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Application of Nonlinear Contraction Condition for Solution of First-order Differential Systems with General Nonlocal Conditions

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Abstract

This paper concerns the existence of solutions to initial-value problems for nonlinear first-order differential systems with nonlocal conditions of functional type. The fixed point principles by Perov, Schauder and Leray-Schauder are applied to a nonlinear integral operator split into two operators, one of Fredholm type and the other of Volterra type. The novelty in this article is combining this approach with the technique that uses convergent to zero matrices and vector norms. We also give some examples in support of our results.

Keywords: Nonlinear differential system; nonlocal initial condition; fixed point; vector norm; matrix convergent to zero.

2020 Mathematics Subject Classification: 34A34, 34A12, 45G10.

1. Introduction

In this article, we study the nonlocal initial-value problem for the first-order differential system

$$\begin{aligned}
x'(t) &= f_1(t, x(t), y(t), z(t)) \\
y'(t) &= f_2(t, x(t), y(t), z(t)) \\
z'(t) &= f_3(t, x(t), y(t), z(t)) \quad \text{a.e. on } [0,1] \\
x(0) &= \alpha[x], \quad y(0) = \beta[y], \quad z(0) = \gamma[z].
\end{aligned} \tag{1}$$

Here, $f_1, f_2, f_3 : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are Carathéodory functions, $\alpha, \beta, \gamma : C[0,1] \to \mathbb{R}$ are linear and continuous functionals. In this paper, problem (1) was studied using as main tools the fixed point principles by Perov, Schauder and Leray-Schauder, together with the technique that uses convergent to zero matrices and vector norms. Note that the m-point boundary condition $x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0$ is

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a particular case of condition $x(0) = \alpha[x]$ when

$$\alpha[x] = -\sum_{k=1}^{m} a_k x(t_k). \tag{2}$$

In [2], the authors studied the nonlocal initial-value problem for first-order differential equations

$$\begin{aligned} x'(t) &= f(t, x(t)) \quad \text{(a.e. on } [0,1]) \\ x(0) + \sum_{k=1}^{m} a_k x(t_k) &= 0, \end{aligned}$$

assuming that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, t_k are given points with $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m < 1$ and a_k, $\widetilde{a}_k$ are real numbers with $1 + \sum_{k=1}^{m} a_k \neq 0$ and $1 + \sum_{k=1}^{m} \widetilde{a}_k \neq 0$. The main idea there was to rewrite the problem as a fixed point problem, involving a sum of two operators, one of Fredholm type whose values depend only on the restrictions of functions to $[0, t_m]$, and the other one, a Volterra type operator depending on the restrictions to $[t_m, 1]$. The same strategy was adapted in [4] for the first-order differential system

$$\begin{aligned} x'(t) &= f(t, x(t), y(t), z(t)) \\ y'(t) &= g(t, x(t), y(t), z(t)) \\ z'(t) &= h(t, x(t), y(t), z(t)) \quad \text{a.e. on } [0,1] \end{aligned}$$
$$x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0, \quad y(0) + \sum_{k=1}^{m} \widetilde{a}_k y(t_k) = 0, \quad z(0) + \sum_{k=1}^{m} \widetilde{\widetilde{a}}_k z(t_k) = 0.$$

In this article, the nonlocal conditions are expressed by means of linear continuous functionals on $C[0,1]$, as in the works by Webb-Lan [9]. Our main assumption on functionals α, β, γ extends to the general case the specific property of the particular functional (2) of depending only on the points from a proper subinterval $[0, t_0]$ of $[0,1]$, namely $[0, t_m]$ (taking $t_0 := t_m$). More exactly, we require the following property:

$$x|_{[0,t_0]} = y|_{[0,t_0]} \text{ implies } \alpha[x - y] = 0, \text{ whenever } x, y \in C[0,1]. \tag{3}$$

Therefore, (3) reads that the value of functional α on any function x only depends on the restriction of x to the fixed subinterval $[0, t_0]$. The key property of functional α satisfying (3) is that

$$\alpha[u] \leq \|\alpha\| \cdot |u|_{C[0,t_0]}, \tag{4}$$

for every $u \in C[0,1]$. Normally, for a given functional

$$\alpha : C[0,1] \to \mathbb{R},$$

we have

$$|\alpha[g]| \leq \|\alpha\| \cdot |g|_{C[0,1]}.$$

However, if α satisfies condition (3), then

$$|\alpha[g]| \leq \|\alpha\| \cdot |g|_{C[0,t_0]}.$$

Indeed, for each $g \in C[0,1]$, if we let $\widetilde{g} \in C[0,1]$ be defined by

$$\widetilde{g}(t) = \begin{cases} g(t), & \text{if } t \in [0,t_0] \\ g(t_0), & \text{if } t \in [t_0,1], \end{cases}$$

then

$$|\alpha[g]| = |\alpha[\widetilde{g}]| \leq \|\alpha\| \cdot |\widetilde{g}|_{C[0,1]} = \|\alpha\| \cdot |g|_{C[0,t_0]}.$$

The goal of this work is to revisite system (1) under the assumption that both functionals α and β satisfy (3), using the strategy from [4]. Problem (1) is equivalent to the following integral system in $C[0,1]^3$:

$$\begin{aligned} x(t) &= \frac{1}{1-\alpha[1]}\alpha[g_1] + \int_0^t f_1(s,x(s),y(s),z(s))ds \\ y(t) &= \frac{1}{1-\beta[1]}\beta[g_2] + \int_0^t f_2(s,x(s),y(s),z(s))ds, \\ z(t) &= \frac{1}{1-\gamma[1]}\gamma[g_3] + \int_0^t f_3(s,x(s),y(s),z(s))ds, \end{aligned}$$

where

$$\begin{aligned} g_1(t) &:= \int_0^t f_1(s,x(s),y(s),z(s))ds, \\ g_2(t) &:= \int_0^t f_2(s,x(s),y(s),z(s))ds, \\ g_3(t) &:= \int_0^t f_3(s,x(s),y(s),z(s))ds. \end{aligned}$$

This can be viewed as a fixed point problem in $C[0,1]^3$ for the completely continuous operator $T : C[0,1]^3 \to C[0,1]^3$, $T = (T_1, T_2, T_3)$, where T_1, T_2 and T_3 are given by

$$\begin{aligned} T_1(x,y,z)(t) &= \frac{1}{1-\alpha[1]}\alpha[g_1] + \int_0^t f_1(s,x(s),y(s),z(s))ds, \\ T_2(x,y,z)(t) &= \frac{1}{1-\beta[1]}\beta[g_2] + \int_0^t f_2(s,x(s),y(s),z(s))ds, \\ T_3(x,y,z)(t) &= \frac{1}{1-\gamma[1]}\gamma[g_3] + \int_0^t f_3(s,x(s),y(s),z(s))ds. \end{aligned}$$

