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Degree of Approximation of Function of Class $Z^{\omega}(\alpha,\gamma)$ by Matrix - Cesaro Means of its Fourier Series

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Abstract

In this paper, we investigate the degree of approximation of a function h belonging to generalized Zygmund class $Z^{\omega}(\alpha,\gamma)$ by Matrix - Cesaro means of its Fourier Series.

Keywords: Summability means; degree of approximation; Zygmund class $Z^{\omega}(\alpha,\gamma)$; Fourier Series.

1. Introduction

Summability theory aims to generalize the concept of the limit of a sequence or series particularly when conventional convergence criteria are inadequate. It explores broder approaches and methods to understands, determine and define the behaviour and convergence of series and sequence beyond traditional limits. The degree of approximation of function belonging to different Lipschitzs class such as $Lip\alpha$, $Lip(\alpha,p)$, $Lip(\xi(t),p)$ and $W(L_p,\xi(t))$, Holder space and different Zygmund class by using single and different double summability means has been discussed by many reasearcher. In the recent past various product summability techniques have been used to obtained better and generalized results. A lots of researcher like Khan [2], Chandra [1], Mittal [7], Lal [4], Lal and Mishra [5] investigated the degree of approximation of function belonging to $Lip(\alpha,\gamma)$ by applying various summability means of its Fourier Series. Recently kim [3] and Sinha [10] obtained the degree of approximation of function of class $Z^{\omega}(\alpha,\gamma)$ by Cesaro means and (N p q) C_1 means respectively. In this paper we obtained the degree of approximation of function of class $z^{\omega}(\alpha,\gamma)$ by Matrix - Cesaro means of its Fourier Series.

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2. Definition and Notations

Let $\sum_{k=0}^{\infty} u_k$ be the infinite series and its n^th^ partial sum $\{s_n\}$ is given by

$$s_n = \sum_{k=0}^{n} u_k$$

Then Cesaro means (C1) of sequence $\{s_n\}$ is given by

$$\sigma_{n=\frac{1}{n+1}} \sum_{k=0}^{n} s_k \tag{1}$$

if $\sigma_n \to$ s as $n \to \infty$ then sequence $\{s_n\}$ or Infinite series $\sum_{k=0}^{\infty} u_k$ is said to be summable by Cesaro means (C 1) to finite number s. Let $T = (a_{nk})$ be an infinite lower triangular matrix satisfying the Silverman Toeplitz conditions of regularity

- $\sum_{k=0}^{n} a_{nk} \to 1$ as $n \to \infty$
- $a_{nk} = 0$ for $k > n$
- $\sum_{k=0}^{\infty} |a_{nk}| \leq N, N$ is finite constant

Matrix means of sequence $\{s_n\}$ is given by

$$t_n^T = \sum_{k=0}^{n} a_{n,n-k} s_{n-k} \tag{2}$$

If $t_n^T \to s$ as $n \to \infty$ then sequence $\{s_n\}$ or Infinite series $\sum_{k=0}^{\infty} u_k$ is said to be summable by Matrix means to finite number s. If Matrix means is superimposed on Cesaro means then Matrix - Cesaro means TC_1 is obtained and TC_1 means of the sequence $\{s_n\}$ is given by

$$t_n^{TC_1} = \sum_{k=0}^{n} a_{n(n-k)} \sigma_{n-k} = \sum_{k=0}^{n} a_{n,n-k} \frac{1}{(n-k+1)} \sum_{v=0}^{n-k} s_v \tag{3}$$

If $t_n^{TC_1} \to s$ as $n \to \infty$ then sequence $\{s_n\}$ or Infinite series $\sum_{k=0}^{\infty} u_k$ is said to be summable by Matrix - Cesaro means to s. Let h be the 2π periodic function integrable in the sense of Lebesgue then the Fourier series of h at any point x is given by

$$h(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{4}$$

The degree of approximation $E_n(h)$ of a function $h : R \to R$ by trignometric polynomial P_n of order n is defined by $\|P_n - h\| = \sup\{|P_n(x) - h(x)| : x \in R\}$ and the degree of approximation $E_n(h)$ is defined by

$$E_n(h) = \min \|P_n - h\|_n \tag{5}$$

where P_n is trigonometric polynomial of degree n.

