varphi\left(\int_a^x F(x, t, w_{n-1}(t), v_{n-1}(t), u_{n-1}(t))dt\right)\right| \\
&\quad |w_{n+1}(x) - w_n(x)| \\
&\leq \|\varphi\| \left(\frac{1}{\|\varphi\|^2(b-a)}\right)^{\frac{1}{2}} \left(\int_a^x |w_n(x) - w_{n-1}(x)|^2 dt\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
|w_{n+1}(x) - w_n(x)| &\leq \left(\frac{1}{(b-a)} \int_a^x |w_n(t_1) - w_{n-1}(t_1)|^2 dt_1\right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{(b-a)^2} \int_a^x \int_a^{t_1} |w_n(t_2) - w_{n-1}(t_2)|^2 dt_2 dt_1\right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{1}{(b-a)^3}\int_a^x\int_a^{t_1}\int_a^{t_2}|w_n(t_3)-w_{n-2}(t_3)|^2dt_3dt_2dt_1\right)^{\frac{1}{2}}\\
&\vdots\\
&\leq \left(\frac{1}{(b-a)^n}\int_a^x\int_a^{t_1}\int_a^{t_2}\cdots\int_a^{t_{n-1}}|w_n(t_3)-w_{n-2}(t_3)|^2dt_n\ldots dt_3dt_2dt_1\right)^{\frac{1}{2}}\\
&\leq \left(\frac{1}{(b-a)^n}d^2(T(w_0,v_0,u_0),u_0)\int_a^x\int_a^{t_1}\int_a^{t_2}\cdots\int_a^{t_{n-1}}dt_n\ldots dt_3dt_2dt_1\right)^{\frac{1}{2}}\\
&\leq \left(\frac{1}{(b-a)^n}\frac{(x-a)^n}{(n)!}d^2(T(w_0,v_0,u_0),u_0)\right)^{\frac{1}{2}}\\
&\leq \frac{d(T(w_0,v_0,u_0),u_0)}{(n!)^{\frac{1}{2}}}.
\end{aligned}$$

Let $(\alpha,\beta,\gamma)\in X^3$ and

$$\begin{aligned}
d(T(\alpha,\beta,\gamma),\alpha) &\leq \epsilon,\\
d(T(\beta,\alpha,\beta),\beta) &\leq \epsilon\\
d(T(\gamma,\beta,\alpha),\gamma) &\leq \epsilon.
\end{aligned}$$

In the previous section, we proved that

$$\begin{aligned}
u(t) &= \lim_{n\to\infty}T^n(\alpha(t),\beta(t),\gamma(t))\\
v(t) &= \lim_{n\to\infty}T^n(\beta(t),\alpha(t),\beta(t))\\
w(t) &= \lim_{n\to\infty}T^n(\gamma(t),\beta(t),\alpha(t))
\end{aligned}$$

are an exact solution of the equation $T(x,y,z)=x$, $T(y,x,y)=y$ and $T(z,y,x)=z$. Clearly there is N with $d(T^N(\alpha,\beta,\gamma),u)\leq\epsilon$, $d(T^N(\beta,\alpha,\beta),v)\leq\epsilon$ and $d(T^N(\gamma,\beta,\alpha),z)\leq\epsilon$, because $T^N(\alpha,\beta,\gamma)$ is uniformly convergent to u, $T^N(\beta,\alpha,\beta)$ is uniformly convergent to v and $T^N(\gamma,\beta,\alpha)$ is uniformly convergent to w as $n\to\infty$. Without loss of generality, for sufficiently large odd number N, we have