In fact, under assumption (3) on α, β and γ operators T_1, T_2 and T_3 appear as sums of three integral operators, one of Fredholm type, whose values depend only on the restrictions of functions to $[0, t_0]$, and the other one, a Volterra type operator depending on the restrictions to $[t_0, 1]$, as this was pointed out in [2]. Thus, T_1 can be rewritten as $T_1 = T_{F_1} + T_{V_1}$, where

$$T_{F_1}(x,y,z)(t) = \begin{cases} \frac{1}{1-\alpha[1]}\alpha[g_1] + \int_0^t f_1(s,x(s),y(s),z(s))ds, & \text{if } t < t_0 \\ \frac{1}{1-\alpha[1]}\alpha[g_1] + \int_0^{t_0} f_1(s,x(s),y(s),z(s))ds, & \text{if } t \geq t_0; \end{cases}$$

$$T_{V_1}(x,y,z)(t) = \begin{cases} 0, & \text{if } t < t_0 \\ \int_{t_0}^t f_1(s,x(s),y(s),z(s))ds, & \text{if } t \geq t_0. \end{cases}$$

Similarly, $T_2 = T_{F_2} + T_{V_2}$, where

$$T_{F_2}(x,y,z)(t) = \begin{cases} \frac{1}{1-\beta[1]}\beta[g_2] + \int_0^t f_2(s,x(s),y(s),z(s))ds, & \text{if } t < t_0 \\ \frac{1}{1-\beta[1]}\beta[g_2] + \int_0^{t_0} f_2(s,x(s),y(s),z(s))ds, & \text{if } t \geq t_0; \end{cases}$$

$$T_{V_2}(x,y,z)(t) = \begin{cases} 0, & \text{if } t < t_0 \\ \int_{t_0}^t f_2(s,x(s),y(s),z(s))ds, & \text{if } t \geq t_0. \end{cases}$$

and $T_3 = T_{F_3} + T_{V_3}$, where

$$T_{F_3}(x,y,z)(t) = \begin{cases} \frac{1}{1-\gamma[1]}\gamma[g_3] + \int_0^t f_3(s,x(s),y(s),z(s))ds, & \text{if } t < t_0 \\ \frac{1}{1-\gamma[1]}\gamma[g_3] + \int_0^{t_0} f_3(s,x(s),y(s),z(s))ds, & \text{if } t \geq t_0; \end{cases}$$

$$T_{V_2}(x,y,z)(t) = \begin{cases} 0, & \text{if } t < t_0 \\ \int_{t_0}^t f_3(s,x(s),y(s),z(s))ds, & \text{if } t \geq t_0. \end{cases}$$

This allows us to split the growth condition on the nonlinear terms $f_1(t,x,y,z)$, $f_2(t,x,y,z)$ and $f_3(t,x,y,z)$ into two parts, one for $t \in [0,t_0]$ and another one for $t \in [t_0,1]$, in such way that one re-obtains the classical growth when $t_0 = 0$, that is for the local initial condition $x(0) = 0$. We conclude this introductory part by some notation, notions and basic results that are used in the next sections. The symbol $|x|_{C[a,b]}$ stands for the max-norm on $C[a,b]$,

$$|x|_{C[a,b]} = \max_{t\in[a,b]} |x(t)|,$$

while $\|x\|_{C[a,b]}$ denotes the Bielecki norm

$$\|x\|_{C[a,b]} = |x(t)e^{-\theta(t-a)}|_{C[a,b]}$$

for some suitable $\theta > 0$. In the next sections, three fixed point principles will be used to prove the existence of solutions for the semilinear problem, namely the fixed point theorems by Perov, Schauder and Leray-Schauder (see [7]). In all three cases a key role will be played by the so called convergent to zero matrices. A square matrix M with nonnegative elements is said to be *convergent to zero* if

$$M^k \to 0 \quad \text{as } k \to \infty.$$

It is known that the property of being convergent to zero is equivalent to each of the following three conditions (for details see [7,8]):

(a) $I - M$ is nonsingular and $(I-M)^{-1} = I + M + M^2 + \dots$, where I stands for the unit matrix of the same order as M;

(b) the eigenvalues of M are located in the interior of e the unit disc of the complex plane;

(c) $I - M$ is nonsingular and $(I-M)^{-1}$ has nonnegative elements.

The following lemma whose proof is immediate from characterization (b) of convergent to zero matrices will be used in the sequel:

Lemma 1.1. *If A is a square matrix that converges to zero and the elements of an other square matrix B are small enough, then $A + B$ also converges to zero.*

We finish this introductory section by recalling (see [1,7]) three fundamental results which will be used in the next sections. Let X be a nonempty set. By a *vector-valued metric* on X we mean a mapping $d : X \times X \to \mathbb{R}^n_+$ such that

(i) $d(u,v) \geq 0$ for all $u, v \in X$ and if $d(u,v) = 0$ then $u = v$;

(ii) $d(u,v) = d(v,u)$ for all $u, v \in X$;

(iii) $d(u,v) \leq d(u,w) + d(w,v)$ for all $u, v, w \in X$.

Here, for $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, \dots, n$. We call the pair (X, d) a *generalized metric space*. For such a space convergence and completeness are similar to those in usual metric spaces. An operator $T : X \to X$ is said to be *contractive* (with respect to the vector-valued metric d on X) if there exists a convergent to zero matrix M such that

$$d(T(u), T(v)) \leq M d(u,v) \quad \text{for all } u, v \in X.$$

Theorem 1.2 (Perov)**.** *Let (X, d) be a complete generalized metric space and $T : X \to X$ a contractive operator with Lipschitz matrix M. Then T has a unique fixed point u^* and for each $u_0 \in X$ we have*

$$d(T^k(u_0), u^*) \leq M^k (I-M)^{-1} d(u_0, T(u_0)) \quad \textit{for all } k \in \mathbf{N}.$$

Theorem 1.3 (Schauder). *Let X be a Banach space, $D \subset X$ a nonempty closed bounded convex set and $T : D \to D$ a completely continuous operator (i.e., T is continuous and $T(D)$ is relatively compact). Then T has at least one fixed point.*

Theorem 1.4 (Leray-Schauder). *Let $(X, \|\cdot\|_X)$ be a Banach space, $R > 0$ and $T : \overline{B}_R(0; X) \to X$ a completely continuous operator. If $\|u\|_X < R$ for every solution u of the equation $u = \lambda T(u)$ and any $\lambda \in (0,1)$, then T has at least one fixed point.*

Throughout the paper we shall assume that the following conditions are satisfied:

(H1) $1 - \alpha[1] \neq 0, 1 - \beta[1] \neq 0$ and $1 - \gamma[1] \neq 0$.

(H2) $f_1, f_2, f_3 : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are such that $f_1(., x, y, z), f_2(., x, y, z), f_3(., x, y, z)$ are measurable for each $(x, y, z) \in \mathbb{R}^2$ and $f_1(t, ., ., .), f_2(t, ., ., .), f_3(t, ., ., .)$ are continuous for almost all $t \in [0,1]$.

