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Two Modified Iterative Methods for Solving Linear Systems by M-matrix

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Abstract

In the present work, to provide the preconditioning effect on each row of the coefficient matrix of a linear system, we consider a new preconditioner $(I + C + P_2)$ which is formed by combining the preconditioners $(I + C)$ [3] and $(I + P_2)$ [2] and applied to two classical iterative methods namely Jacobi method and Gauss-Seidel method. We provide some comparison theorems and numerical examples to illustrate the efficiency of the two new methods. The comparison theorems and numerical experiments show that both the new methods are better than the respective classical iterative methods and finally the results also show that the new Gauss-Seidel method is faster than the new Jacobi iterative method.

Keywords: Preconditioned linear system, M-matrix, Comparison theorem, Modified iterative method.

1. Introduction

We consider the following preconditioned linear system

$$PAx = Pb$$

Where, $A = (a_{ij})_{n\times n}$ is a known nonsingular M-matrix, P is called a preconditioner, b is a known $n \times 1$ and x is an unknown $n \times 1$ vectors. Throughout the present paper, without loss of generality, we always assume that the coefficient matrix A has a splitting of the form $A = I - L - U$; where, I is the identity matrix, L and U are strictly lower triangular and strictly upper triangular matrices obtained from A, respectively. To improve the convergence rate of the iterative methods, many researchers proposed different modified iterative methods with different preconditioners. Some of the preconditioners are stated below:

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In 1991, Gunawardena et al. [1] first proposed the preconditioner $P_s = I + S$, where S is defined as:

$$S = (s_{ij}) = \begin{cases} -a_{i,i+1}, & 1 \le i \le n-1 \\ 0, & \text{otherwise} \end{cases}$$

In 2001, D. J. Evans et al. [2] proposed the preconditioners $P = I + P_1$ and $= I + P_2$; where P_1 and P_2 are defined as:

$$P_1 = (p_{nj}) = \begin{cases} -a_{nj}, & j = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$P_2 = (p_{in}) = \begin{cases} -a_{in}, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

In 1987, J. P. Milaszewicz [3] proposed the preconditioner $P_c = I + C$, where C is defined as:

$$C = (c_{ij}) = \begin{cases} -a_{ij}, & 2 \le i \le n, \ j = 1 \\ 0, & \text{otherwise} \end{cases}$$

In 2003, Morimoto et al. [4] proposed a preconditioned method with the preconditioner $P_R = I + R$, where R is defined by

$$R = (r_{ij}) = \begin{cases} -a_{ij}, & i = n, \ 1 \le j \le n-1 \\ 0, & \text{otherwise} \end{cases}$$

In 2008, Niki et al. [5] considered a preconditioner $P = I + S + R$, where S and R are defined above. In 2000, Jing-yu ZHAO et al. [6] considered a preconditioned Gauss-Seidel method with the preconditioner $P = I + S + P_1$, where S and P_1 are mentioned above. In 2002, Kotakemori et al. [7] proposed the preconditioner P_m as follows:

$$P_m = I + S_{\max}, \text{where } S_{\max} \text{ is defined as}$$

$$S_{\max} = (s_{ij}^m) = \begin{cases} -a_{i,k_i}, & i = 1, 2, \cdots, n-1, \ j > i \\ 0, & \text{otherwise} \end{cases}$$

$$k_i = \min\left\{ j : \max_j |a_{ij}|, \ i < n \right\}.$$

In 2004, M. Morimoto et al. [8] considered a preconditioner P_{sm} as follows:

$$P_{sm} = I + S + S_m$$

where S is mentioned above and S_m is defined as

$$S_m = ((s_m)_{ij}) = \begin{cases} -a_{i,k_i}, & i = 1, 2, \cdots, n-2, \quad j > i+1 \\ 0, & \text{otherwise} \end{cases}$$

