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The Shilov Boundary and Peak Points of the Discrete Beurling Algebras on $\mathbb{Z}_+^2$ and $\mathbb{Z}^2$

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Abstract

Let ω be a weight function on $\mathbb{Z}_+^2$ or $\mathbb{Z}^2$. The Gel'fand spaces of the discrete Beurling algebras $l^1(\mathbb{Z}_+^2, \omega)$ and $l^1(\mathbb{Z}^2, \omega)$ are studied in [2]. In this paper, we study their Shilov boundary, peak points and strong boundary points.

Keywords: Gel'fand space; Gel'fand transform; Shilov boundary; Peak point; Strong boundary Point.

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1. Introduction

Throughout, let $\mathcal{A}$ be a unital, semisimple, commutative Banach algebra. Let $\Delta(\mathcal{A})$ denote the Gel'fand space of $\mathcal{A}$, let $\widehat{a} : \Delta(\mathcal{A}) \longrightarrow \mathbb{C}$; $\varphi \mapsto \varphi(a)$ be the Gel'fand transform of a, let $\Gamma_{\mathcal{A}} : \mathcal{A} \longrightarrow C_0(\Delta(\mathcal{A}))$; $a \mapsto \widehat{a}$ be the *Gel'fand representation* of $\mathcal{A}$, and $\widehat{A} = \Gamma_{\mathcal{A}}(\mathcal{A}) = \{\widehat{a} : a \in \mathcal{A}\}$. Then $\Delta(\mathcal{A})$ is a compact Hausdorff space equipped with the weak* topology, $\Gamma_{\mathcal{A}}$ is a norm decreasing, one-to-one, algebra homomorphism, and $\widehat{A}$ is a unital subalgebra of $C(\Delta(\mathcal{A}))$ which strongly separates the points of $\Delta(\mathcal{A})$, i.e. if $\varphi \neq \psi$, then there exists $a \in \mathcal{A}$ such that $\widehat{a}(\varphi) = 0$ and $f(\psi) = 1$ [3]. By [4], there always exists a smallest closed subset $\partial\mathcal{A}$ of $\Delta(\mathcal{A})$ such that $|\widehat{a}|_{\partial\mathcal{A}} = |\widehat{a}|_{\Delta(\mathcal{A})}$ $(a \in \mathcal{A})$, where $|\widehat{a}|_F = \sup\{|\varphi(a)| : \varphi \in F\}$; such a closed set $\partial\mathcal{A}$ is called the *Shilov boundary* of $\mathcal{A}$ [4]. A point $\varphi \in \Delta(\mathcal{A})$ is a *peak point* for $\mathcal{A}$ if there exists $a \in A$ such that $\widehat{a}(\varphi) = 1$ and $|\widehat{a}(\psi)| < 1$ for all $\varphi \neq \psi \in \Delta(\mathcal{A})$ [4]. Let $S_0(\mathcal{A})$ denote the set of all peak points for $\mathcal{A}$. A point $\varphi \in \Delta(\mathcal{A})$ is a *strong boundary point* for $\mathcal{A}$ if, for each neighbourhood U of φ, there exists $a \in A$ with $\widehat{a}(\varphi) = |\widehat{a}|_{\Delta(\mathcal{A})} = 1$ and $|\widehat{a}|_{\Delta(\mathcal{A})\setminus U} < 1$ [3]. Let $T_0(\mathcal{A})$ denote the set of all strong boundary points for $\mathcal{A}$. Then it is always true that $S_0(\mathcal{A}) \subset T_0(\mathcal{A}) \subset \partial\mathcal{A}$.

Let S be the semigroup $(\mathbb{Z}_+^2, +)$ or the group $(\mathbb{Z}^2, +)$. A *weight* ω on S is a strictly positive function $\omega : S \longrightarrow (0, \infty)$ satisfying $\omega(s + t) \leq \omega(s)\omega(t)$ for all $s, t \in S$. Let $l^1(S, \omega)$ be the set of all functions $f : S \longrightarrow \mathbb{C}$ such that $\sum\{|f(s)|\omega(s) : s \in S\} < \infty$. For $f, g \in l^1(S, \omega)$, the *convolution product* $f * g$ of f

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and g is defined as

$$f * g(s) = \sum\{f(u)g(v) : u, v \in S \text{ and } u + v = s\} \qquad (s \in S).$$

Then $l^1(S, \omega)$ is a unital, commutative Banach algebra with pointwise linear operations, the convolution product, and the norm $\|f\|_\omega = \sum\{|f(s)|\omega(s) : s \in S\}$. The reader should refer to [3] for the definition and some basic properties.

The Gel'fand theory of the Beurling algebras $l^1(\mathbb{Z}_+, \omega)$ and $l^1(\mathbb{Z}, \omega)$ are very well understood. For example: (i) their Gel'fand space can be identified with some closed disc $\mathbb{D}_r$ with center zero and radius r and the closed annulus $\Gamma(r_1, r_2)$ with center zero and radii $0 < r_1 \leq r_2$, respectively; (ii) their Shilov boundaries are just the topological boundaries of $\mathbb{D}_r$ and $\Gamma(r_1, r_2)$, respectively. So it is natural to study the Gel'fand theory and various boundaries of the algebras $l^1(\mathbb{Z}_+^2, \omega)$ and $l^1(\mathbb{Z}^2, \omega)$. Their Gel'fand spaces are studied in [2]. In this paper, we study their geometrical properties; more specifically, we study Shilov boundary, peak points, strong boundary points, and polynomial convexity. It is already proved that $\Delta(l^1(\mathbb{Z}_+^2, \omega)) \cong \mathbb{G}(\omega)$ and $\Delta(l^1(\mathbb{Z}^2, \omega)) \cong \mathbb{T}(\omega)$, where $\mathbb{G}(\omega)$ and $\mathbb{T}(\omega)$ are union of product of closed discs and closed annuli in $\mathbb{C}^2$, respectively [2]. Here, we shall prove that $\mathbb{G}(\omega)$ is always polynomially convex, while $\mathbb{T}(\omega)$ is never.