Zygmund class Z is defined as

$$Z=\{h\in[-\pi,\pi]:|h(x+t)+h(x-t)-2h(x)|=O(|t|)\} \tag{6}$$

In this paper we introduce generalized Zygmund class $Z^{\omega}(\alpha,\gamma)$, defined as

$$Z^{\omega}(\alpha,\gamma)=\left\{h\in[-\pi,\pi]:\left(\int_{-\pi}^{\pi}|h(x+t)+h(x-t)-2h(x)|^{\gamma}dx)^{1/\gamma}=O(|t|^{\alpha}\omega\,|t|\right)\right\} \tag{7}$$

where $\alpha\geq 0$, $\gamma\geq 1$ and ω is continuous non - negative and non- decreasing function. If we take $\alpha=1$, $\omega=$constant and $\gamma\to\infty$ then $Z^{\omega}(\alpha,\gamma)$ class reduces to Zygmund class. Throughout this paper we use the following notations

$$\emptyset_x(t)=h(x+t)+h(x-t)-2h(x) \tag{8}$$

$$K_n(t)=\frac{1}{2\pi}\sum_{k=0}^{n}\frac{a_{n,n-k}}{(n-k+1)}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}} \tag{9}$$

3. Known Theorem

Jaeman Kim [3] has studied the degree of approximation of function of belonging to $Z^{\omega}(\alpha,\gamma)$ by Cesaro means of Fourier series and proved the following theorem.

Theorem 3.1. *Let f be a 2π periodic continuous belonging to $Z^{\omega}(\alpha,\gamma)$ then the degree of approximation of f by Cesaro means of it Fourier series is given by*

$$\|\sigma_n-f\|_{\gamma}=O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right) \tag{10}$$

Sinha [10] Find the degree of approximation of function of class $Z^{\omega}(\alpha,\gamma)$ by $(N,p,q)C_1$ means of Fourier series and established following result.

Theorem 3.2. *Let f be a 2π periodic continuous belonging to $Z^{\omega}(\alpha,\gamma)$ then the degree of approximation of f by $(N,p,q)C_1$ means of its Fourier series is given by*

$$\left\|t_n^{p,q,c_1}-f\right\|_{\gamma}=O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right) \tag{11}$$

provided $\{p_n\}$ and $\{q_n\}$ are two sequence of positive real constant of regular Norlund method (N,p,q) such that

$$\sum_{k=0}^{n}\frac{p_kq_{n-k}}{n-k+1}=O\left(\frac{R_n}{n+1}\right)\ \ \forall\ n\geq 0$$

4. Main Theorem

In this paper we prove the following theorem.

Theorem 4.1. *Let h be a 2π periodic continuous belonging to $4Z^{\omega}(\alpha,\gamma)$ then the degree of approximation of h by Matrix - Cesaro means of its Fourier series is given by*

$$\left\|t_n^{TC_1} - h\right\|_{\gamma} = O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right)$$

5. Required Lemmas

To prove the main theorem following lemmas are required

Lemma 5.1. *For $0 < t \leq \frac{1}{n+1}$, we have*

$$|K_n(t)| = O(n+1) \tag{12}$$

Proof. For $0 < t \leq \frac{1}{n+1}$, we have

$$\begin{aligned}
\sin(n+1)\frac{t}{2} &\leq (n+1)\sin\frac{t}{2}\\
|K_n(t)| &= \left|\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{1}{(n-k+1)}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\right|\\
&\leq \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{1}{(n-k+1)}\frac{(n-k+1)^2\sin^2\frac{t}{2}}{\sin^2\frac{t}{2}}\\
&= \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}(n-k+1)\\
&= \frac{(n+1)}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\\
&= \frac{(n+1)}{2\pi}\\
&= O(n+1)
\end{aligned}$$