$$k_i = \min\left\{ j : \max_j |a_{ij}|, \; i < n-1, \; j > i+1 \right\}.$$

In 2009, Bing Zheng et al. [9] considered the two preconditioned Gauss-Seidel methods with the preconditioners $P_{\max} = I + S_{\max} + R_{\max}$ and $P_R = I + S_{\max} + R$, where $R_{\max}$ is defined as

$$R_{\max} = (r_{ij}^m) = \begin{cases} -a_{n,k_n}, & i = n, \; j = k_n \\ 0, & \text{otherwise} \end{cases}$$

and $k_n = \min\{j : |a_{n,j}| = \max\{|a_{n,l}|, \, l = 1, 2, \cdots, n-1\}\}$. In 2013, Zhouji Chen [10] proposed the preconditioner $P_{s\max}$ as follows:

$$P_{s\max} = I + S + S_m + R_{\max},$$

where S, S_m and $R_{\max}$ are defined above and so on. For our convenience, we use index G for solving the linear systems by preconditioned Gauss-Seidel method and for solving by preconditioned Jacobi method, we use index J. For the preconditioner $P_{c2} = I + C + P_2$, we have $A_{c2} = P_{c2}A = (I + C + P_2)\,A$ which can be split as

$$\begin{aligned} A_{c2_G} &= M_{c2_G} - N_{c2_G} \\ &= \left(I - L + C - D' - E' - D''\right) - (U - P_2 + F' + F'') \\ A_{c2_J} &= M_{c2_J} - N_{c2_J} \\ &= \left(I - D' - D''\right) - \left(L - C + E' + U - P_2 + F' + F''\right) \end{aligned}$$

where D', E' and F' are the diagonal, strictly lower and strictly upper triangular parts of CU, D'' and F'' are the diagonal and strictly upper triangular parts of P_2L, respectively and also $CL = 0, P_2U = 0$. It can be notice that when $0 < a_{ij}a_{ji} < 1$ for $i = 1$ and $j = 2, 3, \cdots, n$; then both $M_{c2_J}^{-1}$ and $M_{c2_G}^{-1}$ are well defined. Then the modified Jacobi and the modified Gauss-Seidel iteration matrices for A_{c2_J} and A_{c2_G} are respectively defined as $T_{c2_J} = M_{c2_J}^{-1} N_{c2_J}$ and $T_{c2_G} = M_{c2_G}^{-1} N_{c2_G}$. We organize the remaining portion of the paper as follows: Section 2 is the preliminaries. We discuss the convergence property and some comparison theorems of the proposed methods in section 3. Three simple numerical examples are given in section 4 to verify our theoretical analysis. At the end, in section 5, conclusion is drawn.

2. Preliminary Notes

Suppose $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$, then we can say that $A \geq B$ if $a_{ij} \geq b_{ij}$ holds for all $i, j = 1, 2, \cdots, n$ and $A \geq 0$ (called nonnegative) if $a_{ij} \geq 0$ for all $i, j = 1, 2, \cdots, n$, where 0 is an $n \times n$ zero matrix. For the $n \times 1$ vectors $a, b; a \geq b$ and $a \geq 0$ can also be defined in the similar manner.

Definition 2.1. *Let $A = (a_{ij})$ be an $n \times n$ matrix. Then the maximum of the module of the eigenvalues of A is called the spectral radius of A and is denoted by $\rho(A)$ i.e.*

$$\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

Definition 2.2. *Let A be a real matrix. Then the representation $A = M - N$ is called a splitting of A if M is a nonsingular matrix. The splitting is called*

(1) convergent if $\rho\left(M^{-1}N\right) < 1$;

(2) regular if $M^{-1} \geq 0$ and $N \geq 0$;

(3) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$;

(4) nonnegative if $M^{-1}N \geq 0$;

(5) M-splitting if M is a nonsingular M-matrix and $N \geq 0$.

Definition 2.3. *The splitting $A = M - N$ is called the Jacobi splitting of A if $M = I$ is nonsingular and $N = L + U$. In addition, the splitting is called*

(1) Jacobi convergent if $\rho\left(M^{-1}N\right) < 1$;

(2) Jacobi regular if $M^{-1} = I^{-1} \geq 0$ and $N = (L + U) \geq 0$.