It follows from [1] that, unlike $l^1(\mathbb{Z}_+^2, \omega)$, the algebra $l^1(\mathbb{Z}^2, \omega)$ is always semisimple. Under some condition on the weight ω, the algebra $l^1(\mathbb{Z}_+^2, \omega)$ is semisimple too.

2. Main Section

To make this paper self instructed, we state here some notations, definitions and results from [2] without proofs.

Notations: Let ω be a weight on $\mathbb{Z}_+^2$. Then we set the following notations.

$$\begin{aligned}
\mathbb{D}_r &= \{z \in \mathbb{C} : |z| \leq r\};\ \Gamma(r) = \{z \in \mathbb{C} : |z| = r\};\\
\Gamma(r, s) &= \{z \in \mathbb{C} : r \leq |z| \leq s\};\\
\mu &= \inf\{\omega(m, n)^{\frac{1}{m+n}} : (m, n) \in \mathbb{Z}_+^2 \setminus \{(0, 0)\}\};\\
\beta_{1,0} &= \inf\{\omega(m, 0)^{\frac{1}{m}} : (m \in \mathbb{N})\};\\
\beta_{2,0} &= \inf\{\omega(0, n)^{\frac{1}{n}} : (n \in \mathbb{N})\};\\
\mathbb{G}(\omega) &= \cup_{i \in \Lambda}(\mathbb{D}_{r_i} \times \mathbb{D}_{s_i}),\ \text{where } r_i, s_i \in \mathbb{R}_+;\\
\mathbb{G}_{db}(\omega) &= \cup_{i \in \Lambda}(\Gamma(r_i) \times \Gamma(s_i));\\
\xi &= \sup\{|zw| : (z, w) \in \mathbb{G}_{db}(\omega)\};\\
\mathbb{G}_p(\omega) &= \{(z, w) \in \mathbb{G}(\omega) : |z||w| = \xi\};
\end{aligned}$$

Let ω be a weight on $\mathbb{Z}^2$. Then we set the following notations.

$$\begin{aligned}
\mathbb{T}(\omega) &= \cup_{r\in\Lambda}(\Gamma(r)\times\Gamma(s_r,t_r)), \text{ where } r,s_r,t_r\in\mathbb{R}_+;\\
\mathbb{T}_{ab}(\omega) &= [\cup_{r\in\Lambda}(\Gamma(r)\times\Gamma(s_r))]\cup[\cup_{r\in\Lambda}(\Gamma(r)\times\Gamma(t_r))];\\
\xi_1 &= \sup\{|zw| : (z,w)\in\cup_{r\in\Lambda}(\Gamma(r)\times\Gamma(t_r))\};\\
\xi_2 &= \inf\{|zw| : (z,w)\in\cup_{r\in\Lambda}(\Gamma(r)\times\Gamma(s_r))\};\\
\xi_3 &= \sup\{|z^{-1}w| : (z,w)\in\cup_{r\in\Lambda}(\Gamma(r)\times\Gamma(t_r))\};\\
\xi_4 &= \inf\{|z^{-1}w| : (z,w)\in\cup_{i\in\Lambda}(\Gamma(r)\times\Gamma(s_r))\};\\
\mathbb{T}_{p_i}(\omega) &= \{(z,w)\in\mathbb{T}(\omega) : |z||w| = \xi_i\} \text{ for } i=1 \text{ or } 2;\\
\mathbb{T}_{p_i}(\omega) &= \{(z,w)\in\mathbb{T}(\omega) : |z^{-1}w| = \xi_i\} \text{ for } i=3 \text{ or } 4;
\end{aligned}$$

Definition 2.1. *Let ω_1 and ω_2 be weights on $\mathbb{Z}_+$ or $\mathbb{Z}$. Then ω is called a* product weight *on $\mathbb{Z}_+^2$ or $\mathbb{Z}^2$ if $\omega(m,n)=\omega_1(m)\omega_2(n)$ $(m,n\in\mathbb{Z}_+$ or $\mathbb{Z})$.*

The next result is proved in [2, Theorem 3.3].

Theorem 2.2. *Let ω be a weight on $\mathbb{Z}_+^2$. Then there is a subset $\{(r_i,s_i) : i\in\Lambda\}$ of $\mathbb{R}_+^2$ such that*

$$\Delta(l^1(\mathbb{Z}_+^2,\omega))\cong\mathbb{G}(\omega)=\cup_{i\in\Lambda}(\mathbb{D}_{r_i}\times\mathbb{D}_{s_i})$$

in such a way that, for each $\varphi\in\Delta(l^1(\mathbb{Z}_+^2,\omega))$ there exists $(z,w)\in\mathbb{G}(\omega)$ such that $\widehat{f}(\varphi)=\widehat{f}(\phi_{z,w})=\widehat{f}(z,w)=\sum\{f(m,n)z^m w^n : m,n\in\mathbb{Z}_+\}$ for all $f\in l^1(\mathbb{Z}_+^2,\omega)$. In particular, if $m\in\mathbb{Z}_+$ and if $p=\sum\{\alpha_{kl}\delta_{e_1}^k\delta_{e_2}^l : k+l\le m\}\in l^1(\mathbb{Z}_+^2,\omega)$, then $\widehat{p}(z,w)=\sum\{\alpha_{kl}z^k w^l : k+l\le m\}$.

