International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **12(4)**(2024), 27–32
Available Online: http://ijmaa.in

Domination Parameters on Regular Graphs of the Commutative Ring Z_n

D. Sunitha[1,*], A. Swetha[1]

[1]*Academic Consultant, Department of Mathematics, Dravidian University, Kuppam, Andhra Pradesh, India*

Abstract

In 1891 the Danish mathematician Julius Petersen (1839-1910) published a paper on the factorization of regular graphs. This was the first paper in the history of mathematics to contain fundamental results explicitly in graph theory. A regular graph is a graph where each vertex has the same number of neighbors; i.e. every vertex has the same degree or valency. A regular graph with vertices of degree k is called a k-regular graph or regular graph of degree k. In this paper our main aim is to find out the dominating parameters of Regular graph obtained on the Commutative ring of type Z_n. A subset D of V is said to be a dominating set of G if every vertex in $V - D$ is adjacent to a vertex in D. We determine the domination parameters i.e., domination number, dominating set and Minimum domination number of Regular graphs of the commutative ring Z_n.

Keywords: Commutating ring Z_n; regular graph; dominating set; minimum dominating number.

1. Introduction

Graph theory is an important branch of Mathematics. A regular graph is a fundamental concept in graph theory, a branch of mathematics that deals with the study of networks and interconnected structures. In a regular graph, all its vertices (nodes) have the same degree, meaning that every vertex has an equal number of edges connected to it. Mathematically, if a graph is k-regular, it means that every vertex in the graph has a degree of 'k', where 'k' is a non-negative integer. Regular graphs are often used to model various real-world systems where each component (represented by a vertex) interacts or connects with the same number of other components. Domination is an area in graph theory with an extensive research activity. The theory of domination in graphs introduced by Ore [1] and Berge [2] is an emerging area of research in graph theory today. For the first time in 1962, the concepts were entitled 'dominating set and dominating number by Ore. In 1977, E. J. Cockayne and S. T. Hedetniemi [3] conducted a commendable and broad survey on the outcomes of the existing concepts of dominating set in graphs at that time. The notation $\gamma(G)$ for the domination number of a graph was applied by the pair and was accepted widely since then.

*Corresponding author (sunidoddala@gmail.com)

2. Basic Concepts and Preliminaries

Definition 2.1. *A non-empty set R together with two binary operations + and · is called a ring if the following conditions are satisfied.*

1. $(R, +)$ *is an abelian group*
2. $(R, \cdot)$ *is a semi group.*
3. *The operation* · *is distributive over* +, *i.e.,* $a \cdot (b + c) = a \cdot b + a \cdot c$ *and* $(a + b) \cdot c = a \cdot c + b \cdot c$.

Definition 2.2. *The ring* $(R, +, \cdot)$ *is called a commutative ring if it satisfies commutative property i.e.,* $a \cdot b = b \cdot a$ *for all* $a, b \in R$.

Definition 2.3. *A graph G is said to be k-regular, if* $d(v) = k$, *for some positive integer k and for every v in* $V(G)$. *A regular graph is one that is k-regular for some positive integer k.*

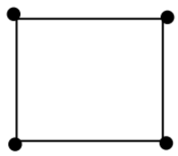

Figure 1: 2-regular

Definition 2.4. *A dominating set D of a graph G is a connected dominating set if the induced sub graph* $\langle V - D \rangle$ *is connected domination number* $\gamma_c(G)$ *of the graph G is the minimum cardinality of the connected dominating set.*

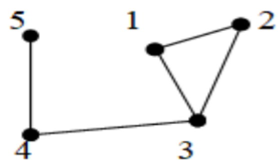

Figure 2: Graph G

Definition 2.5. *A dominating set D of a Graph G is called a minimum dominating set if no proper subset of D is a dominating set.*

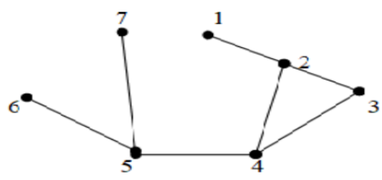

Figure 3: Graph G

In this graph G, the dominating sets are $\{2, 5\}$, $\{1, 4, 5\}$, $\{1, 4, 6, 7\}$. *The minimum dominating set* $= \{2, 5\}$.