2. Nonlinearities with the Lipschitz Property: Application of Perov's Fixed Point Theorem

Here we show that the existence of solutions to problem (1) follows from Perov's fixed point theorem when f_1, f_2, f_3 satisfy Lipschitz conditions in x and y:

$$|f_1(t,x,y,z) - f_1(t,\overline{x},\overline{y},\overline{z})| \leq \begin{cases} a_1^1|x-\overline{x}| + b_1^1|y-\overline{y}| + c_1^1|z-\overline{z}|, & \text{if } t \in [0,t_0] \\ a_2^1|x-\overline{x}| + b_2^1|y-\overline{y}| + c_2^1|z-\overline{z}|, & \text{if } t \in [t_0,1], \end{cases} \tag{5}$$

$$|f_2(t,x,y,z) - f_2(t,\overline{x},\overline{y},\overline{z})| \leq \begin{cases} a_1^2|x-\overline{x}| + b_1^2|y-\overline{y}| + c_1^2|z-\overline{z}|, & \text{if } t \in [0,t_0] \\ a_2^2|x-\overline{x}| + b_2^2|y-\overline{y}| + c_2^2|z-\overline{z}|, & \text{if } t \in [t_0,1], \end{cases} \tag{6}$$

$$|f_3(t,x,y,z) - f_3(t,\overline{x},\overline{y},\overline{z})| \leq \begin{cases} a_1^3|x-\overline{x}| + b_1^3|y-\overline{y}| + c_1^3|z-\overline{z}|, & \text{if } t \in [0,t_0] \\ a_2^3|x-\overline{x}| + b_2^3|y-\overline{y}| + c_2^3|z-\overline{z}|, & \text{if } t \in [t_0,1], \end{cases} \tag{7}$$

for all $x, y, z, \overline{x}, \overline{y}, \overline{z} \in \mathbb{R}$.
In what follows we denote by

$$A_\alpha := \frac{\|\alpha\|}{|1-\alpha[1]|} + 1, \qquad B_\beta := \frac{\|\beta\|}{|1-\beta[1]|} + 1, \text{ and } C_\gamma := \frac{\|\gamma\|}{|1-\gamma[1]|} + 1.$$

Theorem 2.1. *If f_1, f_2, f_3 satisfy the Lipschitz conditions* (5), (6),(7) *and the matrix*

$$M_0 := \begin{bmatrix} a_1^1 t_0 A_\alpha & b_1^1 t_0 A_\alpha & c_1^1 t_0 A_\alpha \\ a_1^2 t_0 B_\beta & b_1^2 t_0 B_\beta & c_1^2 t_0 B_\beta \\ a_1^3 t_0 C_\gamma & b_1^3 t_0 C_\gamma & c_1^3 t_0 C_\gamma \end{bmatrix} \tag{8}$$

converges to zero, then problem (1) *has a unique solution.*

Proof. We shall apply Perov's fixed point theorem in $C[0,1]^3$ endowed with the vector norm $\|\cdot\|$ defined by

$$\|u\| = (\|x\|, \|y\|, \|z\|)$$

for $u = (x, y, z)$, where for $w \in C[0,1]$, we let

$$\|w\| = \max\{|w|_{C[0,t_0]}, \|w\|_{C[t_0,1]}\}.$$

We have to prove that T is contractive, more exactly that

$$\|T(u) - T(\overline{u})\| \leq M_\theta \|u - \overline{u}\|$$

for all $u = (x, y, z), \overline{u} = (\overline{x}, \overline{y}, \overline{z}) \in C[0,1]^3$ and some matrix M_θ converging to zero. To this end, let $u = (x, y, z), \overline{u} = (\overline{x}, \overline{y}, \overline{z})$ be any elements of $C[0,1]^3$. For $t \in [0, t_0]$, we have

$$\begin{aligned}|T_1(x,y,z)(t) - T_1(\overline{x},\overline{y},\overline{z})(t)| &= \bigg| \frac{1}{1-\alpha[1]}\alpha[g_1] + \int_0^t f_1(s, x(s), y(s), z(s))ds \\ &\quad - \frac{1}{1-\alpha[1]}\alpha[\overline{g}_1] - \int_0^t f_1(s, \overline{x}(s), \overline{y}(s), \overline{z}(s))ds \bigg| \\ &\leq |\frac{1}{1-\alpha[1]}||\alpha[g_1 - \overline{g}_1]| + \int_0^t |f_1(s, x(s), y(s), z(s)) - f_1(s, \overline{x}(s), \overline{y}(s), \overline{z}(s))|ds.\end{aligned}$$

Thus, using (3),

$$\alpha[g_1 - \overline{g}_1] \leq \|\alpha\| \cdot |g_1 - \overline{g}_1|_{C[0,t_0]}$$

and therefore by (4), we obtain the following evaluation:

$$\begin{aligned}|T_1(x,y,z)(t) - T_1(\overline{x},\overline{y},\overline{z})(t)| &\leq \frac{\|\alpha\|}{|1-\alpha[1]|}|g_1 - \overline{g}_1|_{C[0,t_0]} \\ &\quad + \int_0^t (a_1^1|x(s) - \overline{x}(s)| + b_1^1|y(s) - \overline{y}(s)| + c_1^1|z(s) - \overline{z}(s)|)ds.\end{aligned}$$

Now, taking the supremum, we have

$$\begin{aligned}|T_1(x,y,z) - T_1(\overline{x},\overline{y},\overline{z})|_{C[0,t_0]} &\leq \frac{\|\alpha\|}{|1-\alpha[1]|}|g_1 - \overline{g}_1|_{C[0,t_0]} + a_1^1 t_0|x - \overline{x}|_{C[0,t_0]} \\ &\quad + b_1^1 t_0 |y - \overline{y}|_{C[0,t_0]} + c_1^1 t_0 |z - \overline{z}|_{C[0,t_0]}.\end{aligned}$$

Also

$$\begin{aligned}|g_1(t) - \overline{g}_1(t)| &\leq \int_0^t |f_1(s, x(s), y(s), z(s)) - f_1(s, \overline{x}(s), \overline{y}(s), \overline{z}(s))|ds \\ &\leq \int_0^t (a_1^1|x(s) - \overline{x}(s)| + b_1^1|y(s) - \overline{y}(s)| + c_1^1|z(s) - \overline{z}(s)|)ds \\ &\leq a_1^1 t_0|x - \overline{x}|_{C[0,t_0]} + b_1^1 t_0|y - \overline{y}|_{C[0,t_0]} + c_1^1 t_0|z - \overline{z}|_{C[0,t_0]},\end{aligned}$$

which gives

$$|g_1 - \overline{g}_1|_{C[0,t_0]} \leq a_1^1 t_0 |x - \overline{x}|_{C[0,t_0]} + b_1^1 t_0 |y - \overline{y}|_{C[0,t_0]} + c_1^1 t_0 |z - \overline{z}|_{C[0,t_0]}. \tag{9}$$

From (8) and (9), we obtain

$$\begin{aligned} |T_1(x,y,z) - T_1(\overline{x},\overline{y},\overline{z})|_{C[0,t_0]} &\leq \left(\frac{\|\alpha\|}{|1-\alpha[1]|} + 1 \right) (a_1^1 t_0 |x - \overline{x}|_{C[0,t_0]} + b_1^1 t_0 |y - \overline{y}|_{C[0,t_0]} + c_1^1 t_0 |z - \overline{z}|_{C[0,t_0]}) \\ &= A_\alpha a_1^1 t_0 |x - \overline{x}|_{C[0,t_0]} + A_\alpha b_1^1 t_0 |y - \overline{y}|_{C[0,t_0]} + A_\alpha c_1^1 t_0 |z - \overline{z}|_{C[0,t_0]}. \end{aligned} \tag{10}$$