Next Lemma will be required at later stage.

Lemma 2.3. *Let ω be a weight on $\mathbb{Z}_+^2$. Then the Beurling algebra $l^1(\mathbb{Z}_+^2,\omega)$ is semisimple iff $\mu>0$.*

Proof. Let ω be semisimple. Suppose, if possible, that $\mu=0$. Let $0<\epsilon<1$. Since $\mu=0$, there exists $(m,n)\in\mathbb{Z}_+^2\setminus\{(0,0)\}$ such that $\omega(m,n)^{\frac{1}{m+n}}<\epsilon$. This implies that, for each $k\in\mathbb{N}$,

$$\omega(km,kn)^{\frac{1}{k}}\le\omega(m,n)<\epsilon^{m+n}<\epsilon.$$

Therefore, $\inf\{\omega(km,kn)^{\frac{1}{k}} : k\in\mathbb{N}\}=0$, which is a contradiction. Hence $\mu>0$.
Conversely, let $\mu>0$. Then, for any $(m,n)\in\mathbb{Z}_+^2\setminus\{(0,0)\}$, it is clear that $\inf\{\omega(km,kn)^{\frac{1}{k}} : k\in\mathbb{N}\}\ge\mu^{m+n}>0$. Hence ω is semisimple. □

Definition 2.4 ([4]). *Let $n\in\mathbb{N}$. A compact subset K of $\mathbb{C}^n$, is said to be* polynomially convex *if, for every $z\in\mathbb{C}^n\setminus K$, there exists a polynomial p in n-variables such that $p(z)=1$ and $|p(w)|<1$ for all $w\in K$.*

Lemma 2.5. *The Gel'fand set $\mathbb{G}(\omega)$ is polynomially convex.*

Proof. Let $\delta_e=\delta_{(0,0)}$ denote the identity of $l^1(\mathbb{Z}_+^2,\omega)$ and let $(z_1,w_1)\notin\mathbb{G}(\omega)$. Then, for any $\varphi\in\Delta(l^1(\mathbb{Z}_+^2,\omega))$, either $\varphi(\delta_{e_1})\neq z_1$ or $\varphi(\delta_{e_2})\neq w_1$. Consider the ideal $I=\{(\delta_{e_1}-z_1\delta_e)*g_1+(\delta_{e_2}-$

$w_1\delta_e) * g_2 : g_1, g_2 \in l^1(\mathbb{Z}_+^2,\omega)\}$ of $l^1(\mathbb{Z}_+^2,\omega)$. Then we must have $I = l^1(\mathbb{Z}_+^2,\omega)$. If possible, suppose that, $I \neq l^1(\mathbb{Z}_+^2,\omega)$. Then there exists $\varphi \in \Delta(l^1(\mathbb{Z}_+^2,\omega))$ such that $I \subset ker\varphi \subset l^1(\mathbb{Z}_+^2,\omega)$. Since $\delta_{e_1} - z_1\delta_e, \delta_{e_2} - w_1\delta_e \in I$, so $\varphi(\delta_{e_1}) - z_1 = \varphi(\delta_{e_1} - z_1\delta_e) = 0 = \varphi(\delta_{e_2} - w_1\delta_e) = \varphi(\delta_{e_2}) - w_1$, which is a contradiction. Thus $I = l^1(\mathbb{Z}_+^2,\omega)$, and hence there exist $g_1, g_2 \in l^1(\mathbb{Z}_+^2,\omega)$ such that

$$\delta_e = (\delta_{e_1} - z_1\delta_e) * g_1 + (\delta_{e_2} - w_1\delta_e) * g_2.$$

Choose $\delta > 0$ such that $\delta(\|\delta_{e_1} - z_1\delta_e\|_\omega + \|\delta_{e_2} - w_1\delta_e)\|_\omega) < 1$. Since $l^1(\mathbb{Z}_+^2,\omega)$ is generated by δ_{e_1} and δ_{e_2}, there exist $m_0, n_0 \in \mathbb{Z}_+$ such that $p_1 = \sum_{k+l\leq m_0} \alpha_{kl}\delta_{e_1}^k\delta_{e_2}^l$, $p_2 = \sum_{k+l\leq n_0} \beta_{kl}\delta_{e_1}^k\delta_{e_2}^l$, and $\|p_j - g_j\| \leq \delta$ for $j = 1, 2$. By substituting the values of δ_e above, it follows that

$$\begin{aligned}\|\delta_e - ((\delta_{e_1} - z_1\delta_e) * p_1 + (\delta_{e_2} - w_1\delta_e) * p_2)\|_\omega &\leq \|\delta_{e_1} - z_1\delta_e\|_\omega\|g_1 - p_1\|_\omega + \|\delta_{e_2} - w_1\delta_e)\|_\omega\|g_2 - p_2\|_\omega \\ &< 1.\end{aligned}$$