$$\begin{aligned}
d(\alpha,u) &\leq d(\alpha,T^N(\alpha,\beta,\gamma))+d(T^N(\alpha,\beta,\gamma),u)\\
&\leq d(\alpha,T(\alpha,\beta,\gamma))+d(T(\alpha,\beta,\gamma),T^2(\alpha,\beta,\gamma))+d(T^2(\alpha,\beta,\gamma),T^3(\alpha,\beta,\gamma))\\
&\quad+\cdots+d(T^{N-1}(\alpha,\beta,\gamma),T^N(\alpha,\beta,\gamma))+d(T^N(\alpha,\beta,\gamma),u)\\
&\leq d(\alpha,T(\alpha,\beta,\gamma))+\frac{d(\alpha,T(\alpha,\beta,\gamma))}{(1!)^{\frac{1}{2}}}+\frac{d(\alpha,T(\alpha,\beta,\gamma))}{(2!)^{\frac{1}{2}}}\\
&\quad+\cdots+\frac{d(\alpha,T(\alpha,\beta,\gamma))}{((N-1)!)^{\frac{1}{2}}}+d(T^N(\alpha,\beta,\gamma),u)\\
&\leq d(\alpha,T(\alpha,\beta,\gamma))\left(1+\frac{1}{(1!)^{\frac{1}{2}}}+\frac{1}{(2!)^{\frac{1}{2}}}+\cdots+\frac{1}{((N-1)!)^{\frac{1}{2}}}\right)+\epsilon\\
&\leq \epsilon\left(\left\{1+1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{6}}\right\}+\left\{\left(\frac{1}{2^4}\right)^{\frac{1}{2}}+\left(\frac{1}{2^5}\right)^{\frac{1}{2}}+\cdots+\left(\frac{1}{2^{N-1}}\right)^{\frac{1}{2}}\right\}\right)+\epsilon
\end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon\left(\left\{1+1+1+\frac{1}{2}\right\}+\left\{\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{\frac{N-1}{2}}}\right\}\right.\\
&\quad\left.+\left\{\frac{1}{2^{2.5}}+\frac{1}{2^{3.5}}+\frac{1}{2^{4.5}}+\cdots+\frac{1}{2^{\frac{N-2}{2}}}\right\}\right)+\epsilon\\
&\leq \epsilon\left(\frac{7}{2}+\left\{\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{\frac{N-1}{2}}}\right\}+\left\{\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{\frac{N-1}{2}}}\right\}\right)+\epsilon\\
&\leq \epsilon\left(\frac{7}{2}+\left\{1-\frac{1}{2^{\frac{N-3}{2}}}\right\}\right)+\epsilon\\
&= \left(\frac{11}{2}-\frac{1}{2^{\frac{N-3}{2}}}\right)=K\epsilon,
\end{aligned}
$$

$$
\begin{aligned}
d(\beta,v) &\leq d(\beta,T^N(\beta,\alpha,\beta))+d(T^N(\beta,\alpha,\beta),v)\\
&\leq d(\beta,T(\beta,\alpha,\beta))+d(T(\beta,\alpha,\beta),T^2(\beta,\alpha,\beta))+d(T^2(\beta,\alpha,\beta),T^3(\beta,\alpha,\beta))\\
&\quad+\cdots+d(T^{N-1}(\beta,\alpha,\beta),T^N(\beta,\alpha,\beta))+d(T^N(\beta,\alpha,\beta),v)\\
&\leq d(\beta,T(\beta,\alpha,\beta))+\frac{d(\beta,T(\beta,\alpha,\beta))}{(1!)^{\frac{1}{2}}}+\frac{d(\beta,T(\beta,\alpha,\beta))}{(2!)^{\frac{1}{2}}}\\
&\quad+\cdots+\frac{d(\beta,T(\beta,\alpha,\beta))}{((N-1)!)^{\frac{1}{2}}}+d(T^N(\beta,\alpha,\beta),v)\\
&\leq d(\beta,T(\beta,\alpha,\beta))\left(1+\frac{1}{(1!)^{\frac{1}{2}}}+\frac{1}{(2!)^{\frac{1}{2}}}+\cdots+\frac{1}{((N-1)!)^{\frac{1}{2}}}\right)+\epsilon\\
&\leq \epsilon\left(\left\{1+1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{6}}\right\}+\left\{\left(\frac{1}{2^4}\right)^{\frac{1}{2}}+\left(\frac{1}{2^5}\right)^{\frac{1}{2}}+\cdots+\left(\frac{1}{2^{N-1}}\right)^{\frac{1}{2}}\right\}\right)+\epsilon\\
&\leq \epsilon\left(\left\{1+1+1+\frac{1}{2}\right\}+\left\{\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{\frac{N-1}{2}}}\right\}\right.\\
&\quad\left.+\left\{\frac{1}{2^{2.5}}+\frac{1}{2^{3.5}}+\frac{1}{2^{4.5}}+\cdots+\frac{1}{2^{\frac{N-2}{2}}}\right\}\right)+\epsilon\\
&\leq \epsilon\left(\frac{7}{2}+\left\{\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{\frac{N-1}{2}}}\right\}+\left\{\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{\frac{N-1}{2}}}\right\}\right)+\epsilon\\
&\leq \epsilon\left(\frac{7}{2}+\left\{1-\frac{1}{2^{\frac{N-3}{2}}}\right\}\right)+\epsilon\\
&= \left(\frac{11}{2}-\frac{1}{2^{\frac{N-3}{2}}}\right)=K\epsilon,
\end{aligned}
$$