Definition 2.4. *A splitting of matrix A i.e., $A = M - N$ is called a Gauss-Seidel splitting if $M = I - L$ is nonsingular and $N = U$. In addition, the splitting is called*

(1) Gauss-Seidel convergent if $\rho\left(M^{-1}N\right) < 1$;

(2) Gauss-Seidel regular if $M^{-1} = (I - L)^{-1} \geq 0$ and $N = U \geq 0$;

(3) Gauss-Seidel weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

Definition 2.5 ([11]). *A matrix $A = (a_{ij})_{n\times n}$ is an L-matrix if $a_{ii} > 0$, $1 \leq i \leq n$ and $a_{ij} \leq 0$; $1 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j$. A nonsingular L-matrix A is said to be a nonsingular M-matrix if $A^{-1} \geq 0$.*

Definition 2.6. *A Z-matrix A is called an M-matrx, if all the diagonal entries of A are positive, all the real eigenvalues of A are positive and the real part of any eigenvalue of A is positive.*

Definition 2.7. *Preconditioning is a procedure that transforms a given system into a form which is much more suitable for numerical solution and it is mainly related to reduce the condition number of the problem or to reduce the spectral radius of the iteration matrix.*

Lemma 2.8 ([12])**.** *Let $A = M - N$ be an M-splitting of A. Then $\rho\left(M^{-1}N\right) < 1$ if and only if A is a nonsingular M-matrix.*

Lemma 2.9 ([13])**.** *Let A be a nonsingular M-matrix and let $A = M_1 - N_1 = M_2 - N_2$ be the two convergence splitting, the first one weak regular and second one regular if $M_1^{-1} \geq M_2^{-1}$, then*

$$\rho\left(M_1^{-1}N_1\right) \leq \rho\left(M_2^{-1}N_2\right) < 1.$$

Lemma 2.10 ([14])**.** *Let A be a nonsingular L-matrix. Then A is called a nonsingular M-matrix if and only if there exists a positive vector y such that $Ay > 0$.*

Lemma 2.11 ([12])**.** *Let A be irreducible, $A = M - N$ be an M-splitting. Then there is a positive vector x such that $M^{-1}Nx = \rho\left(M^{-1}N\right)x$ and $\rho\left(M^{-1}N\right) > 0$.*

Lemma 2.12 ([15])**.** *Let $A = (a_{ij}) \in R^{n\times n}$ be an irreducible M-matrix with $a_{i,i+1} \neq 0$ for $1 \leq i \leq n-1$, and let $A_s = (I+S)A = M_s - N_s$ be the Gauss-Seidel splitting of A_s. Then $M_s^{-1}N_s$ has a positive perron vector and $\rho\left(M_s^{-1}N_s\right) > 0$.*

Lemma 2.13 ([1])**.** *Let A be a nonnegative matrix. Then*

(1) If $\alpha x \leq Ax$ for some nonnegative vector x, $x \neq 0$, then $\alpha \leq \rho(A)$.

(2) If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is a positive vector.

Lemma 2.14 ([16])**.** *Let A be an M-matrix and let $A_s = (I+S)A = M_s - N_s$ be the Gauss-Seidel splitting of A_s. If $\rho\left(M_s^{-1}N_s\right) > 0$, then $Ax \geq 0$ for any nonnegative perron vector of $M_s^{-1}N_s$.*

Lemma 2.15 ([17])**.** *Let A be a nonnegative nxn nonzero matrix, then*

(1) $\rho(A)$, the spectral radius of A, is an eigenvalue;

(2) A has a nonnegative eigenvector corresponding to $\rho(A)$;

(3) $\rho(A)$ is a simple eigenvalue of A;

(4) $\rho(A)$ increases when any entry of A increases.