Let $p = \delta_e - (\delta_{e_1} - z_1\delta_e) * p_1 - (\delta_{e_2} - w_1\delta_e) * p_2 \in l^1(\mathbb{Z}_+^2,\omega)$. Then $\widehat{p}(z,w) = 1 - (z - z_1)\widehat{p_1}(z,w) - (w - w_1)\widehat{p_2}(z,w)$ $((z,w) \in \mathbb{G}(\omega))$ and $\widehat{p}$ is a polynomial in two variables z and w. Clearly $\widehat{p}(z_1,w_1) = 1$. Let $(z,w) \in \mathbb{G}(\omega)$ be arbitrary. Then, by Theorem 2.2,

$$\begin{aligned}|\widehat{p}(z,w)| &= |\widehat{p}(\varphi(\delta_{e_1}), \varphi(\delta_{e_2}))| \\ &= |1 - (\varphi(\delta_{e_1}) - z_1)\widehat{p_1}(\varphi(\delta_{e_1}), \varphi(\delta_{e_2})) - (\varphi(\delta_{e_2}) - w_1)\widehat{p_2}(\varphi(\delta_{e_1}), \varphi(\delta_{e_2}))| \\ &= |\varphi(\delta_e) - \varphi(\delta_{e_1} - z_1\delta_e)\varphi(p_1) - \varphi(\delta_{e_2} - w_1\delta_e)\varphi(p_2)| \\ &\leq \|\delta_e - (\delta_{e_1} - z_1\delta_e) * p_1 - (\delta_{e_2} - w_1\delta_e) * p_2\|_\omega \\ &< 1.\end{aligned}$$

This proves that $\mathbb{G}(\omega)$ is polynomially convex. □

Lemma 2.6 **([3]).** *Let $\mathcal{A}$ be a semisimple commutative Banach algebra. Then $S_0(\mathcal{A}) \subset T_0(\mathcal{A}) \subset \partial\mathcal{A}$. If $\Delta(\mathcal{A})$ is metrizable, then $\partial(\mathcal{A}) = T_0(\mathcal{A}) = S_0(\mathcal{A})$.*

Lemma 2.7. *Let $(z_0,w_0) \in \mathbb{G}(\omega)$ and let $S = S_0(l^1(\mathbb{Z}_+^2,\omega))$ or $T_0(l^1(\mathbb{Z}_+^2,\omega))$ or $\partial l^1(\mathbb{Z}_+^2,\omega)$. If $(z_0,w_0) \in S$, then $\Gamma(|z_0|) \times \Gamma(|w_0|) \subset S$.*

Proof. Since $\Delta(l^1(\mathbb{Z}_+^2,\omega)) \cong \mathbb{G}(\omega)$ is metrizable, it is sufficient to prove only for $S_0(l^1(\mathbb{Z}_+^2,\omega))$ due to Lemma 2.6 above. Assume that $(z_0,w_0) \in S_0(l^1(\mathbb{Z}_+^2,\omega))$. Then there exists $f \in l^1(\mathbb{Z}_+^2,\omega)$ such that $\widehat{f}(z_0,w_0) = 1$ and $|\widehat{f}(z,w)| < 1$ for all $(z,w) \neq (z_0,w_0)$. Let $(z_1,w_1) \in \Gamma(|z_0|) \times \Gamma(|w_0|)$ be arbitrary and let $\theta_1, \theta_2 \in \mathbb{R}$ such that $z_0 = e^{i\theta_1}z_1$ and $w_0 = e^{i\theta_2}w_1$. Take $g(m,n) = e^{i(m\theta_1+n\theta_2)}f(m,n)$. Then $g \in l^1(\mathbb{Z}_+^2,\omega)$ and $\widehat{g}(z,w) = \widehat{f}(e^{i\theta_1}z, e^{i\theta_2}w)$. Hence $\widehat{g}(z_1,w_1) = \widehat{f}(e^{i\theta_1}z_1, e^{i\theta_2}w_1) = \widehat{f}(z_0,w_0) = 1$ and $|\widehat{g}(z,w)| = |\widehat{f}(e^{i\theta_1}z, e^{i\theta_2}w)| < 1$ for all $(z,w) \neq (z_1,w_1)$. Thus $(z_1,w_1) \in S_0(l^1(\mathbb{Z}_+^2,\omega))$. □

Theorem 2.8. *Let ω be a weight on $\mathbb{Z}_+^2$. Then $\partial l^1(\mathbb{Z}_+^2,\omega) \subset \mathbb{G}_{db}(\omega)$.*