and

$$
\begin{aligned}
d(\gamma,w) &\leq d(\gamma,T^N(\gamma,\beta,\alpha))+d(T^N(\gamma,\beta,\alpha),w)\\
&\leq d(\gamma,T(\gamma,\beta,\alpha))+d(T(\gamma,\beta,\alpha),T^2(\gamma,\beta,\alpha))+d(T^2(\gamma,\beta,\alpha),T^3(\gamma,\beta,\alpha))\\
&\quad+\cdots+d(T^{N-1}(\gamma,\beta,\alpha),T^N(\gamma,\beta,\alpha))+d(T^N(\gamma,\beta,\alpha),w)\\
&\leq d(\gamma,T(\gamma,\beta,\alpha))+\frac{d(\gamma,T(\gamma,\beta,\alpha))}{(1!)^{\frac{1}{2}}}+\frac{d(\gamma,T(\gamma,\beta,\alpha))}{(2!)^{\frac{1}{2}}}\\
&\quad+\cdots+\frac{d(\gamma,T(\gamma,\beta,\alpha))}{((N-1)!)^{\frac{1}{2}}}+d(T^N(\gamma,\beta,\alpha),u)\\
&\leq d(\gamma,T(\gamma,\beta,\alpha))\left(1+\frac{1}{(1!)^{\frac{1}{2}}}+\frac{1}{(2!)^{\frac{1}{2}}}+\cdots+\frac{1}{((N-1)!)^{\frac{1}{2}}}\right)+\epsilon
\end{aligned}
$$

$$
\begin{aligned}
&\leq \epsilon\left(\left\{1+1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{6}}\right\}+\left\{\left(\frac{1}{2^4}\right)^{\frac{1}{2}}+\left(\frac{1}{2^5}\right)^{\frac{1}{2}}+\cdots+\left(\frac{1}{2^{N-1}}\right)^{\frac{1}{2}}\right\}\right)+\epsilon \\
&\leq \epsilon\left(\left\{1+1+1+\frac{1}{2}\right\}+\left\{\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{\frac{N-1}{2}}}\right\}\right. \\
&\quad \left.+\left\{\frac{1}{2^{2.5}}+\frac{1}{2^{3.5}}+\frac{1}{2^{4.5}}+\cdots+\frac{1}{2^{\frac{N-2}{2}}}\right\}\right)+\epsilon \\
&\leq \epsilon\left(\frac{7}{2}+\left\{\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{\frac{N-1}{2}}}\right\}+\left\{\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{\frac{N-1}{2}}}\right\}\right)+\epsilon \\
&\leq \epsilon\left(\frac{7}{2}+\left\{1-\frac{1}{2^{\frac{N-3}{2}}}\right\}\right)+\epsilon \\
&= \left(\frac{11}{2}-\frac{1}{2^{\frac{N-3}{2}}}\right)=K\epsilon,
\end{aligned}
$$

which complete the proof. □

4. Examples

In this section we present some example of classical integral and functional equations which are particular case of equation 1 and consequently, the existence, uniqueness and stability of there solutions can be established using Theorem 2.1 and 3.1.

Example 4.1. *Consider the following linear Volterra integral equation*

$$
\begin{aligned}
u(x) &= x^5-\frac{x^8}{7}+\int_0^x (1-t-x)\,(u(t))\,dt, \\
v(x) &= x^6-\frac{x^9}{10}+\int_0^x (1-t-x)^2\,(v(t))^2\,dt, \qquad (13)\\
w(x) &= x^7-\frac{x^10}{13}+\int_0^x (1-t-x)^3\,(w(t))^3\,dt, \quad x,t\in[0,1]. \qquad (14)
\end{aligned}
$$

We have

$$
\begin{aligned}
|F(x,t,u_1,v_1,w_1)-F(x,t,u_2,v_2,w_2)| &= |(t-x)(u_1+v_1+w_1)-(t-x)(u_2+v_2+w_2)| \\
&= |(t-x)|(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|) \\
&= \left(\frac{|t-x|}{\mu}\right)(\mu(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|)),
\end{aligned}
$$

where $\frac{1}{\sqrt{3}}\leq\mu<1$. *Now we put* $p(x,t)=\frac{t-x}{\mu}$ *and* $\phi(t)=\mu t$. *Because*

$$
\sup_{x\in[0,1]}\int_0^1 p^2(x,t)dt=\frac{1}{6\mu^2}\leq 1,
$$

then by applying the result obtained in Theorem 2.1 and 3.1, we deduce that the equation 13 has a stable unique solution in Banach space $C[0,1]$.