3. Main Result

Result 3.1. *Let $R = Z_n$ be the commutative ring of integers modulo n, where $n = 2^r$ $(r \geq 1)$. Let S be the set of all nilpotent elements of R and $G(R)$ be the graph of R. If the edge set of R is defined as $E(R) = \{x, y \in R/x$ and y are adjacent to each other iff $x - y$ or $y - x \in S$ and $x \neq y\}$, then the graph $G(R)$ is a Regular graph.*

Proof. Let $R = Z_n$ be the commutative ring of integers modulo n where $n = 2^r$ $(r \geq 1)$ and let S be the set of all nilpotent elements of R i.e., $S = \{a \in R/a^k = 0$ for some positive integer k and $a \neq 0\}$. Then $S \subseteq R$. Now for any $s \in S$, $s - 0 \in S$, i.e., every element of S is adjacent to 0. Again since $n = 2^r$, i.e., n is even and the set of all nilpotent elements in s are only the even integers of R. Because the elements of S are divisible by the even number n, therefore S contains all non zero even integers of R. Define $E(R) = \{x, y \in R/$ x and y are adjacent to each other iff $x - y$ or $y - x \in S$ and $x \neq y\}$. Let $x, y \in G(R) \Leftrightarrow x$ is adjacent to those elements $y \in R$ such that $x - y$ or $y - x \in S$. Now for $0 \in R$ and for every non zero element $s \in S$, $s - 0$ or $0 - s \in S$. Hence 0 is adjacent to every element of S. Now for any $x \in R$, $x \neq 0$, if x is even (odd) then x is adjacent to some element $y \in R$ such that $x - y$ or $y - x \in S$ then y is also even (odd). So that $x - y \in S$, since the set S of nilpotent elements in R is all non zero even integers. Since the number of elements in R is 2^r, and so there are exactly $2^r/2$ even integers (including). Therefore the number of nilpotent elements in R is equal to $(2^r/2) - 1 = 2^{r-1} - 1$ (Since zero is not a nilpotent element) $\Leftrightarrow$ There are exactly $|S|$ elements adjacent to each $x \in R \Leftrightarrow |V(R)| = |R|$ and every vertex in $G(R)$ is of degree $2^{r-1} - 1 \Leftrightarrow$ Hence $G(R)$ is an $2^{r-1} - 1$-regular graph. □

Illustration 3.2. *Let $R = Z_n$ be a commutative ring of integers modulo n, where $n = 2^r$ $(r \geq 1)$ and let S be the set of all nilpotent elements in R, i.e., $S = \{a \in R/a^k = 0, a \neq 0$ for some positive integer $k\}$:*

Case (i): *If $r = 1$, then $n = 2$ and $R = \{0, 1\} \Rightarrow S = \phi$. Therefore no two elements of R are adjacent and the graph $G(R)$ is a null graph.*

Case(ii): *If $r = 2$, then $n = 2^2$ and $R = \{0, 1, 2, 3\}$ and $S = \{2\}$. Now vertices 0 and 2 are even integers and 1 and 3 are odd integers. Then the difference of 0 and 2, 1 and 3 is in S. Therefore 0 and 2, 1 and 3 are adjacent. $V(R) = \{0, 1, 2, 3\}$ and $E(R) = \{e_1, e_2\}$, where $e_1 = (0, 2)$, $e_2 = (1, 3)$. Hence the graph $G(R)$ is a 1- regular graph.*

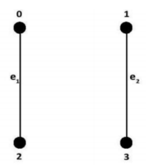

Figure 4: 1-Regular Graph

Case(iii): *If $r = 3$, then $n = 2^3$ and $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $S = \{2, 4, 6\}$. Now vertices 0, 2, 4, 6 are even integers and 1, 3, 5, 7 are odd integers, $V(R) = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and*

$E(R) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}$, where $e_1 = (0,2)$, $e_2 = (0,4)$, $e_3 = (0,6)$, $e_4 = (2,4)$, $e_5 = (2,6)$, $e_6 = (4,6)$, $e_7 = (1,3)$, $e_8 = (1,5)$, $e_9 = (1,7)$, $e_{10} = (3,5)$, $e_{11} = (3,7)$, $e_{12} = (5,7)$. Then $\{0,2,4,6\}$ are adjacent to each other similarly $\{1,3,5,7\}$ are adjacent to each other. Hence the graph $G(R)$ is a 3-regular graph.