Proof. Let $f \in l^1(\mathbb{Z}_+^2, \omega)$. Since $\widehat{f}$ is continuous and $\mathbb{G}(\omega)$ is compact, there exists $(z_0, w_0) \in \mathbb{G}(\omega)$ such that $|\widehat{f}|_{\Delta(l^1(\mathbb{Z}_+^2,\omega))} = |\widehat{f}(z_0, w_0)|$. Since $\mathbb{G}(\omega) = \cup_{i\in\Lambda}\mathbb{D}_{r_i} \times \mathbb{D}_{s_i}$, there exists $i \in \Lambda$ such that $(z_0, w_0) \in \mathbb{D}_{r_i} \times \mathbb{D}_{s_i}$. Define $g_1 : \mathbb{D}_{r_i} \longrightarrow \mathbb{C}$ as $g_1(z) = \widehat{f}(z, w_0)$ $(z \in \mathbb{D}_{r_i})$. Since $\widehat{f}$ is continuous on $\mathbb{D}_{r_i} \times \mathbb{D}_{s_i}$ and analytic on $int(\mathbb{D}_{r_i} \times \mathbb{D}_{s_i})$, g_1 is continuous on $\mathbb{D}_{r_i}$ and analytic on $int(\mathbb{D}_{r_i})$. So by the maximum modulus principle, there exists $z_1 \in \Gamma(r_i)$ such that $|\widehat{f}(z, w_0)| = |g_1(z)| \leq |g_1(z_1)| = |\widehat{f}(z_1, w_0)|$. Next define $g_2 : \mathbb{D}_{s_i} \longrightarrow \mathbb{C}$ as $g_2(w) = \widehat{f}(z_1, w)$ $(w \in \mathbb{D}_{s_i})$. As above argument, there exists $w_1 \in \Gamma(s_i)$ such that $|\widehat{f}(z_1, w)| = |g_2(w)| \leq |g_2(w_1)| = |\widehat{f}(z_1, w_1)|$ $(w \in \mathbb{D}_{s_i})$. Hence $|\widehat{f}|_{\mathbb{G}(\omega)} = |\widehat{f}(z_0, w_0)| \leq |\widehat{f}(z_1, w_0)| \leq |\widehat{f}(z_1, w_1)| \leq |\widehat{f}|_{\Gamma(r_i)\times\Gamma(s_i)}$. Thus $\cup_{i\in\Lambda}(\Gamma(r_i) \times \Gamma(s_i))$ is a boundary of $l^1(\mathbb{Z}_+^2, \omega)$. Hence we get $\partial l^1(\mathbb{Z}_+^2, \omega) \subset \mathbb{G}_{db}(\omega)$. □

We believe that the two sets $\partial l^1(\mathbb{Z}_+^2, \omega)$ and $\mathbb{G}_{db}(\omega)$ should be identical. Unfortunately, we could not prove it. However, if the weight ω is a product weight, then it is true; this is proved in the next result.

Theorem 2.9. *Let ω be a weight on $\mathbb{Z}_+^2$.*

(I) If $\mu = 0$, then $\mathbb{G}(\omega) = (\mathbb{D}_{\beta_{1,0}} \times \{0\}) \cup (\{0\} \times \mathbb{D}_{\beta_{2,0}})$ and following holds.

(a) If $\beta_{1,0} = \beta_{2,0} = 0$, then $\partial l^1(\mathbb{Z}_+^2, \omega) = \{(0, 0)\}$;

(b) If $\beta_{1,0} = 0$ and $\beta_{2,0} > 0$, then $\partial l^1(\mathbb{Z}_+^2, \omega) = \{0\} \times \Gamma(\beta_{2,0})$;

(c) If $\beta_{1,0} > 0$ and $\beta_{2,0} = 0$, then $\partial l^1(\mathbb{Z}_+^2, \omega) = \Gamma(\beta_{1,0}) \times \{0\}$;

(d) If $\beta_{1,0}, \beta_{2,0} > 0$, then $\partial l^1(\mathbb{Z}_+^2, \omega) = (\Gamma(\beta_{1,0}) \times \{0\}) \cup (\{0\} \times \Gamma(\beta_{2,0}))$.

(II) If $\mu > 0$, then $\mathbb{G}_p(\omega) \subset \partial l^1(\mathbb{Z}_+^2, \omega)$;

(III) If ω is a product weight, then $\partial l^1(\mathbb{Z}_+^2, \omega) = \mathbb{G}_{db}(\omega)$.

Proof. (I) If $\mathbb{D}_r \times \mathbb{D}_s \subset \mathbb{G}(\omega)$ for some $r, s > 0$, then $\mu \geq \min\{r, s\} > 0$. So, by [2], $\mathbb{G}(\omega) = (\mathbb{D}_{\beta_{1,0}} \times \{0\}) \cup (\{0\} \times \mathbb{D}_{\beta_{2,0}})$. If $\beta_{1,0} = \beta_{2,0} = 0$, then $\mathbb{G}(\omega) = \{(0, 0)\}$. Clearly $\partial l^1(\mathbb{Z}_+^2, \omega) = \{(0, 0)\}$. If $\beta_{1,0} = 0$ and $\beta_{2,0} > 0$, then $\mathbb{G}(\omega) = \{0\} \times \mathbb{D}_{\beta_{2,0}}$. Take $f = \frac{1}{2}(\delta_{(0,0)} + \beta_{2,0}^{-1}\delta_{(0,1)}) \in l^1(\mathbb{Z}_+^2, \omega)$. Then the Gel'fand transform $\widehat{f}$ of f will be $\widehat{f}(0, w) = \frac{1}{2}(1 + \beta_{2,0}^{-1}w)$ such that $\widehat{f}(0, \beta_{2,0}) = 1$ and $|\widehat{f}(0, w)| < 1$ for all $(0, w) \neq (0, \beta_{2,0})$. Thus $(0, \beta_{2,0})$ is a peak point and hence by Lemma 2.7, $\{0\} \times \Gamma(\beta_{2,0}) \subset \partial l^1(\mathbb{Z}_+^2, \omega)$. But in this case, $\mathbb{G}_{db}(\omega) = \{0\} \times \Gamma(\beta_{2,0})$. So by Theorem 2.9, $\partial l^1(\mathbb{Z}_+^2, \omega) = \{0\} \times \Gamma(\beta_{2,0})$. This proves (b). The proof of c follows by the similar arguments given in (b) above. Now assume that $\beta_{1,0} > 0$ and $\beta_{2,0} > 0$. Let $f = \frac{1}{2}(\delta_{(0,0)} + \beta_{1,0}^{-1}\delta_{(1,0)} + \frac{\beta_{2,0}^{-1}}{2}\delta_{(0,1)})$. Then the Gel'fand transform $\widehat{f}$ of f is given by $\widehat{f}(z, w) = \frac{1}{2}(1 + \beta_{1,0}^{-1}z + \frac{\beta_{2,0}^{-1}}{2}w)$. Let $(z, w) \in \mathbb{G}(\omega)$ be such that $(z, w) \neq (\beta_{1,0}, 0)$. Then either $z = 0$ or $w = 0$. Hence $\widehat{f}(\beta_{1,0}, 0) = 1$ and $|\widehat{f}(z, w)| < 1$ for all $(z, w) \neq (\beta_{1,0}, 0)$. Thus $(\beta_{1,0}, 0) \in \partial l^1(\mathbb{Z}_+^2, \omega)$. So we have $\Gamma(\beta_{1,0}) \times \{0\} \subset \partial l^1(\mathbb{Z}_+^2, \omega)$. Similarly, we can show that $\{0\} \times \Gamma(\beta_{2,0}) \subset \partial l^1(\mathbb{Z}_+^2, \omega)$. This proves (d).