Example 4.2. *Consider the following linear Volterra integral equation*

$$\begin{aligned} u(x) &= \frac{1}{10}\sin\left(\frac{1}{1+x}\right)+\frac{x}{9}\int_0^x \frac{\arctan(x^2t)}{(1+xt)^2}\sin(u(t))dt \\ v(x) &= \frac{1}{10}\cos\left(\frac{1}{1+x}\right)+\frac{x}{9}\int_0^x \frac{\arctan(x^2t)}{(1+xt)^2}\cos(u(t))dt \\ w(x) &= \frac{1}{20}\sin\left(\frac{2}{1+x}\right)+\frac{x}{18}\int_0^x \frac{\arctan(x^2t)}{(1+xt)^2}\sin 2(u(t))dt. \end{aligned} \tag{15}$$

We take

$$\begin{aligned} |F(x,t,u_1,v_1,w_1) &- F(x,t,u_2,v_2,w_2)| = \left|\frac{x}{9}\frac{\arctan(x^2t)}{(1+xt)^2}(u_1+v_1+w_1)-\frac{x}{9}\frac{\arctan(x^2t)}{(1+xt)^2}(u_2+v_2+w_2)\right| \\ &= \left|\frac{1}{9}\frac{x\arctan(x^2t)}{(1+xt)^2}\,|\sin(u_1)-\sin(u_2)|+\frac{1}{9}\frac{x\arctan(x^2t)}{(1+xt)^2}\,|\cos(v_1)-\cos(v_2)|\right. \\ &\quad \left.+\frac{1}{18}\frac{x\arctan(x^2t)}{(1+xt)^2}\,|\sin(w_1-w_2)|\right| \\ &\le \left(\left|\frac{x\arctan(x^2t)}{(1+xt)^2}\right|\right)\frac{1}{18}(2|\sin(u_1)-\sin(u_2)| \\ &\quad +2|\cos(v_1)-\cos(v_2)|+|\sin(w_1-w_2)\cos(w_1) \\ &\le \left(\left|\frac{x\arctan(x^2t)}{(1+xt)^2}\right|\right)\frac{1}{18}(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|), \end{aligned}$$

Take $p(x,t)=\frac{\arctan(x^2t)}{(1+xt)^2}$ *and* $\phi(t)=\frac{t}{18}$. *Since* $\sup_{x\in[0,1]}\int_0^1 p^2(x,t)dt \le 1$, *then by applying the result obtained in Theorem 2.1 and 3.1, we deduce that the equation 15 has a stable unique solution in Banach space* $C[0,1]$.

Example 4.3. *Consider the following linear Volterra integral equation*

$$\begin{aligned} u(x) &= f(x)+\lambda\int_0^x (x-t)^{-\alpha}u(t)dt \\ v(x) &= g(x)+\lambda\int_0^x (x-t)^{-\alpha}v(t)dt \\ w(x) &= h(x)+\lambda\int_0^x (x-t)^{-\alpha}w(t)dt \end{aligned} \tag{16}$$

where $|\lambda|<1$ *and* $0<\alpha<\frac{1}{3}$. *Then*

$$\begin{aligned} |F(x,t,u_1,v_1,w_1) &- F(x,t,u_2,v_2,w_2)| = |(x-t)^{-\alpha}\lambda(u_1+v_1+w_1)-(x-t)^{-\alpha}\lambda(u_2+v_2+w_2)| \\ &= |(x-t)^{-\alpha}\lambda|u_1-u_2|+|(x-t)^{-\alpha}\lambda|v_1-v_2|+|(x-t)^{-\alpha}\lambda|w_1-w_2| \\ &\le |(x-t)|^{-\alpha}|\lambda(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|) \\ &\le \left(\left|\frac{x\arctan(x^2t)}{(1+xt)^2}\right|\right)\frac{1}{18}(2|\sin(u_1)-\sin(u_2)|+2|\cos(v_1)-\cos(v_2)| \\ &\quad +|\sin 2(u_1)-\sin 2(u_2)|) \\ &\le \left(\left|\frac{x\arctan(x^2t)}{(1+xt)^2}\right|\right)\frac{1}{18}(2|u_1-u_2|+2|v_1-v_2|+|w_1-w_2|), \end{aligned}$$

Take $p(x,t) = (x-t)^{-\alpha}$ *and* $\phi(t) = \lambda t$. *Since* $\sup_{x\in[0,1]} \int_0^1 p^2(x,t)dt \leq 1$, *then by applying the result obtained in Theorem 2.1 and 3.1, we deduce that the equation 15 has a stable unique solution in Banach space* $C[0,1]$.