3. Convergence Property and Comparison Theorems

In this section, we establish some comparison theorems for the preconditioned Jacobi and the preconditioned Gauss-Seidel methods with the preconditioner $P_{c2} = I + C + P_2$ and also discuss the convergence property of the two preconditioned methods. We also assume that, $A = (a_{ij})_{n\times n}$ is a nonsingular M-matrix with $a_{n,1} \neq 0$ and $a_{i,i+1} \neq 0$ for $1 \leq i \leq n-1$.

Theorem 3.1. *Let A be a nonsingular M-matrix and we assume that $0 < a_{i,j}a_{j,i} < 1$ for $i = 1$ and $j = 2, 3, \cdots, n$. Then $A_{c2_J} = M_{c2_J} - N_{c2_J}$ is a regular and Jacobi convergent splitting.*

Proof. We observe that when $0 < a_{i,j}a_{j,i} < 1$, $i = 1$, $j = 2, 3, \cdots, n$; then the diagonal elements of A_{c2_J} are positive and ${M_{c2_J}}^{-1}$ is well defined. It is well known by Lemma 2.10. (see [14]) that, an L-matrix A is a nonsingular M-matrix if and only if there exists a positive vector y such that $Ay > 0$. By taking such y, the fact that $I + C + P_2 \geq 0$ gives $A_{c2_J}y = (I + C + P_2)\, Ay > 0$ and hence the L-matrix A_{c2_J} is a nonsingular M-matrix which means that ${A_{c2_J}}^{-1} \geq 0$. Thus by Lemma 2.8, we have $\rho({M_{c2_J}}^{-1}N_{c2_J}) < 1$ i.e. $\rho(T_{c2_J}) < 1$.
Again, we notice that when $0 < a_{i,j}a_{j,i} < 1$, $i = 1$, $j = 2, 3, \cdots, n$; we have $D' + D'' < I$ so that $(I - D' - D'') \geq 0$. Therefore

$$\begin{aligned}
{M_{c2_J}}^{-1} &= (I - D' - D'')^{-1} \\
&= [I - (D' + D'')]^{-1} \\
&= I + \left(D' + D''\right) + (D' + D'')^2 + \cdots \geq 0
\end{aligned}$$

and $N_{c2_J} = L - C + E' + U - P_2 + F' + F'' \geq 0$, since $L \geq C \geq 0$, $U \geq P_2 \geq 0$ and $E' + F' + F'' \geq 0$. Hence $A_{c2_J} = M_{c2_J} - N_{c2_J}$ is a regular Jacobi convergent splitting by Definition 2.3 and Lemma 2.8. □

Theorem 3.2. *Let A be a nonsingular M-matrix. Then under the assumption of the Theorem 3.1, the following inequality holds:*

$$\rho(T_{c2_J}) \leq \rho(T_J) < 1$$

Proof. From Theorem 3.1, we know that $A_{c2_J} = P_{c2}A = M_{c2_J} - N_{c2_J}$, where $M_{c2_J} = I - D' - D''$ and $N_{c2_J} = L - C + E' + U - P_2 + F' + F''$, is a regular Jacobi convergent splitting. On the other hand, the iteration matrix of the classical Jacobi method for A is $T_J = I^{-1}(L + U)$. Since A is a nonsingular M-matrix and hence the classical Jacobi splitting $A = M_J - N_J = I - (L + U)$ of A is clearly regular and convergent i.e., $\rho(T_J) < 1$.
Now, let us consider the following splitting of A:

$$A_{c2_J} = P_{c2}A = (I + C + P_2)A = M_{c2_J} - N_{c2_J}$$

Then

$$A = (I + C + P_2)^{-1} M_{c2_J} - (I + C + P_2)^{-1} N_{c2_J}$$

We let, $M_1 = (I + C + P_2)^{-1} M_{c2_J}$ and $N_1 = (I + C + P_2)^{-1} N_{c2_J}$. Then, it can be easily seen that ${M_1}^{-1} N_1 = {M_{c2_J}}^{-1} N_{c2_J}$ and hence $\rho({M_1}^{-1} N_1) < 1$.