(II) Assume that $\mu > 0$. Since $\mathbb{G}_{db}(\omega)$ is compact, the set $\mathbb{G}_p(\omega) = \{(z, w) : |zw| = \xi\}$ is non-empty.

Let $(z_0, w_0) \in \mathbb{G}_p(\omega)$. Consider, $f = \frac{1}{2}(\delta_{(0,0)} + z_0^{-1}w_0^{-1}\delta_{(1,1)}) \in l^1(\mathbb{Z}_+^2, \omega)$. The Gel'fand transform of f is given by

$$\widehat{f}(z,w) = \frac{1}{2}(1 + z_0^{-1}w_0^{-1}zw).$$

Then $\widehat{f}(z_0, w_0) = 1$ and $|\widehat{f}(z,w)| < 1$ for all $(z,w) \neq (z_0, w_0)$. Hence (z_0, w_0) is a peak point. Therefore $\mathbb{G}_p(\omega) \subset S_0(l^1(\mathbb{Z}_+^2, \omega))$.

(III) If ω is a product weight, then by [2, Theorem 3.7(i)], we have $\mathbb{G}(\omega) = \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$. If $\beta_{i,0} = 0$ for $i = 1$ or 2. Then, by the similar arguments given in (II) above, the result follows. Now let $\beta_{i,0} \neq 0$ $(i = 1, 2)$ and let $(z_0, w_0) \in \Gamma(\beta_{1,0}) \times \Gamma(\beta_{2,0})$. Consider $f = \frac{1}{4}(\delta_{(0,0)} + z_0^{-1}\delta_{e_1} + w_0^{-1}\delta_{e_2} + z_0^{-1}w_0^{-1}\delta_{(1,1)})$. Then the Gel'fand transform $\widehat{f}$ of f is given by

$$\widehat{f}(\varphi) = \widehat{f}(\phi_{z,w}) = \widehat{f}(z,w) = \frac{1}{4}(1 + z_0^{-1}z)(1 + w_0^{-1}w) \ ((z,w) \in \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}).$$

Then $\widehat{f}(z_0, w_0) = 1$ and $|\widehat{f}(z,w)| < 1$ $(z,w) \neq (z_0, w_0)$. Hence $\Gamma(\beta_{1,0}) \times \Gamma(\beta_{2,0}) \subset \partial l^1(\mathbb{Z}_+^2, \omega)$. Note that, for any $f \in l^1(\mathbb{Z}_+^2, \omega)$, the Gel'fand transform $\widehat{f}$ is continuous on $\mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$ and analytic on the interior of $\mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$. Hence, by the maximum modulus principle, $\Gamma(\beta_{1,0}) \times \Gamma(\beta_{2,0}) \subset \partial l^1(\mathbb{Z}_+^2, \omega)$. Clearly, $\mathbb{G}_{db}(\omega) = \Gamma(\beta_{1,0}) \times \Gamma(\beta_{2,0})$. Hence, by Theorem 2.9, $\partial l^1(\mathbb{Z}_+^2, \omega) = \Gamma(\beta_{1,0}) \times \Gamma(\beta_{2,0})$. □

Theorem 2.10 ([2]). *Let ω be any weight on $\mathbb{Z}^2$. Then there exists a subset Λ of $\mathbb{R}_+^\bullet$ and a subset $\{(s_r, t_r) : r \in \Lambda\}$ of $\mathbb{R}_+^{\bullet 2}$ such that*

$$\Delta(l^1(\mathbb{Z}^2, \omega)) \cong \cup_{r \in \Lambda}(\Gamma(r) \times \Gamma(s_r, t_r)).$$

The following proof can be compared with the proof of Lemma 2.5.