5. Acknowledgements

The authors express their sincere gratitude to anonymous referee for insightful reading the manuscript and constructive comments that led to the improvement the quality of paper.

References

[1] R. P. Agrawal, D. O'Regan and P. J. Y. Wang, *Positive Solution of Differential. Difference and Integral Equation, Dordrecht*, Kluwer Academic, (1999).

[2] R. P. Agarwal and D. O'Regan, *Existence of Solutions to Singular Integral Equations*, Comput. Math. Appl., 37(1999), 25-29.

[3] J. Banas and B. Rzepka, *On existence and asymptotic stability of solutions of a nonlinear integral equation*, J. Math. Anal. Appl., 284(2003), 165-173.

[4] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, (2004).

[5] J. Brzdek, *On a method of proving the Hyers–Ulam stability of functional equations on restricted domains*, Aust. J. Math. Anal. Appl., 6(2009), 1-10.

[6] L. P. Castro and A. Ramos, *Stationary Hyers-Ulam-Rassias stability for a class of nonlinear Volterra integral equations*, Banach J. Math. Anal., 3(2009), 36-43.

[7] Deepmala and H. K. Pathak, *A study on some problems on existence of solution for nonlinear functional integral equations*, Acta Mathematica Scientia, 33B(5)(2013), 1305-1313.

[8] Deepmala, V. N. Mishra, H. R. Marasi, H. Shabanian and M. Nosraty, *Solutions of volterra fredholm integro differential equations using cheyshev collection method*, Global journal of Tech. optim., 210(8)(2017),

[9] G. B. Folland, *Real Analysis Modern Techniques and Their Application*, University of Washington, (1984).

[10] M. Gachpazan and O. Baghani, *Hyers-Ulam stability of Volterra integral equation*, Int. J. Nonlinear Anal. Appl., 1(2010), 19-25.

[11] M. Gachpazan and O. Baghani, *Hyers-Ulam stability of nonlinear integral equation*, Fixed Point Theory and Applications, 2010(2010).

[12] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA, 27(1941), 222-224.

[13] S. M. Jung, *A fixed point approach to the stability of a Volterra integral equation*, Fixed Point Theory and Applications, 2007(2007).

[14] K. Maleknejad, K. Nouri and R. Mollapourasl, *Existence of solutions for some nonlinear integral equations*, Commun. Nonlinear Sci. Numer. Simulat., 14(2009), 2559-2564.

[15] L. N. Mishra, R. P. Agrawal and M. Sen, *Solvability and asymptotics behaviour for some nonlinear quadratic integral equation involving Erdélyi-Kober fractional integrals on the unbounded integral*, Progress in Fractional Differentiation and applications, 3(2016), 153-168.

[16] L. N. Mishra and R. P. Agrawal, *On existence theorems for some nonlinear functional-integral equations*, Dynamic systems and Applications, 25(2016), 303-320.

[17] L. N. Mishra and M. Sen, *On the concept of existence and local attractivity solutions for some quadratic volterra integral equation of functional order*, Applied Mathematics and Computations, 285(2016), 174-183.

[18] M. Meehan and D. O'Regan, *Existence theory for nonlinear Volterra integro-differential and integral equations*, Nonlinear Anal. Theor., 31(1998), 317-341.

[19] Z. Moszner, *On the stability of functional equations*, Aequationes Math., 77(2009), 33-88.

[20] D. O'Regan and M. Meehan, *Existence Theory for Nonlinear Integral and Integro differential Equations*, Dordrecht, Kluwer Academic, (1998).

[21] B. Paneah, *A new approach to the stability of linear functional operators*, Aequationes Math., 78(2009), 45-61.

[22] W. Prager and J. Schwaiger, *Stability of the multi-Jensen equation*, Bull. Korean Math. Soc., 45(2008), 133-142.

[23] S. M. Ulam, *Problems in Modern Mathematics*, Chapter VI. Wiley, New York, (1960).

[24] G. Wang, M. Zhou and L. Sun, *Hyers–Ulam stability of linear differential equation of first order*, Appl. Math. Lett., 21(2008), 1024-1028.