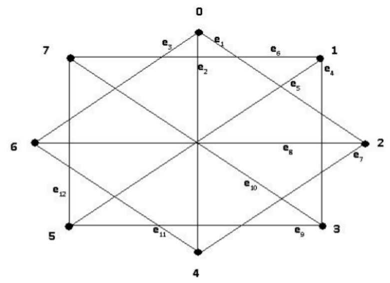

Figure 5: 3-Regular Graph

From the following graph we say that graph $G(R)$ is a 3- regular graph. Continuing like this if $n = 2^r$, where $r = 4, 5, \ldots,$ then the graph $G(R)$ is a $7, 15, \ldots,$ regular graph respectively. The following Table shows the relation between n and $G(R)$

Number of vertices $n = 2^r$ $(r \geq 1)$	$G(R) = (2^{r-1} - 1)$-Regular graph
2^1	Null
2^2	1-Regular graph
2^3	3-Regular graph
2^4	7-Regular graph
2^5	15-Regular graph
2^6	31-Regular graph
.	.
.	.

Table 1:

Conclusion: *From the above illustration we observe that every vertex in $G(R)$ has degree $2^{r-1} - 1$, where $n = 2^r$ $(r \geq 1)$ and $G(R)$ is $(2^{r-1} - 1)$-regular graph.*

Result 3.3. *Let $R = Z_n$ be a commutative ring of integers modulo n, where $n = 2^r$ $(r \geq 1)$. Let S be the set of all nilpotent elements of R. If the graph $G(R)$ with the edge set $E(R) = \{x, y \in R/x \text{ and } y \text{ are adjacent iff } x - y \in S \text{ and } x \neq y\}$, then the domination number of the graph $G(R)$ is two.*

Proof. Let R be the commutative ring of integers modulo n, where $n = 2^r$ $(r \geq 1)$ and let S be the set of all nilpotent elements of R, i.e., $S = \{a \in R/a^k = 0, a \neq 0 \text{ for some positive integer k}\}$. Then $S \subseteq R$. Let $G(R)$ denote the graph of R such that the edge set $E(R)$ of $G(R)$ is defined by, $E(R) = \{x, y \in R/\text{x and y are adjacent iff } x - y \in S \text{ and } x \neq y\}$. Then by the Theorem ? $G(R)$ is a $(2^{r-1} - 1)$-Regular graph. Now we want to find the domination number $\gamma(G)$ of the graph $G(R)$. First we construct the dominating set for the regular graph $G(R)$, where V is the vertex set of $G(R)$. Let D be any non empty sub set of the vertex set V of $G(R)$. Then D will become the dominating set, if every vertex in $V - D$

is adjacent to some vertex in D. Since $n = 2^r$ where $r = 1$, then $|R| = 2^1 = 2$ and the $G(R)$ is a null graph and hence there does not exist dominating set for $G(R)$. If $r \geq 2$, then the graph $G(R)$ is a $(2^{r-1} - 1)$-Regular graph. That is every vertex in $G(R)$ has degree $(2^{r-1} - 1)$. If every vertex in $V - D$ is adjacent to some vertex in D then $|D| \leq 2^{r-1}$. If $|D| = 1$ then $V - D$ consist of $2^r - 1$ elements and has degree $2^{r-1} - 1$ such that every vertex of $V - D$ is not adjacent to the vertex in D. Hence if $|D| = 1$ then D is not a dominating set. So let $|D| = 2$. If $D = \{v_1, v_2\}$ where the v_1 is adjacent to $(2^{r-1} - 1)$ vertices of $G(R)$ and the other vertex v_1 in such a way that v_2 is adjacent to those vertices which are not adjacent to v_1 in $G(R)$. Then every vertex in $V - D$ is adjacent to one vertex in D. Hence D is the dominating set for $G(R)$. Similarly we find the other dominating sets of G Now we want to find the domination number $\gamma(G)$ of the graph $G(R)$. Among all the dominating sets of $G(R)$, $D = \{v_1, v_2\}$ is the minimum dominating set of $G(R)$. So $|D| = 2$. Hence the domination number of $G(R)$ is two, i.e., $\gamma(G) = 2$. □