Again, we note that

$$\begin{aligned}
{M_1}^{-1} &= {M_{c2_J}}^{-1} (I + C + P_2) \\
&= \left(I - D' - D''\right)^{-1} (I + C + P_2) \\
&\geq (I - D' - D'')^{-1} I \qquad \text{[Since } I + C + P_2 \geq I\text{]} \\
&= [I - \left(D' + D''\right)]^{-1} \\
&\geq I^{-1} \\
&= {M_J}^{-1}
\end{aligned}$$

i.e., ${M_1}^{-1} \geq {M_J}^{-1}$ and also $A = M_1 - N_1 = M_J - N_J$ are the two convergent and regular splitting. Therefore, it is obvious from Lemma 2.9, that

$$\rho({M_1}^{-1} N_1) \leq \rho({M_J}^{-1} N_J) < 1$$

Thus, $\rho({M_{c2_J}}^{-1} N_{c2_J}) \leq \rho({M_J}^{-1} N_J) < 1$ i.e., $\rho(T_{c2_J}) \leq \rho(T_J) < 1$. □

Theorem 3.3. *Let A be a nonsingular M-matrix. Then under the assumption of the Theorem 3.1, $A_{c2_G} = M_{c2_G} - N_{c2_G}$ is a regular and Gauss-Seidel convergent splitting.*

Proof. Under the assumption of the Theorem 3.1, the diagonal elements of A_{c2_G} are positive and thus ${M_{c2_G}}^{-1}$ is well defined. Using Lemma 2.10, (see [14]) that an L-matrix A is a nonsingular M-matrix if and only if there exists a positive vector y such that $Ay > 0$. By taking such y, the fact that $I + C + P_2 \geq 0$ gives $A_{c2_G} y = (I + C + P_2)\, Ay > 0$. Thus the L-matrix A_{c2_G} is a nonsingular M-matrix i.e., ${A_{c2_G}}^{-1} \geq 0$ and so by Lemma 2.8, we get $\rho({M_{c2_G}}^{-1} N_{c2_G}) < 1$ i.e., $\rho(T_{c2_G}) < 1$.

Clearly, $L + E' - C \geq 0$, since $L \geq C \geq 0$. Under the assumption of the Theorem 3.1, we have $D' + D'' < I$ so that $(I - D' - D'') \geq 0$. Hence

$$\begin{aligned}
{M_{c2_G}}^{-1} &= (I - L + C - D' - E' - D'')^{-1} \\
&= \Big[(I - D' - D'') - (L + E' - C)\Big]^{-1} \\
&= \Big[I - (I - D' - D'')^{-1} (L + E' - C)\Big]^{-1} \left(I - D' - D''\right)^{-1} \\
&= \Big\{ I + \Big[\left(I - D' - D''\right)^{-1} (L + E' - C)\Big] + \Big[\left(I - D' - D''\right)^{-1} (L + E' - C)\Big]^2 + \cdots
\end{aligned}$$

$$\cdots + \left[(I-D'-D'')^{-1}(L+E'-C)\right]^{n-1}\Big\}(I-D'-D'')^{-1} \geq 0$$

and $N_{c2_G} = U - P_2 + F' + F'' \geq 0$, since $U \geq P_2 \geq 0$ and $F' + F'' \geq 0$. Therefore, it follows from Definition 2.4 and Lemma 2.8, that $A_{c2_G} = M_{c2_G} - N_{c2_G}$ is a regular and Gauss-Seidel convergent splitting. □