For $t \in [t_0, 1]$ and any $\theta > 0$, we have

$$\begin{aligned} |T_1(x,y,z)(t) - T_1(\overline{x},\overline{y},\overline{z})(t)| &\leq \left| \frac{1}{1-\alpha[1]} \right| |\alpha[g_1 - \overline{g}_1]| + \int_0^t |f_1(s,x(s),y(s),z(s)) - f_1(s,\overline{x}(s),\overline{y}(s),\overline{z}(s))| ds \\ &+ \int_{t_0}^t |f_1(s,x(s),y(s),z(s)) - f_1(s,\overline{x}(s),\overline{y}(s),\overline{z}(s))| ds. \end{aligned}$$

Hence, (4) gives

$$\begin{aligned} |T_1(x,y,z)(t) - T_1(\overline{x},\overline{y},\overline{z})(t)| &\leq \big(\frac{\|\alpha\|}{|1-\alpha[1]|} + 1 \big) \big(a_1^1 t_0 |x - \overline{x}|_{C[0,t_0]} + b_1^1 t_0 |y - \overline{y}|_{C[0,t_0]} + c_1^1 t_0 |z - \overline{z}|_{C[0,t_0]} \big) \\ &+ \int_{t_0}^t |f_1(s,x(s),y(s),z(s)) - f_1(s,\overline{x}(s),\overline{y}(s),\overline{z}(s))| ds. \end{aligned}$$

The last integral can be further estimated as follows:

$$\begin{aligned} \int_{t_0}^t |f_1(s,x(s),y(s),z(s)) - f_1(s,\overline{x}(s),\overline{y}(s),\overline{z}(s))| ds &\leq \int_{t_0}^t (a_2^1 |x(s) - \overline{x}(s)| + b_2^1 |y(s) - \overline{y}(s)| + c_2^1 |z(s) - \overline{z}(s)|) ds \\ &= a_2^1 \int_{t_0}^t |x(s) - \overline{x}(s)| \cdot e^{-\theta(s-t_0)} \cdot e^{\theta(s-t_0)} ds \\ &\quad + b_2^1 \int_{t_0}^t |y(s) - \overline{y}(s)| \cdot e^{-\theta(s-t_0)} \cdot e^{\theta(s-t_0)} ds \\ &\quad + c_2^1 \int_{t_0}^t |z(s) - \overline{z}(s)| \cdot e^{-\theta(s-t_0)} \cdot e^{\theta(s-t_0)} ds \\ &\leq \frac{a_2^1}{\theta} e^{\theta(t-t_0)} \|x - \overline{x}\|_{C[t_0,1]} + \frac{b_2^1}{\theta} e^{\theta(t-t_0)} \|y - \overline{y}\|_{C[t_0,1]} \\ &+ \frac{c_2^1}{\theta} e^{\theta(t-t_0)} \|z - \overline{z}\|_{C[t_0,1]}. \end{aligned}$$

Thus

$$\begin{aligned} |T_1(x,y,z)(t) - T_1(\overline{x},\overline{y},\overline{z})(t)| &\leq A_\alpha a_1^1 t_0 |x - \overline{x}|_{C[0,t_0]} + A_\alpha b_1^1 t_0 |y - \overline{y}|_{C[0,t_0]} + A_\alpha c_1^1 t_0 |z - \overline{z}|_{C[0,t_0]} \\ &+ \frac{a_2^1}{\theta} e^{\theta(t-t_0)} \|x - \overline{x}\|_{C[t_0,1]} + \frac{b_2^1}{\theta} e^{\theta(t-t_0)} \|y - \overline{y}\|_{C[t_0,1]} \\ &+ \frac{c_2^1}{\theta} e^{\theta(t-t_0)} \|z - \overline{z}\|_{C[t_0,1]}. \end{aligned}$$

Dividing by $e^{\theta(t-t_0)}$ and taking the supremum when $t \in [t_0, 1]$, we obtain

$$\begin{aligned} \|T_1(x,y,z) - T_1(\overline{x},\overline{y},\overline{z})\|_{C[t_0,1]} \leq A_\alpha a_1^1 t_0 |x-\overline{x}|_{C[0,t_0]} + A_\alpha b_1^1 t_0 |y-\overline{y}|_{C[0,t_0]} + A_\alpha c_1^1 t_0 |z-\overline{z}|_{C[0,t_0]} \\ + \frac{a_2^1}{\theta}\|x-\overline{x}\|_{C[t_0,1]} + \frac{b_2^1}{\theta}\|y-\overline{y}\|_{C[t_0,1]} + \frac{c_2^1}{\theta}\|z-\overline{z}\|_{C[t_0,1]}. \end{aligned} \tag{11}$$

Now (10) and (11) imply

$$\|T_1(x,y,z) - T_1(\overline{x},\overline{y},\overline{z})\| \leq \left(A_\alpha a_1^1 t_0 + \frac{a_2^1}{\theta}\right)\|x-\overline{x}\| + \left(A_\alpha b_1^1 t_0 + \frac{b_2^1}{\theta}\right)\|y-\overline{y}\| + \left(A_\alpha c_1^1 t_0 + \frac{c_2^1}{\theta}\right)\|z-\overline{z}\| \tag{12}$$

Similarly,

$$\|T_2(x,y,z) - T_2(\overline{x},\overline{y},\overline{z})\| \leq \left(A_\alpha a_1^2 t_0 + \frac{a_2^2}{\theta}\right)\|x-\overline{x}\| + \left(A_\alpha b_1^2 t_0 + \frac{b_2^2}{\theta}\right)\|y-\overline{y}\| + \left(A_\alpha c_1^2 t_0 + \frac{c_2^2}{\theta}\right)\|z-\overline{z}\| \tag{13}$$

$$\|T_3(x,y,z) - T_3(\overline{x},\overline{y},\overline{z})\| \leq \left(A_\alpha a_1^3 t_0 + \frac{a_2^3}{\theta}\right)\|x-\overline{x}\| + \left(A_\alpha b_1^3 t_0 + \frac{b_2^3}{\theta}\right)\|y-\overline{y}\| + \left(A_\alpha c_1^3 t_0 + \frac{c_2^3}{\theta}\right)\|z-\overline{z}\| \tag{14}$$

Using the vector norm we can put both inequalities (12), (13), (14) under the vector inequality

$$\|T(u) - T(\overline{u})\| \leq M_\theta \|u-\overline{u}\|,$$

where

$$M_\theta = \begin{bmatrix} A_\alpha a_1^1 t_0 + \frac{a_2^1}{\theta} & A_\alpha b_1^1 t_0 + \frac{b_2^1}{\theta} & A_\alpha c_1^1 t_0 + \frac{c_2^1}{\theta} \\ A_\alpha a_1^2 t_0 + \frac{a_2^2}{\theta} & A_\alpha b_1^2 t_0 + \frac{b_2^2}{\theta} & A_\alpha c_2^2 t_0 + \frac{c_2^2}{\theta} \\ A_\alpha a_1^3 t_0 + \frac{a_2^3}{\theta} & A_\alpha b_1^3 t_0 + \frac{b_2^3}{\theta} & A_\alpha c_2^3 t_0 + \frac{c_2^3}{\theta} \end{bmatrix}. \tag{15}$$