□

Lemma 5.2. *For $\frac{1}{n+1} < t \leq \pi$*

$$|K_n(t)| = O\left(\frac{1}{(n+1)\,t^2}\right) \tag{13}$$

Proof. For $\frac{1}{n+1} < t \leq \pi$, we have $\frac{t}{\pi} \leq \sin\frac{t}{2}$ and $\sin(n+1)\frac{t}{2} \leq 1$

$$\begin{aligned}
|K_n(t)| &= \left|\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{1}{(n-k+1)}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\right|\\
&\leq \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{1}{(n-k+1)}\frac{\pi^2}{t^2} \quad \text{[By Jordan lemma]}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{2t^2}\sum_{k=0}^{n} a_{n,n-k}\frac{1}{(n-k+1)}\\
&= \frac{\pi}{2t^2}\frac{1}{(n+1)}\sum_{k=0}^{n} a_{n,n-k}\\
&= \frac{\pi}{2t^2}\frac{1}{(n+1)}\\
&= O\left(\frac{1}{(n+1)t^2}\right)
\end{aligned}$$

□

6. Proof of the Main Theorem

Proof. The n$^{\text{th}}$ partial sum $\{s_n\}$ of Fourier Series is given by

$$s_n(x) - h(x) = \frac{1}{2\pi}\int_0^{\pi} \varnothing(t)\,\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt \tag{14}$$

(C 1) transform of $\{s_n\}$ is given by

$$\frac{1}{(n+1)}\sum_{k=0}^{n}\{s_n(x)-h(x)\} = \frac{1}{2\pi(n+1)}\int_0^{\pi}\varnothing(t)\left(\sum_{k=0}^{n}\frac{\sin(k+\frac{1}{2})t}{\sin\frac{t}{2}}\right)dt$$

$$\sigma_n(x) - h(x) = \frac{1}{2\pi(n+1)}\int_0^{\pi}\varnothing(t)\frac{sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}}dt \tag{15}$$

The Matrix - Cesaro transform TC_1 of $\{s_n\}$ is given by

$$\sum_{k=0}^{n} a_{n,n-k}(\sigma_{n-k}(x) - h(x)) = \frac{1}{2\pi}\int_0^{\pi}\varnothing(t)\left[\sum_{k=0}^{n}\frac{a_{n,n-k}}{(n-k+1)}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\right]dt \tag{16}$$

Then

$$t_n^{TC_1}(x) - h(x) = \int_0^{\pi} K_n(t)\varnothing(t)\,dt \tag{17}$$

Using generalized Minkowski inequality for above integral we have

$$\begin{aligned}
\left\|t_n^{TC_1} - h\right\|_{\gamma} &= \left(\int_{-\pi}^{\pi}\left|t_n^{TC_1}(x) - h(x)\right|^{\gamma}dx\right)^{1/\gamma}\\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\left|\int_0^{\pi}\varnothing(x,t)\,K_n(t)dt\right|^{\gamma}dx\right)^{1/\gamma}\\
&= \frac{1}{2\pi}\int_0^{\pi}\left(\int_{-\pi}^{\pi}|\varnothing(x,t)\,dx|^{\gamma}\right)^{\frac{1}{\gamma}}|K_n(t)|\,dt\\
&= \frac{1}{2\pi}\int_0^{\pi}O(t^{\alpha}\omega(t))\,|K_n(t)|\,dt\\
&= O\left(\int_0^{\frac{1}{n+1}}(t^{\alpha}\omega(t))\,|K_n(t)|\,dt\right) + O\int_{\frac{1}{n+1}}^{\pi}(t^{\alpha}\omega(t))\,|K_n(t)|\,dt\\
&= J_1 + J_2
\end{aligned} \tag{18}$$

$$J_1 = O\left(\int_0^{\frac{1}{n+1}} t^{\alpha}\omega\left(t\right)\ \left|K_n(t)\right| dt\right) \tag{19}$$

By Lemma 5.1 we have

$$J_1 = O\left(\int_0^{\frac{1}{n+1}} \left(t^{\alpha}\omega\left(t\right)\right)\left(n+1\right) dt\right)$$

since $t^{\alpha}\omega\left(t\right)$ is non negative increasing function using mean value theorem for integral we obtained

$$\begin{aligned} J_1 &= O\left(\left(\frac{1}{n+1}\right)^{\alpha}\omega\left(\frac{1}{n+1}\right)\int_0^{\frac{1}{n+1}}(n+1)dt\right) \\ &= O\left(\left(\frac{1}{n+1}\right)^{\alpha}\omega\left(\frac{1}{n+1}\right)(n+1)\frac{1}{(n+1)}\right) \\ &= O\left(\left(\frac{1}{n+1}\right)^{\alpha}\omega\left(\frac{1}{n+1}\right)\right) \end{aligned}$$