Theorem 3.4. *Let A be a nonsingular M-matrix. Then under the assumption of the Theorem 3.1, the following inequality holds:*

$$\rho(T_{c2_G}) \leq \rho(T_G) < 1$$

Proof. For the M-matrix A, the iteration matrix of the classical Gauss-Seidel method is $T_G = (I-L)^{-1}U$. As A is a nonsingular M-matrix and so the classical Gauss-Seidel splitting $A = M_G - N_G = (I-L) - U$ of A is clearly regular and convergent i.e., $\rho(T_G) < 1$. From Theorem 3.3, we get, $A_{c2_G} = P_{c2}A = M_{c2_G} - N_{c2_G}$, where $M_{c2_G} = I - L + C - D' - E' - D''$ and $N_{c2_G} = U - P_2 + F' + F''$ is the regular Gauss-Seidel convergent splitting i.e., $\rho\left(M_{c2_G}{}^{-1}N_{c2_G}\right) < 1$ i.e.m $\rho\left(T_{c2_G}\right) < 1$.
Now, let us consider the following splitting of A:

$$A_{c2_G} = P_{c2}A = (I + C + P_2)A = M_{c2_G} - N_{c2_G}$$

Or,

$$A = (I + C + P_2)^{-1}M_{c2_G} - (I + C + P_2)^{-1}N_{c2_G}$$

We assume, $M_2 = (I + C + P_2)^{-1}M_{c2_G}$ and $N_2 = (I + C + P_2)^{-1}N_{c2_G}$. One can easily verify that $M_2{}^{-1}N_2 = M_{c2_G}{}^{-1}N_{c2_G}$ and thus $\rho(M_2{}^{-1}N_2) < 1$. Also, we can notice that

$$\begin{aligned}
M_2{}^{-1} &= M_{c2_G}{}^{-1}(I + C + P_2) \\
&= (I - L + C - D' - E' - D'')^{-1}(I + C + P_2) \\
&= \left[(I - D' - D'') - (L + E' - C)\right]^{-1}(I + C + P_2) \\
&= \left[I - (I - D' - D'')^{-1}(L + E' - C)\right]^{-1}(I + C + P_2)(I - D' - D'')^{-1} \\
&\geq \left[I - (I - D' - D'')^{-1}(L + E' - C)\right]^{-1}(I + C + P_2)I^{-1} \\
&= \left[I - (I - D' - D'')^{-1}(L + E' - C)\right]^{-1}(I + C + P_2) \\
&\geq \left[I - (I - D' - D'')^{-1}(L + E' - C)\right]^{-1}(I + C) \\
&= \Big\{I + \left[(I - D' - D'')^{-1}(L + E' - C)\right] + \left[(I - D' - D'')^{-1}(L + E' - C)\right]^2 + \cdots \\
&\quad \cdots + \left[(I - D' - D'')^{-1}(L + E' - C)\right]^{n-1}\Big\}(I + C) \\
&= (I + C) + \left[(I - D' - D'')^{-1}(L + E' - C)\right](I + C) + \left[(I - D' - D'')^{-1}(L + E' - C)\right]^2(I + C) + \cdots
\end{aligned}$$

$$\cdots + \left[(I - D' - D'')^{-1}(L + E' - C)\right]^{n-1}(I + C)$$
$$\geq I + L + L^2 + \cdots + L^{n-1}$$
$$= (I - L)^{-1}$$
$$= {M_G}^{-1}$$

i.e. ${M_2}^{-1} \geq {M_G}^{-1}$ and also $A = M_2 - N_2 = M_G - N_G$ be the two convergent and regular splitting. Therefore, it follows from Lemma 2.9, that $\rho({M_2}^{-1}N_2) \leq \rho({M_G}^{-1}N_G) < 1$ i.e., $\rho({M_{c2_G}}^{-1}N_{c2_G}) \leq \rho({M_G}^{-1}N_G) < 1$ i.e., $\rho(T_{c2_G}) \leq \rho(T_G) < 1$. □