Lemma 2.11. *Let ω be a weight on $\mathbb{Z}^2$. Then*

(i) *$\mathbb{T}(\omega)$ is never polynomially convex;*

(ii) *Let $z_1, w_1 \neq 0$ be such that $(z_1, w_1) \notin \mathbb{T}(\omega)$. Then there exists $f \in l^1(\mathbb{Z}^2, \omega)$ such that $\widehat{f}(z_1, w_1) = 1$ and $|\widehat{f}(z,w)| < 1$ for all $(z,w) \in \mathbb{T}(\omega)$;*

Proof. (i) Assume that $\mathbb{T}(\omega)$ is polynomially convex. Since for each $r \in \Lambda$ the point $(0, t_r) \notin \mathbb{T}(\omega)$, we get a polynomial p_r such that $p_r(0, t_r) = 1$ and $|p_r(z,w)| < 1$ for all $(z,w) \in \mathbb{T}(\omega)$. Fix some $r \in \Lambda$ and define a polynomial $q : \mathbb{C} \longrightarrow \mathbb{C}$ by $q(z) = p_r(z, t_r)$. Then $|q(z)| < 1$ for all $z \in \mathbb{C}$ with $|z| = r$ and $q(0) = p_r(0, t_r) = 1$, which is a contradiction to the maximum modulus principle.

(ii) Let $(z_1, w_1) \notin \mathbb{T}(\omega)$, $\varphi \in \Delta(l^1(\mathbb{Z}^2, \omega))$, and let $ker(\varphi) = M$. Then $\varphi(\delta_{e_1}) \neq z_1$ or $\varphi(\delta_{e_2}) \neq w_1$. Take a set $I = \{(\delta_{e_1} - z_1\delta_e) * g_1 + (\delta_{e_2} - w_1\delta_e) * g_2 : g_1, g_2 \in l^1(\mathbb{Z}^2, \omega)\}$ of $l^1(\mathbb{Z}^2, \omega)$. Then clearly, I is an ideal. Also the elements $\delta_{e_1} - z_1\delta_e$ and $\delta_{e_2} - w_1\delta_e \in I$ but atleast one of these two elements does not belong to $ker(\varphi)$ for each $\varphi \in \Delta(l^1(\mathbb{Z}^2, \omega))$. This implies that $I \subset ker\varphi$ for any $\varphi \in \Delta(l^1(\mathbb{Z}^2, \omega))$ is not possible. Thus $I = l^1(\mathbb{Z}^2, \omega)$. Hence there exists $g_1, g_2 \in l^1(\mathbb{Z}^2, \omega))$ such that

$$\delta_e = (\delta_{e_1} - z_1\delta_e) * g_1 + (\delta_{e_2} - w_1\delta_e) * g_2.$$

Choose $\delta > 0$ such that $\delta(\|\delta_{e_1} - z_1\delta_e\|_\omega + \|\delta_{e_2} - w_1\delta_e)\|_\omega) < 1$. Since $l^1(\mathbb{Z}^2, \omega))$ is generated by $\delta_{e_1}, \delta_{-e_1}, \delta_{e_2}$, and δ_{-e_2}, there exists polynomials p_1, p_2 in 4-variables such that $\|p_j(\delta_{e_1}, \delta_{-e_1}, \delta_{e_2}, \delta_{e_2}) - g_j\|_\omega \leq \delta$ for $j = 1, 2$. It follows that

$$\begin{aligned}\|\delta_e - (\delta_{e_1} - z_1\delta_e) * p_1(\delta_{e_1}, \delta_{-e_1}, \delta_{e_2}, \delta_{-e_2}) - (\delta_{e_2} - w_1\delta_e) * p_2(\delta_{e_1}, \delta_{-e_1}, \delta_{e_2}, \delta_{-e_2})\|_\omega \\ \leq \|\delta_{e_1} - z_1\delta_e\|_\omega \|g_1 - p_1(\delta_{e_1}, \delta_{-e_1}, \delta_{e_2}, \delta_{-e_2})\|_\omega \\ + \|\delta_{e_2} - w_1\delta_e)\|_\omega \|g_2 - p_2(\delta_{e_1}, \delta_{-e_1}, \delta_{e_2}, \delta_{-e_2})\|_\omega \\ < 1.\end{aligned}$$

Now, define a map $f : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ as follows

$$f = \delta_e - (\delta_{e_1} - z_1\delta_e) * p_1(\delta_{e_1}, \delta_{-e_1}, \delta_{e_2}, \delta_{-e_2}) - (\delta_{e_2} - w_1\delta_e) * p_2(\delta_{e_1}, \delta_{-e_1}, \delta_{e_2}, \delta_{-e_2}).$$

Then $f \in l^1(\mathbb{Z}^2, \omega)$ and the Gel'fand transform of f is given by

$$\widehat{f}(z, w) = 1 - (z - z_1)p_1(z, z^{-1}, w, w^{-1}) - (w - w_1)p_2(z, z^{-1}, w, w^{-1}).$$

Then $\widehat{f}(z_1, w_1) = 1$, and by the similar arguments given in Lemma 2.5, we have $|\widehat{f}(z, w)| < 1$ for every $(z, w) \in \mathbb{T}(\omega)$. □

Theorem 2.12. *Let ω be a weight on $\mathbb{Z}^2$. Then $\partial l^1(\mathbb{Z}^2, \omega) \subset \mathbb{T}_{ab}(\omega)$.*