Clearly the matrix M_θ can be represented as $M_\theta = M_0 + M_1$, where

$$M_1 = \begin{bmatrix} \frac{a_2^1}{\theta} & \frac{b_2^1}{\theta} & \frac{c_2^1}{\theta} \\ \frac{a_2^2}{\theta} & \frac{b_2^2}{\theta} & \frac{c_2^2}{\theta} \\ \frac{a_2^3}{\theta} & \frac{b_2^3}{\theta} & \frac{c_2^3}{\theta} \end{bmatrix}.$$

Since M_0 is assumed to be convergent to zero, from Lemma 1.1 we have that M_θ also converges to zero for large enough $\theta > 0$. The result follows now from Perov's fixed point theorem. □

3. Nonlinearities with Growth at Most Linear: Application of Schauder's Fixed Point Theorem

Here we show that the existence of solutions to problem (1) follows from Schauder's fixed point theorem when f_1, f_2, f_3, instead of the Lipschitz condition, satisfy the more relaxed condition of growth at most linear:

$$|f_1(t,x,y,z)| \le \begin{cases} a_1^1|x| + b_1^1|y| + c_1^1|z| + d_1^1, & \text{if } t \in [0,t_0] \\ a_2^1|x| + b_2^1|y| + c_2^1|z| + d_2^1, & \text{if } t \in [t_0,1], \end{cases} \tag{16}$$

$$|f_2(t,x,y,z)| \le \begin{cases} a_1^2|x| + b_1^2|y| + c_1^2|z| + d_1^2, & \text{if } t \in [0,t_0] \\ a_2^2|x| + b_2^2|y| + c_2^2|z| + d_2^2, & \text{if } t \in [t_0,1], \end{cases} \tag{17}$$

$$|f_3(t,x,y,z)| \le \begin{cases} a_1^3|x| + b_1^3|y| + c_1^3|z| + d_1^3, & \text{if } t \in [0,t_0] \\ a_2^3|x| + b_2^3|y| + c_2^3|z| + d_2^3, & \text{if } t \in [t_0,1], \end{cases} \tag{18}$$

Theorem 3.1. *If f_1, f_2, f_3 satisfy* (16), (17), (18) *and matrix* (8) *converges to zero, then* (1) *has at least one solution.*

Proof. To apply Schauder's fixed point theorem, we look for a nonempty, bounded, closed and convex subset B of $C[0,1]^3$ so that $T(B) \subset B$. Let x, y, z be any elements of $C[0,1]$. For $t \in [0,t_0]$, using (3) and (4), we have

$$\begin{aligned} |T_1(x,y,z)(t)| &= |\frac{1}{1-\alpha[1]}\alpha[g_1] + \int_0^t f_1(s,x(s),y(s),z(s))ds| \\ &\le |\frac{1}{1-\alpha[1]}||\alpha[g_1]| + \int_0^t (a_1^1|x(s)| + b_1^1|y(s)| + c_1^1|z(s)| + d_1)ds \\ &\le \frac{\|\alpha\|}{|1-\alpha[1]|}|g_1|_{C[0,t_0]} + a_1^1 t_0|x|_{C[0,t_0]} + b_1^1 t_0|y|_{C[0,t_0]} + c_1^1 t_0|z|_{C[0,t_0]} + d_1^1 t_0. \end{aligned} \tag{19}$$

Also

$$\begin{aligned} |g_1(t)| &\le \int_0^t |f_1(s,x(s),y(s),z(s))|ds \\ &\le \int_0^t (a_1^1|x(s)| + b_1^1|y(s)| + c_1^1|z(s)| + d_1)ds \\ &\le a_1^1 t_0|x|_{C[0,t_0]} + b_1^1 t_0|y|_{C[0,t_0]} + c_1^1 t_0|z|_{C[0,t_0]} + d_1^1 t_0, \end{aligned}$$

which gives

$$|g_1|_{C[0,t_0]} \le a_1^1 t_0|x|_{C[0,t_0]} + b_1^1 t_0|y|_{C[0,t_0]} + c_1^1 t_0|z|_{C[0,t_0]} + d_1 t_0. \tag{20}$$

From (19) and (20), we obtain

$$\begin{aligned} |T_1(x,y,z)|_{C[0,t_0]} &\le (\frac{\|\alpha\|}{|1-\alpha[1]|} + 1)(a_1^1 t_0|x|_{C[0,t_0]} + b_1^1 t_0|y|_{C[0,t_0]} + c_1^1 t_0|z|_{C[0,t_0]}) + \widetilde{d_1^1} \\ &= a_1^1 t_0 A_\alpha|x|_{C[0,t_0]} + b_1^1 t_0 A_\alpha|y|_{C[0,t_0]} + c_1^1 t_0 A_\alpha|z|_{C[0,t_0]} + \widetilde{d_1^1}, \end{aligned} \tag{21}$$

where $\widetilde{d_1^1} := d_1^1 t_0 A_\alpha$. For $t \in [t_0,1]$ and any $\theta > 0$, we have

$$|T_1(x,y,z)(t)| = a_1^1 t_0 A_\alpha|x|_{C[0,t_0]} + b_1^1 t_0 A_\alpha|y|_{C[0,t_0]} + c_1^1 t_0 A_\alpha|z|_{C[0,t_0]} + \widetilde{d_1^1}$$

$$
\begin{aligned}
&+ \int_{t_0}^{t} (a_2^1|x(s)| + b_2^1|y(s)| + c_2^1|z(s)| + d_2^1\ )ds \\
&\leq a_1^1 t_0 A_\alpha |x|_{C[0,t_0]} + b_1^1 t_0 A_\alpha |y|_{C[0,t_0]} + c_1^1 t_0 A_\alpha |z|_{C[0,t_0]} + \widetilde{d}_1^1 + (1-t_0)d_2^1 \\
&+ a_2^1 \int_{t_0}^{t} |x(s)| \cdot e^{-\theta(s-t_0)} \cdot e^{\theta(s-t_0)} ds \\
&+ b_2^1 \int_{t_0}^{t} |y(s)| \cdot e^{-\theta(s-t_0)} \cdot e^{\theta(s-t_0)} ds \\
&+ c_2^1 \int_{t_0}^{t} |z(s)| \cdot e^{-\theta(s-t_0)} \cdot e^{\theta(s-t_0)} ds \\
&\leq a_1^1 t_0 A_\alpha |x|_{C[0,t_0]} + b_1^1 t_0 A_\alpha |y|_{C[0,t_0]} + c_1^1 t_0 A_\alpha |z|_{C[0,t_0]} + d_0^1 \\
&+ \frac{a_2^1}{\theta} e^{\theta(t-t_0)} \|x\|_{C[t_0,1]} + \frac{b_2^1}{\theta} e^{\theta(t-t_0)} \|y\|_{C[t_0,1]} + \frac{c_2^1}{\theta} e^{\theta(t-t_0)} \|z\|_{C[t_0,1]},
\end{aligned}
$$

where $d_0^1 := \widetilde{d}_1^1 + (1-t_0)d_2^1$. Dividing by $e^{\theta(t-t_0)}$ and taking the supremum, it follows that

$$
\begin{aligned}
\|T_1(x,y,z)\|_{C[t_0,1]} \leq\ & a_1^1 t_0 A_\alpha |x|_{C[0,t_0]} + b_1^1 t_0 A_\alpha |y|_{C[0,t_0]} + c_1^1 t_0 A_\alpha |z|_{C[0,t_0]} \\
&+ \frac{a_2^1}{\theta} e^{\theta(t-t_0)} \|x\|_{C[t_0,1]} + \frac{b_2^1}{\theta} e^{\theta(t-t_0)} \|y\|_{C[t_0,1]} + \frac{c_2^1}{\theta} e^{\theta(t-t_0)} \|z\|_{C[t_0,1]} + d_0^1.
\end{aligned}
\tag{22}
$$