Theorem 3.5. *Let A be a nonsingular M-matrix. Then under the assumption of the Theorem 3.1, the following inequality holds:*

$$\rho\left(T_{c2_G}\right) \leq \rho\left(T_{c2_J}\right) < 1$$

Proof. Since A is a nonsingular M-matrix and hence $A^{-1} \geq 0$. For Jacobi splitting, $A_{c2_J} = P_{c2}A = M_{c2_J} - N_{c2_J}$, we have $M_{c2_J} = I - D' - D''$ and $N_{c2_J} = L - C + E' + U - P_2 + F' + F''$. We know from Theorem 3.1, that $A_{c2_J} = P_{c2}A$ is a regular Jacobi convergent splitting i.e., $\rho\left(M_{c2_J}^{-1}N_{c2_J}\right) < 1$ i.e., $\rho\left(T_{c2_J}\right) < 1$.
For Gauss-Seidel splitting, $A_{c2_G} = P_{c2}A = M_{c2_G} - N_{c2_G}$, we have $M_{c2_G} = I - L + C - D' - E' - D''$ and $N_{c2_G} = U - P_2 + F' + F''$. From Theorem 3.3, we know that $A_{c2_G} = P_{c2}A$ is a regular Gauss-Seidel convergent splitting i.e., $\rho\left(M_{c2_G}^{-1}N_{c2_G}\right) < 1$ i.e., $\rho\left(T_{c2_G}\right) < 1$. In order to show the required inequality, we consider the following splitting of A:

$$A_{c2_G} = P_{c2}A = (I + C + P_2)\,A = M_{c2_G} - N_{c2_G}$$

Or,

$$A = (I + C + P_2)^{-1}M_{c2_G} - (I + C + P_2)^{-1}N_{c2_G}$$

We assume, $M_3 = (I + C + P_2)^{-1}M_{c2_G}$ and $N_3 = (I + C + P_2)^{-1}N_{c2_G}$. Again,

$$A_{c2_J} = P_{c2}A = (I + C + P_2)\,A = M_{c2_J} - N_{c2_J}$$

Then

$$A = (I + C + P_2)^{-1}\,M_{c2_J} - (I + C + P_2)^{-1}\,N_{c2_J}$$

We suppose, $M_4 = (I + C + P_2)^{-1}M_{c2_J}$ and $N_4 = (I + C + P_2)^{-1}N_{c2_J}$. One can easily obtain that, $M_3^{-1}N_3 = M_{c2_G}^{-1}N_{c2_G}$ and $M_4^{-1}N_4 = M_{c2_J}^{-1}N_{c2_J}$ and also $A = M_3 - N_3 = M_4 - N_4$ are the two regular and convergent splitting. Hence $\rho\left(M_3^{-1}N_3\right) < 1$ and $\rho\left(M_4^{-1}N_4\right) < 1$. We also can notice that

$$M_3^{-1} = M_{c2_G}^{-1}(I + C + P_2)$$

$$
\begin{aligned}
&= (I - L + C - D' - E' - D'')^{-1}(I + C + P_2) \\
&= \left[(I - D' - D'') - (L + E' - C)\right]^{-1}(I + C + P_2) \\
&= \left[I - (I - D' - D'')^{-1}(L + E' - C)\right]^{-1}(I - D' - D'')^{-1}(I + C + P_2) \\
&\geq I^{-1}(I - D' - D'')^{-1}(I + C + P_2) \\
&= I(I - D' - D'')^{-1}(I + C + P_2) \\
&= (I - D' - D'')^{-1}(I + C + P_2) \\
&= M_{c2_J}^{-1}(I + C + P_2) \\
&= \left[(I + C + P_2)^{-1}M_{c2_J}\right]^{-1} \\
&= M_4^{-1} \\
M_3^{-1} &\geq M_4^{-1}
\end{aligned}
$$

Thus, from Lemma 2.9, we have

$$\rho\left(M_3^{-1}N_3\right) \leq \rho\left(M_4^{-1}N_4\right) < 1$$

i.e.,

$$\rho\left(M_{c2_G}^{-1}N_{c2_G}\right) \leq \rho\left(M_{c2_J}^{-1}N_{c2_J}\right) < 1$$

i.e.,

$$\rho\left(T_{c2_G}\right) \leq \rho\left(T_{c2_J}\right) < 1$$

□

4. Numerical Examples

In this section, we give three simple numerical examples to confirm our theoretical analysis given in Section 3. The spectral radii for the iteration matrices of the two modified iterative methods are computed using MATLAB R12.