Illustration 3.4. *Let R be a commutative ring of integers modulo n, where $n = 2^r$ ($r \geq 1$).*
Case (i): *If $r = 1$, then $n = 2^1$ and $R = \{0, 1\}$, then the set of all nilpotent elements in R is empty. Therefore the graph $G(R)$ is a null graph, then there does not exist the dominating set for $G(R)$.*
Case(ii): *If $r = 2$, then $R = \{0, 1, 2, 3\}$ and $S = \{2\}$. Therefore the graph $G(R)$ is a 1-regular graph.*

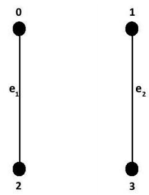

Figure 6: 1-Regular Graph

Let $D = \{0, 1\}$, then $V - D = \{2, 3\}$, $V = \{0, 1, 2, 3\}$, $E(R) = \{e_1, e_2\}$, where $e_1 = (0, 2)$, $e_2 = (1, 3)$ and every vertex of $V - D$ is adjacent to some vertex in D. Therefore $D = \{0, 1\}$ is the dominating set. Similarly we can see that $D = \{1, 2\}$, $\{0, 3\}$ and $\{2, 3\}$ etc are dominating set for $G(R)$. Hence the minimum dominating set of $G(R)$ is one of the set $\{0, 1\}$, $\{1, 2\}$, $\{0, 3\}$ and $\{2, 3\}$. Therefore the domination number $\gamma(G) = 2$.
Case(iii): *If $r = 3$, then $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $S = \{2, 4, 6\}$. Therefore the graph $G(R)$ is a 3-regular graph.*

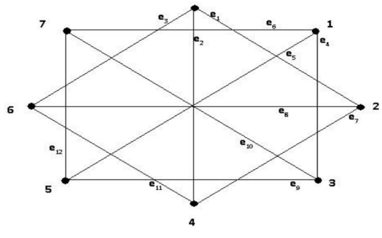

Figure 7: 3-Regular Graph

Let $D = \{0, 1\}$, then $V - D = \{2, 3, 4, 5, 6, 7\}$, $V = \{0, 1, 2, 3, 4, 5, 6, 7\}$,

$E(R) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}\}$, where $e_1 = (0,2)$, $e_2 = (0,4)$, $e_3 = (0,6)$, $e_4 = (1,3)$, $e_5 = (1,5)$, $e_6 = (1,7)$, $e_7 = (2,4)$, $e_8 = (2,6)$, $e_9 = (3,5)$, $e_{10} = (3,7)$, $e_{11} = (4,6)$, $e_{12} = (5,7)$ and every vertex of $V - D$ is adjacent to some vertex in D. Therefore $D = \{0,1\}$ is the dominating set. Similarly we see that $D = \{2,3\}$, $\{5,6\}$ and $\{6,7\}$ etc are dominating set for $G(R)$. It is observed that if D contains at least two consecutive vertices, then D forms a dominating set. Hence the domination number of $G(R)$ is two. Continuing like this if $n = 2^r$ and $r = 4, 5, \ldots$ then the graph $G(R)$ is a $7, 15, \ldots$ Regular graph respectively. In this case also we find that the domination number is two. Therefore the domination number of $G(R)$ is two, i.e., $\gamma(G) = 2$.

Conclusion: *From the above Illustration we observe that the domination number for the regular graph $G(R)$ is two.*

References

[1] O. Ore, *Theory of Graph*, Ann. Math. Sco. Colloq. Publ., 38(1962).

[2] Berge C Holland, *Graphs and Hyperactive Graphs*, North Holland, Amsterdram in Graps, Networks, 10(1980), 211-215.

[3] T. W. Haynes, S. T. Hedetniemi and P. S. Slater, *Domination in Graphs: Advanced Topic*, Marcel Dekker Inc., New York, (1998).

[4] P. B. Bhattacharya, S. K. Jain and Nagpul, *Basic Abstract Algebra*, Plika Press Pvt. Lt Red, New Delhi, (2007).

[5] R. P. Grimaldi, *Graphs from rings*, Proceedings of the 20th South-eastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1989). Congv. Numer. Vol. 71(1990), 95-103.

[6] G. Chartrand and O. R. Oellermann, *Applied and Algorithmic Graph Theory*, McGraw-Hill Inc., New York, (1993).

[7] Frank Harary, *Graph theory*, Narosa publishing house, New Delhi, (2001).

[8] T. W. Haynes, S. T. Hedetniemi and P. S. Slater, *Fundementals of domination in graphs*, Marcel Dekker Inc., New York, (1998).