Proof. It is enough to show that the set $\mathbb{T}_{ab}(\omega)$ is boundary for $l^1(\mathbb{Z}^2, \omega)$. Let $f \in l^1(\mathbb{Z}^2, \omega)$ and let $(z_0, w_0) \in \mathbb{T}(\omega)$ such that $|\widehat{f}|_{\mathbb{T}(\omega)} = |\widehat{f}(z_0, w_0)|$. Since $\mathbb{T}(\omega) = \cup_{r\in\Lambda}(\Gamma(r) \times \Gamma(s_r, t_r))$, $(z_0, w_0) \in \Gamma(r) \times \Gamma(s_r, t_r)$ for some $r \in \Lambda$. Define $g : \Gamma(s_r, t_r) \longrightarrow \mathbb{C}$ as $g(w) = \widehat{f}(z_0, w)$. Then g is continuous on $\Gamma(s_r, t_r)$ and analytic on its interior. So there exists $w_1 \in \Gamma(s_r) \cup \Gamma(t_r)$ such that $|\widehat{f}(z_0, w)| = |g(w)| \leq |g(w_1)| \leq |\widehat{f}(z_0, w_1)|$ for all $w \in \Gamma(s_r, t_r)$. This implies $|\widehat{f}|_{\mathbb{T}(\omega)} \leq |\widehat{f}(z_0, w_1)|$. Note that $(z_0, w_1) \in (\Gamma(|z_0|) \times \Gamma(s_r)) \cup \Gamma(|z_0|) \times \Gamma(t_r)$. Thus $\mathbb{T}_{ab}(\omega)$ is a boundary for $l^1(\mathbb{Z}^2, \omega)$. Hence $\partial l^1(\mathbb{Z}^2, \omega) \subset \mathbb{T}_{ab}(\omega)$. □

Theorem 2.13. *Let ω be a weight on $\mathbb{Z}^2$. Then*

(i) $\cup_{i=1}^4 \mathbb{T}_{p_i}(\omega) \subset \partial l^1(\mathbb{Z}^2, \omega))$;

(ii) If ω be a product weight, then $\cup_{i=1}^4 \mathbb{T}_{p_i}(\omega) = \partial l^1(\mathbb{Z}^2, \omega) = \mathbb{T}_{ab}(\omega)$.

Proof. (i) Since $\mathbb{T}_{ab}(\omega)$ is compact, each set $\mathbb{T}_{p_i}$ is non-empty. Let $(z_0, w_0) \in \mathbb{T}_{p_1}(\omega)$. Then $|z_0 w_0| = \xi_1$. Take $f = \frac{1}{2}(\delta_{(0,0)} + z_0^{-1}w_0^{-1}\delta_{(1,1)}) \in l^1(\mathbb{Z}^2, \omega)$. Then the Gel'fand transform $\widehat{f}$ of f is given by

$$\widehat{f}(z, w) = \frac{1}{2}(1 + z_0^{-1}w_0^{-1}zw) \ \ ((z, w) \in \mathbb{T}(\omega)).$$

Clearly, $\widehat{f}(z_0, w_0) = 1$ and $|\widehat{f}(z, w)| < 1$ for all $(z, w) \neq (z_0, w_0)$. So (z_0, w_0) is a peak point, and hence $(z_0, w_0) \in \partial l^1(\mathbb{Z}^2, \omega)$ due to Lemma 2.6.

Similarly, if $(z_0, w_0) \in \mathbb{T}_{p_2}$, then $|z_0 w_0| = \xi_2$. Take $f = \frac{1}{2}(\delta_{(0,0)} + z_0 w_0 \delta_{(-1,-1)})$. Then $\widehat{f}(z_0, w_0) = 1$ and $|\widehat{f}(z, w)| < 1$ for all $(z, w) \neq (z_0, w_0)$. Thus $\mathbb{T}_{p_2} \subset S_0(l^1(\mathbb{Z}^2, \omega)) = \partial l^1(\mathbb{Z}^2, \omega)$. If $(z_0, w_0) \in \mathbb{T}_{p_3}$, then take $f = \frac{1}{2}(\delta_{(0,0)} + z_0^{-1} w_0 \delta_{(1,-1)})$; and if $(z_0, w_0) \in \mathbb{T}_{p_4}$, then take $f = \frac{1}{2}(\delta_{(0,0)} + z_0 w_0^{-1} \delta_{(-1,1)})$. Then, as per the above arguments, we can show that $\mathbb{T}_{p_3}, \mathbb{T}_{p_4} \subset \partial l^1(\mathbb{Z}^2, \omega)$.

(ii) Since ω is a product weight, by [2, Theorem 3.7(ii)], the Gel'fand space of $l^1(\mathbb{Z}^2, \omega)$ is homeomorphic to $\Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$. It is clear from the maximum modulus principle that $\partial l^1(\mathbb{Z}^2, \omega) \subset (\Gamma(\alpha_{1,0}) \times \Gamma(\alpha_{2,0})) \cup (\Gamma(\beta_{1,0}) \times \Gamma(\alpha_{2,0})) \cup (\Gamma(\alpha_{1,0}) \times \Gamma(\beta_{2,0})) \cup (\Gamma(\beta_{1,0}) \times \Gamma(\beta_{2,0}))$. Conversely, let $(z_0, w_0) \in \Gamma(\alpha_{1,0}) \times \Gamma(\alpha_{2,0})$, then $|z_0 w_0| = \xi_4$ and hence, by the argument given in (i) above, (z_0, w_0) is a peak point, and hence $(z_0, w_0) \in \partial l^1(\mathbb{Z}^2, \omega)$. Similarly, we can show that other sets are also subsets of $\partial l^1(\mathbb{Z}^2, \omega)$. □

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