Example 4.1. *We consider the following* 4×4 *matrix of the form:*

$$A = \begin{pmatrix} 1 & -0.2 & -0.3 & -0.2 \\ -0.2 & 1 & -0.3 & -0.1 \\ -0.1 & -0.2 & 1 & -0.3 \\ -0.2 & -0.3 & -0.3 & 1 \end{pmatrix}$$

After computation, we obtain

$$\rho\left(T_J\right)=0.6668, \ \ \rho\left(T_G\right)=0.4671, \ \ \rho\left(T_{c2_J}\right)=0.5956, \ \ \rho\left(T_{c2_G}\right)=0.3863$$

Clearly,

$$\rho\left(T_{c2_J}\right)=0.5956<\rho\left(T_J\right)=0.6668$$
$$\rho\left(T_{c2_G}\right)=0.3863<\rho\left(T_G\right)=0.4671$$
$$\rho\left(T_{c2_G}\right)=0.3863<\rho\left(T_{c2_J}\right)=0.5956$$

Example 4.2. *We consider the following* 4×4 *matrix of the form:*

$$A=\begin{pmatrix} 1 & -0.1 & -0.2 & -0.5 \\ -0.1 & 1 & -0.1 & -0.5 \\ -0.3 & -0.1 & 1 & -0.1 \\ -0.4 & -0.3 & -0.1 & 1 \end{pmatrix}$$

After computation, we obtain

$$\rho\left(T_J\right)=0.7332, \ \ \rho\left(T_G\right)=0.5408, \ \ \rho\left(T_{c2_J}\right)=0.6034, \ \ \rho\left(T_{c2_G}\right)=0.3673$$

Obviously,

$$\rho\left(T_{c2_J}\right)=0.6034<\rho\left(T_J\right)=0.7332$$
$$\rho\left(T_{c2_G}\right)=0.3673<\rho\left(T_G\right)=0.5408$$
$$\rho\left(T_{c2_G}\right)=0.3673<\rho\left(T_{c2_J}\right)=0.6034$$

Example 4.3. *We consider the following* 5×5 *matrix of the form:*

$$A=\begin{pmatrix} 1 & -0.2 & -0.3 & -0.1 & -0.2 \\ -0.1 & 1 & -0.1 & -0.3 & -0.1 \\ -0.2 & -0.1 & 1 & -0.1 & -0.2 \\ -0.2 & -0.1 & -0.1 & 1 & -0.3 \\ -0.1 & -0.2 & -0.2 & -0.1 & 1 \end{pmatrix}$$

After computation, we get

$$\rho\left(T_J\right)=0.6551, \ \ \rho\left(T_G\right)=0.4608, \ \ \rho\left(T_{c2_J}\right)=0.5813, \ \ \rho\left(T_{c2_G}\right)=0.3720$$

Obviously,

$$\rho\left(T_{c2_J}\right) = 0.5813 < \rho\left(T_J\right) = 0.6551$$
$$\rho\left(T_{c2_G}\right) = 0.3720 < \rho\left(T_G\right) = 0.4608$$
$$\rho\left(T_{c2_G}\right) = 0.3720 < \rho\left(T_{c2_J}\right) = 0.5813$$

5. Conclusion

In this paper, we have discussed basically two preconditioned iterative methods namely the preconditioned Jacobi and the preconditioned Gauss-Seidel iterative methods for solving linear systems of equations. The comparison theorems and the numerical experiments show that the two preconditioned iterative methods converge faster than the respective classical iterative methods and also the preconditioned Gauss-Seidel method is superior as compared to the preconditioned Jacobi method with the preconditioner $(I + C + P_2)$.

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