Again, using non negative increasing function $t^{\alpha}\omega\left(t\right)$ we have

$$\left(\left(\frac{1}{n+1}\right)^{\alpha}\omega\left(\frac{1}{n+1}\right)\right) \leq \frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)$$

So J_1 can be written as

$$J_1 = O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right) \tag{20}$$

Now $J_2 = O\left(\int_{\frac{1}{n+1}}^{\pi} t^{\alpha}\omega\left(t\right)\left|K_n(t)\right| dt\right)$. By Lemma 5.2, we have

$$\begin{aligned} J_2 &= O\left(\int_{\frac{1}{n+1}}^{\pi}\left(t^{\alpha}\omega\left(t\right)\right)\frac{1}{t^2(n+1)}dt\right) \\ &= O\left(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi}\left(t^{\alpha-2}\omega\left(t\right)\right)\right)dt \end{aligned} \tag{21}$$

Putting $t = \frac{1}{u} \Rightarrow dt = \frac{-1}{u^2}du$, when, $t = \pi$, $u = \frac{1}{\pi}$ and $t = \frac{1}{n+1} \Rightarrow u = n+1$

$$\begin{aligned} J_2 &= O\left(\frac{1}{n+1}\int_{n+1}^{\pi}\left(\frac{1}{u}\right)^{\alpha-2}\left(\frac{-1}{u^2}\right)\omega\left(\frac{1}{u}\right)du\right) \\ &= O\left(\frac{1}{n+1}\int_{\pi}^{n+1}u^{-\alpha}\omega\left(\frac{1}{u}\right)du\right) \end{aligned} \tag{22}$$

Since $t^{\alpha}\omega\left(t\right)$ is non negative increasing function $\left(\frac{1}{u}\right)^{\alpha}\omega\left(\frac{1}{u}\right)$ is now a decreasing function hence we have,

$$O\left(\frac{1}{n+1}\int_{\pi}^{n+1}u^{-\alpha}\omega(\frac{1}{u})du\right) = O\left(\frac{1}{n+1}\pi^{\alpha}\omega\left(\pi\right) + \frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right)$$

It follows from the continuity of $t^{\alpha}\omega(t)$ we can choose a positive constant M such that $M(1^{\alpha}\omega(1)) \geq \pi^{\alpha}\omega(\pi)$. By virtue of a non- negative property of $t^{\alpha}\omega(t)$ we have

$$\begin{aligned} O\left(\frac{1}{n+1}\pi^{\alpha}\omega(\pi)+\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right) &= O\left(\frac{1}{n+1}M\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)+\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right) \\ &= O\left(\frac{M+1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right) \\ &= O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right) \end{aligned}$$

J_2 can be written as

$$J_2 = O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right) \tag{23}$$

Therefore combining J_1, J_2 and (18) we obtain

$$\begin{aligned} \left\|t_n^{TC_1}-h\right\|_{\gamma} &= O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right)+O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right) \\ &= O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right) \end{aligned}$$

This completes the proof of our main theorem. □

As a consequence, the following corollary can be derived from our main theorem.

Corollary 6.1. *Let h be a 2π periodic continuous function belonging to $Z^{\omega}(\alpha,\gamma)$ with $\omega(t) = t^m(m>1)$,then the degree of approximation of h by Matrix - Cesaro means of its Fourier series is given by $\left\|t_n^{TC_1}-h\right\|_{\gamma} = O\left(\frac{1}{n+1}\right)$.*

Proof. We have from the theorem

$$\left\|t_n^{TC_1}-h\right\|_{\gamma} = O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\omega\left(\frac{1}{k}\right)\right)$$

putting $\omega(t) = t^m(m>1)$ we obtain

$$= O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{\alpha}\left(\frac{1}{k}\right)^{m}\right) \tag{24}$$

$$= O\left(\frac{1}{n+1}\sum_{k=1}^{n+1}\left(\frac{1}{k}\right)^{m+\alpha}\right)$$

$$= O\left(\frac{1}{n+1}\right) \tag{25}$$

This completes the proof of corollary. □

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