Clearly, (21) and (22) give

$$
\|T_1(x,y,z)\| \leq (a_1^1 t_0 A_\alpha + \frac{a_2^1}{\theta})\|x\| + (b_1^1 t_0 A_\alpha + \frac{b_2^1}{\theta})\|y\| + (c_1^1 t_0 A_\alpha + \frac{c_2^1}{\theta})\|z\| + \widetilde{d}_0^1, \tag{23}
$$

where $\widetilde{d}_0^1 = \max\left\{\widetilde{d}_1^1, d_0^1\right\}$. Similarly,

$$
\|T_2(x,y,z)\| \leq (a_1^2 t_0 B_\beta + \frac{a_2^2}{\theta})\|x\| + (b_1^2 t_0 B_\beta + \frac{b_2^2}{\theta})\|y\| + (c_1^2 t_0 B_\beta + \frac{c_2^2}{\theta})\|z\| + \widetilde{d}_0^2, \tag{24}
$$

with $\widetilde{d}_0^2 = \max\{\widetilde{d}_1^2, d_0^2\}$, where $\widetilde{d}_1^2 := d_1^2 t_0 B_\beta$ and $d_0^2 := \widetilde{d}_1^2 + (1-t_0)d_2^2$.

$$
\|T_3(x,y,z)\| \leq (a_1^3 t_0 C_\gamma + \frac{a_2^3}{\theta})\|x\| + (b_1^3 t_0 C_\gamma + \frac{b_2^3}{\theta})\|y\| + (c_1^3 t_0 C_\gamma + \frac{c_2^3}{\theta})\|z\| + \widetilde{d}_0^3, \tag{25}
$$

with $\widetilde{d}_0^3 = \max\{\widetilde{d}_1^3, d_0^3\}$, where $\widetilde{d}_1^3 := d_1^3 t_0 C_\gamma$ and $d_0^3 := \widetilde{d}_1^3 + (1-t_0)d_2^3$. Now (23), (24) and (25) can be put together as

$$
\begin{bmatrix} \|T_1(x,y,z)\| \\ \|T_2(x,y,z)\| \\ \|T_3(x,y,z)\| \end{bmatrix} \leq M_\theta \begin{bmatrix} \|x\| \\ \|y\| \\ \|z\| \end{bmatrix} + \begin{bmatrix} \widetilde{d}_0^1 \\ \widetilde{d}_0^2 \\ \widetilde{d}_0^3 \end{bmatrix},
$$

where the matrix M_θ is given by (15) and converges to zero for a large enough $\theta > 0$. Next we look for two positive numbers R_1, R_2 such that if $\|x\| \leq R_1, \|y\| \leq R_2, \|z\| \leq R_3$, then $\|T_1(x,y,z)\| \leq R_1$,

$\|T_2(x,y,z)\| \le R_2$, $\|T_3(x,y,z)\| \le R_3$. To this end it is sufficient that

$$\begin{aligned}
&(a_1^1 t_0 A_\alpha + \frac{a_2^1}{\theta})R_1 + (b_1^1 t_0 A_\alpha + \frac{b_2^1}{\theta})R_2 + (c_1^1 t_0 A_\alpha + \frac{c_2^1}{\theta})R_3 + \widetilde{d^1}_0 \le R_1 \\
&(a_1^2 t_0 B_\beta + \frac{a_2^2}{\theta})R_1 + (b_1^2 t_0 B_\beta + \frac{b_2^2}{\theta})R_2 + (c_1^2 t_0 B_\beta + \frac{c_2^2}{\theta})R_3 + \widetilde{d^2}_0 \le R_2, \\
&(a_1^3 t_0 C_\gamma + \frac{a_2^3}{\theta})R_1 + (b_1^3 t_0 C_\gamma + \frac{b_2^3}{\theta})R_2 + (c_1^3 t_0 C_\gamma + \frac{c_2^3}{\theta})R_3 + \widetilde{d^3}_0 \le R_2,
\end{aligned} \tag{26}$$

or equivalently

$$M_\theta \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} + \begin{bmatrix} \widetilde{d}_0^1 \\ \widetilde{d}_0^2 \\ \widetilde{d}_0^3 \end{bmatrix} \le \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix},$$

whence

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \ge (I - M_\theta)^{-1} \begin{bmatrix} \widetilde{d}_0^1 \\ \widetilde{d}_0^2 \\ \widetilde{d}_0^3 \end{bmatrix}.$$

Note that $I - M_\theta$ is invertible and its inverse $(I - M_\theta)^{-1}$ has nonnegative elements since M_θ converges to zero. Thus, if

$$B = \{(x,y,z) \in C[0,1]^3 : \|x\| \le R_1, \|y\| \le R_2, \|z\| \le R_3\},$$

then $T(B) \subset B$ and Schauder's fixed point theorem can be applied. □

4. More General Nonlinearities: Application of the Leray-Schauder Principle

We now consider that nonlinearlities f_1, f_2, f_3 satisfy more general growth conditions, namely:

$$|f_1(t,u)| \le \begin{cases} \omega_1(t, |u|_e), & \text{if } t \in [0, t_0] \\ \gamma(t)\beta_1(|u|_e), & \text{if } t \in [t_0, 1], \end{cases} \tag{27}$$

$$|f_2(t,u)| \le \begin{cases} \omega_2(t, |u|_e), & \text{if } t \in [0, t_0] \\ \gamma(t)\beta_2(|u|_e), & \text{if } t \in [t_0, 1], \end{cases} \tag{28}$$

$$|f_3(t,u)| \le \begin{cases} \omega_3(t, |u|_e), & \text{if } t \in [0, t_0] \\ \gamma(t)\beta_3(|u|_e), & \text{if } t \in [t_0, 1], \end{cases} \tag{29}$$

for all $u = (x,y,z) \in \mathbb{R}^3$, where by $|u|_e$ we mean the Euclidean norm in $\mathbb{R}^3$. Here $\omega_1, \omega_2, \omega_3$ are Carathéodory functions on $[0, t_0] \times \mathbb{R}_+$, nondecreasing in their second argument, $\gamma \in L^1[t_0, 1]$, while $\beta_1, \beta_2, \beta_3 : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing and $1/\beta_1, 1/\beta_2, 1/\beta_3 \in L^1_{loc}(\mathbb{R}_+)$.

Theorem 4.1. *Assume that* (27), (28) *hold. In addition assume that there exists a positive number* R_0 *such that* *for* $\rho = (\rho_1, \rho_2\rho_3) \in (0, \infty)^3$,

$$\frac{1}{\rho_1}\int_0^{t_0} \omega_1(t, |\rho|_e)dt \geq \frac{1}{A_\alpha}, \frac{1}{\rho_2}\int_0^{t_0} \omega_2(t, |\rho|_e)dt \geq \frac{1}{B_\beta} \text{ and } \frac{1}{\rho_3}\int_0^{t_0} \omega_3(t, |\rho|_e)dt \geq \frac{1}{C_\gamma} \tag{30}$$

imply $|\rho|_e \leq R_0$, *and*

$$\int_{R^*}^{\infty} \frac{d\tau}{\beta_1(\tau) + \beta_2(\tau) + \beta_3(\tau)} > \int_{t_0}^{1} \eta(s)ds, \tag{31}$$

where $R^* = \left[(A_\alpha \int_0^{t_0} \omega_1(t, R_0)dt)^2 + (B_\beta \int_0^{t_0} \omega_2(t, R_0)dt)^2 + (C_\gamma \int_0^{t_0} \omega_3(t, R_0)dt)^2\right]^{1/2}$. *Then* (1) *has at least one solution.*

Proof. The result will follow from the Leray-Schauder fixed point theorem once we have proved the boundedness of the set of all solutions to equation $u = \lambda T(u)$, for $\lambda \in [0, 1]$. Let $u = (x, y, z)$ be such a solution. Then, for $t \in [0, t_0]$, also using condition (3) and (4), we have

$$\begin{aligned} |x(t)| &= |\lambda T_1(x, y, z)(t)| \\ &= \lambda|\frac{1}{1-\alpha[1]}\alpha[g_1] + \int_0^t f_1(s, x(s), y(s), z(s))ds| \\ &\leq \frac{\|\alpha\|}{|1-\alpha[1]|}|g_1|_{C[0,t_0]} + \int_0^t |f_1(s, x(s), y(s), z(s))|ds \\ &\leq (\frac{\|\alpha\|}{|1-\alpha[1]|} + 1)\int_0^{t_0} \omega_1(s, |u(s)|_e)ds \\ &= A_\alpha \int_0^{t_0} \omega_1(s, |u(s)|_e)ds. \end{aligned} \tag{32}$$

Similarly,

$$|y(t)| \leq B_\beta \int_0^{t_0} \omega_2(s, |u(s)|_e)ds. \tag{33}$$

and

$$|z(t)| \leq C_\gamma \int_0^{t_0} \omega_3(s, |u(s)|_e)ds. \tag{34}$$

Let $\rho_1 = |x|_{C[0,t_0]}$, $\rho_2 = |y|_{C[0,t_0]}$,$\rho_3 = |z|_{C[0,t_0]}$. Then from (32), (33), (34), we deduce

$$\begin{aligned} \rho_1 &\leq A_\alpha \int_0^{t_0} \omega_1(s, |u(s)|_e)ds \\ \rho_2 &\leq B_\beta \int_0^{t_0} \omega_2(s, |u(s)|_e)ds \\ \rho_3 &\leq B_\gamma \int_0^{t_0} \omega_3(s, |u(s)|_e)ds. \end{aligned}$$

By (30), this guarantees

$$|\rho|_e \leq R_0. \tag{35}$$

Next we let $t \in [t_0, 1]$. Then

$$|x(t)| = |\lambda T_1(x, y, z)(t)|$$

$$
\begin{aligned}
&\leq A_\alpha \int_0^{t_0} \omega_1(s, R_0) ds + \int_{t_0}^t |f_1(s, x(s), y(s), z(s))| ds \\
&\leq A_\alpha \int_0^{t_0} \omega_1(s, R_0) ds + \int_{t_0}^t \gamma(s) \beta_1(|u(s)|_e) ds \\
&=: \phi_1(t)
\end{aligned}
$$

and similarly

$$|y(t)| \leq B_\beta \int_0^{t_0} \omega_2(s, R_0) ds + \int_{t_0}^t \gamma(s) \beta_2(|u(s)|_e) ds =: \phi_2(t),$$

and

$$|z(t)| \leq C_\gamma \int_0^{t_0} \omega_3(s, R_0) ds + \int_{t_0}^t \gamma(s) \beta_3(|u(s)|_e) ds =: \phi_3(t).$$

Denote $\psi(t) := (\phi_1^2(t) + \phi_2^2(t) + \phi_3^2(t))^{1/2}$. Then

$$
\begin{aligned}
\phi_1'(t) &= \gamma(t) \beta_1(|u(t)|_e) \leq \gamma(t) \beta_1(\psi(t)) \\
\phi_2'(t) &= \gamma(t) \beta_2(|u(t)|_e) \leq \gamma(t) \beta_2(\psi(t))) \\
\phi_3'(t) &= \gamma(t) \beta_3(|u(t)|_e) \leq \gamma(t) \beta_3(\psi(t))).
\end{aligned} \tag{36}
$$

Consequently,

$$
\begin{aligned}
\psi'(t) &= \frac{\phi_1(t)\phi_1'(t) + \phi_2(t)\phi_2'(t) + \phi_3(t)\phi_3'(t)}{\psi(t)} \\
&\leq \gamma(t) \cdot \frac{\phi_1(t)}{\psi(t)} \cdot \beta_1(\psi(t)) + \gamma(t) \cdot \frac{\phi_2(t)}{\psi(t)} \cdot \beta_2(\psi(t)) + \gamma(t) \cdot \frac{\phi_3(t)}{\psi(t)} \cdot \beta_3(\psi(t)) \\
&\leq \gamma(t)[\beta_1(\psi(t)) + \beta_2(\psi(t)) + \beta_3(\psi(t))].
\end{aligned}
$$

It follows that

$$\int_{t_0}^t \frac{\psi'(s)}{\beta_1(\psi(s)) + \beta_2(\psi(s)) + \beta_3(\psi(s))} ds \leq \int_{t_0}^t \gamma(s) ds.$$

Furthermore, using (31) we obtain

$$\int_{\psi(t_0)}^{\psi(t)} \frac{d\tau}{\beta_1(\tau) + \beta_2(\tau) + \beta_3(\tau)} \leq \int_{t_0}^t \gamma(s) ds \leq \int_{t_0}^1 \gamma(s) ds < \int_{R^*}^\infty \frac{d\tau}{\beta_1(\tau) + \beta_2(\tau) + \beta_3(\tau)}. \tag{37}$$

Note that $\psi(t_0) = R^*$. Then from (37) it follows that there exists R_1 such that

$$\psi(t) \leq R_1,$$

for all $t \in [t_0, 1]$. Then $|x(t)| \leq R_1$, $|y(t)| \leq R_1$ and $|z(t)| \leq R_1$, for all $t \in [t_0, 1]$, whence

$$|x|_{C[t_0,1]} \leq R_1, \quad |y|_{C[t_0,1]} \leq R_1 7 \quad |z|_{C[t_0,1]} \leq R_1. \tag{38}$$

Let $R = \max\{R_0, R_1\}$. From (35), (38) we have $|x|_{C[0,1]} \leq R$, $|y|_{C[0,1]} \leq R$ and $|z|_{C[0,1]} \leq R$ as desired. $\square$

Remark 4.2. *If $\omega_1(t,\tau) = \gamma_0(t)\beta_0(\tau)$, then the first inequality in* (30) *implies that $\beta_0(\tau) \le d\tau + d'$ for all $\tau \in R_+$ and some constants d and d'; i.e., the growth of β_0 is at most linear. However, β_1 may have a superlinear growth. Thus we may say that under the assumptions of Theorem 4.1, the growth of $f_1(t,u)$ in u is at most linear for $t \in [0,t_0]$ and can be superlinear for $t \in [t_0,1]$. The same can be said about $f_2(t,u)$ and $f_3(t,u)$.*

In particular, when $\alpha = \beta = \gamma = 0$, problem (1) becomes the classical local initial value problem

$$\begin{aligned} &x' = f_1(t,x,y,z); \qquad y' = f_2(t,x,y,z); \qquad z' = f_3(t,x,y,z) \quad (\text{a.e. } t \in [0,1]) \\ &x(0) = y(0) = z(0) = 0, \end{aligned} \tag{39}$$

and our assumptions reduce to the classical conditions (see [3,6]) and Theorem 4.1 gives the following result.

Corollary 4.3. *Assume that*

$$|f_1(t,u)| \le \gamma(t)\beta_1(|u|_e), \quad |f_2(t,u)| \le \gamma(t)\beta_2(|u|_e) \quad |f_3(t,u)| \le \gamma(t)\beta_3(|u|_e)$$

for $t \in [0,1]$ and $u \in \mathbb{R}^3$, where $\gamma \in L^1[0,1]$, while $\beta_1, \beta_2\beta_3 : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing and $1/\beta_1, 1/\beta_2, 1/\beta_3 \in L^1_{loc}(\mathbb{R}_+)$. In addition assume that

$$\int_0^\infty \frac{d\tau}{\beta_1(\tau) + \beta_2(\tau) + \beta_3(\tau)} > \int_0^1 \eta(s)ds.$$

Then problem (39) *has at least one solution.*

A result similar to the above corollary was given in [5].

Remark 4.4. *Since the trivial solution satisfies the boundary conditions, the solution given by Theorem 4.1 might be zero.*

5. Numerical Examples

In this section, we give some numerical examples to illustrate the existence results from Sections 2 and 3.

Example 5.1. Consider the initial value problem

$$\begin{aligned} x'(t) &= 0.1 + \frac{1}{4}\frac{y^2(t) + z^2(t)}{1 + y^2(t) + z^2(t)} \sin((x(t) + y(t) + z(t)) =: f(x,y,z) \\ y'(t) &= 0.1 + \frac{2}{3}\frac{x^2(t) + z^2(t)}{1 + x^2(t) + z^2(t)} \sin((x(t) + y(t) + z(t)) =: g(x,y,z) \\ z'(t) &= 0.1 + \frac{3}{4}\frac{x^2(t) + y^2(t)}{1 + x^2(t) + y^2(t)} \sin((x(t) + y(t) + z(t)) =: h(x,y,z) \\ x(0) &= \int_0^{1/3} x(s)ds, \quad y(0) = \int_0^{1/3} y(s)ds, \quad z(0) = \int_0^{1/3} z(s)ds, \end{aligned} \tag{40}$$

for $t \in [0,40]$.

We have that

$$\alpha[u] = \int_0^{1/3} u(s)ds \Longrightarrow \alpha[1] = \frac{1}{3} \Longrightarrow \|\alpha\| = \frac{1}{3}.$$

Consequently, $t_0 = 1/3$, $A_\alpha = 3 = B_\beta = C_\gamma$ and

$$M_0 = \begin{pmatrix} a_1^1 & b_1^1 & c_1^1 \\ a_1^2 & b_1^2 & c_1^2 \\ a_1^3 & b_1^3 & c_1^3 \end{pmatrix}.$$

However,

$$\begin{aligned}
&\sup_{\xi,\eta,\tau\in\mathbb{R}} |\frac{\partial f(\xi,\eta,\tau)}{\partial x}| \leq \frac{1}{2} = a_1^1, \quad \sup_{\xi,\eta,\tau\in\mathbb{R}} |\frac{\partial f(\xi,\eta,\tau)}{\partial y}| \leq \frac{3}{32} = b_1^1, \sup_{\xi,\eta,\tau\in\mathbb{R}} |\frac{\partial f(\xi,\eta,\tau)}{\partial z}| \leq \frac{2}{33} = c_1^1, \\
&\sup_{\xi,\eta,\tau\in\mathbb{R}} |\frac{\partial g(\xi,\eta,\tau)}{\partial x}| \leq \frac{1}{4} = a_1^2, \quad \sup_{\xi,\eta,\tau\in\mathbb{R}} |\frac{\partial g(\xi,\eta,\tau)}{\partial y}| \leq \frac{2}{29} = b_1^2, \sup_{\xi,\eta,\tau\in\mathbb{R}} |\frac{\partial g(\xi,\eta,\tau)}{\partial z}| \leq \frac{2}{31} = c_1^2, \\
&\sup_{\xi,\eta,\tau\in\mathbb{R}} |\frac{\partial h(\xi,\eta,\tau)}{\partial x}| \leq \frac{1}{5} = a_1^3, \quad \sup_{\xi,\eta,\tau\in\mathbb{R}} |\frac{\partial h(\xi,\eta,\tau)}{\partial y}| \leq \frac{3}{41} = b_1^3, \sup_{\xi,\eta,\tau\in\mathbb{R}} |\frac{\partial h(\xi,\eta,\tau)}{\partial z}| \leq \frac{3}{35} = c_1^3,
\end{aligned}$$

and then

$$M_0 = \begin{pmatrix} \frac{1}{2} & \frac{3}{32} & \frac{2}{33} \\ \frac{1}{4} & \frac{2}{29} & \frac{2}{31} \\ \frac{1}{5} & \frac{3}{41} & \frac{2}{35} \end{pmatrix}$$

has the eigenvalues $\lambda_1 = 0.002$, $\lambda_2 = 0.072$ and $\lambda_3 = 0.581$. From Theorem 2.1, problem (40) has a unique solution.

Example 5.2. Consider the initial value problem

$$\begin{aligned}
x' &= -0.9x - 1.8\frac{xyz}{2+x^2} + 90 := f(x,y,z) \\
y' &= -0.2y - 1.8\frac{xyz}{2+x^2} + 750 := g(x,y,z) \\
z' &= -0.3z - 1.8\frac{xyz}{2+x^2} + 850 := h(x,y,z) \\
x(0) &= \int_0^{1/3} x(s)ds, \quad y(0) = \int_0^{1/3} y(s)ds, \quad z(0) = \int_0^{1/3} z(s)ds,
\end{aligned} \tag{41}$$

for $t \in [0,1]$.

We consider

$$M_0 = \begin{pmatrix} a_1^1 & b_1^1 & c_1^1 \\ a_1^2 & b_1^2 & c_1^2 \\ a_1^3 & b_1^3 & c_1^3 \end{pmatrix}.$$

We have

$$|\frac{x}{2+x^2}| \leq \frac{\sqrt{2}}{4},$$

so that the matrix

$$M_0 = \begin{pmatrix} 0.9 & 0.6364 & 0.7324 \\ 0 & 0.4364 & 0.2542 \\ 0 & 0 & 0.2457 \end{pmatrix}$$

has the eigenvalues $\lambda_1 = 0.9$, $\lambda_2 = 0.4364$ $\lambda_3 = 0.2457$. From Theorem 3.1, problem (41) has at least one solution.

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