International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **12(3)**(2024), 61–66
Available Online: http://ijmaa.in

Two Inequalities about Means

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Abstract

In this paper we first generalize an inequality that involves Arithmetic Mean, Geometric Mean, and Harmonic Mean. We then generalize a similar inequality that involves Square Mean, Arithmetic Mean, and Geometric Mean.

Keywords: AM-GM-HM Inequality; Power Mean Inequality; Muirhead's Inequality.
2020 Mathematics Subject Classification: 26D15.

1. Introduction

Inequality is an important topic, especially in Analysis, see [3]. Since Polya first systematically introduced this topic in [2], there are more and more people studying this topic. Among all the inequalities, those that involve different kind of means are applied very frequently, especially in many Math competitions, like International Mathematical Olympiad, see [1]. While teaching this subject, we found some patterns among the inequalities of Arithmetic Mean, Geometric Mean, and Harmonic Mean. The product of $(n-1)$ Arithmetic Means along with a Harmonic Mean is always greater than or equal to the product of n Geometric Means. We then focused on the not-so-commonly-used Quadratic Mean, and found that there is a similar pattern. Instead of products, the sum of $(n-1)$ Geometric Means and a Quadratic Mean is always less than or equal to the sum of n Arithmetic Means. We will provide our proofs in the next section. Next, we will introduce the definitions and the properties that will be used in our proofs.

Theorem 1.1 (AM-GM-HM Inequality). *Let $a_1, a_2, \cdots, a_n$ be positive real numbers. The numbers*

$$AM = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad GM = \sqrt[n]{a_1 a_2 \cdots a_n}, \quad HM = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

are called the arithmetic mean, geometric mean, and harmonic mean for the numbers $a_1, a_2, \cdots, a_n$, respectively,

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and we have

$$AM \geq GM \geq HM.$$

Equalities occur if and only if $a_1 = a_2 = \cdots = a_n$.

Theorem 1.2 (QM-AM Inequality). *Let* $a_1, a_2, \cdots, a_n$ *be positive real numbers, and* AM *be their arithmetic mean. The number*

$$QM = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}$$

is called the quadratic mean for the numbers $a_1, a_2, \cdots, a_n$*, and we have*

$$QM \geq AM.$$

Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

Please notice that, both Theorem 1.1 and Theorem 1.2 are special cases of the Power Mean Inequality. Interested readers may refer to [1] or [2] for proofs and applications.

Definition 1.3. *Let* $x_1, x_2, \cdots, x_n$ *be a sequence of positive real numbers, and let* $a_1, a_2, \cdots, a_n$ *be arbitrarily real numbers. Consider* $F(x_1, x_2, \cdots, x_n) = x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n}$*. The notation* $T[a_1, a_2, \cdots, a_n]$ *is defined as the sum of all possible products* $F(x_1, x_2, \cdots, x_n)$ *over all permutations of* $a_1, a_2, \cdots, a_n$.

Before moving on, let us see some examples.

Example 1.4. *For* $F(x, y, z)$,

$$\begin{aligned} T[3,0,0] &= 2x^3 + 2y^3 + 2z^3, \\ T[2,1,0] &= x^2y + x^2z + y^2x + y^2z + z^2x + z^2y, \\ T[1,1,1] &= 6xyz. \end{aligned}$$

Example 1.5. *For* $F(x_1, x_2, \cdots, x_n)$*, with* AM *and* GM *being the arithmetic mean and the geometric mean of* $x_1, x_2, \cdots, x_n$ *respectively,*

$$\begin{aligned} T[1,0,\cdots,0] &= (n-1)!\,(x_1 + x_2 + \cdots + x_n) = n! \cdot AM, \\ T[1,1,\cdots,1] &= n!\,(x_1 x_2 \cdots x_n) = n! \cdot GM^n. \end{aligned}$$

We now move on to the next definition.

Definition 1.6. *We say the sequence* $(b_i)_{i=1}^n$ *is majorized by* $(a_i)_{i=1}^n$*, denoted* $(b_i) \prec (a_i)$*, if we can rearrange the terms of the sequences in such a way as to satisfy the following:*

(1) $b_1 + b_2 + \cdots + b_n = a_1 + a_2 + \cdots + a_n$

(2) $b_1 \geq b_2 \geq \cdots \geq b_n$ *and* $a_1 \geq a_2 \geq \cdots \geq a_n$

(3) $b_1 + b_2 + \cdots + b_s \leq a_1 + a_2 + \cdots a_s$ *for any* $1 \leq s \leq n$.

Example 1.7.

$$(2,2,1) \prec (3,1,1),$$
$$(1,1,1,1) \prec (2,1,1,0) \prec (3,1,0,0) \prec (4,0,0,0).$$

With all the preparation above, we now introduce the Muirhead's Inequality. Again, interested readers may refer to [1] for its proof and applications.

Theorem 1.8 (Muirhead's Inequality)**.** *Let* $x_1, x_2, \cdots, x_n$ *be a sequence of non-negative real numbers and let* (a_i) *and* (b_i) *be sequences of positive real numbers such that* $(b_i) \prec (a_i)$. *Then*

$$T[b_i] \leq T[a_i].$$

Equality occurs if and only if $(a_i) = (b_i)$ *or* $x_1 = x_2 = \cdots = x_n$.

Example 1.9. *Let* (x, y, z) *be a sequence of positive variables.*

$$T[2,2,1] \leq T[3,1,1],$$
$$2(x^2y^2z + x^2z^2y + y^2z^2x) \leq 2(x^3yz + y^3zx + z^3yx).$$

In the next section, we will introduce the new inequalities we discovered.

2. Main Results

To make the content easier to read, in this section we will use just single letters Q, A, G, and H to indicate the quadratic mean, arithmetic mean, geometric mean, and harmonic mean for a given sequence respectively. We start with two examples.

Example 2.1. *For positive real numbers* a *and* b*, we notice that*

$$ab = \frac{a+b}{2} \cdot \frac{2ab}{a+b}.$$

Let $A = \frac{a+b}{2}$ *denote the arithmetic mean,* $G = \sqrt{ab}$ *denote the geometric mean, and* $H = \frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b}$ *denote the harmonic mean. The above identity can be written as:*

$$G^2 = A \cdot H.$$

Example 2.2. *For positive real numbers a, b and c, we notice that*

$$3(ab+bc+ca) \leq (a+b+c)^2 .$$

This is equivalent to

$$abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \leq 3\left(\frac{a+b+c}{3}\right)^2 .$$

Again, let A, G, H denote the arithmetic mean, geometric mean, and harmonic mean respectively. The above can be written as:

$$G^3 \leq A^2 \cdot H.$$

As we can see, when the number of the sequence increases, the power of their geometric mean and the arithmetic mean also change accordingly with certain pattern. We now generalize it to the following result.

Theorem 2.3. *For any n positive real numbers, with A, G, H being their arithmetic mean, geometric mean, and harmonic mean respectively, we have*

$$G^n \leq A^{n-1} \cdot H.$$

Proof. Let $a_1, a_2, \cdots, a_n$ be positive real numbers with A, G, H being their arithmetic mean, geometric mean, and harmonic mean respectively. We want to prove that

$$G^n \leq A^{n-1} \cdot H,$$

or equivalently,

$$n^{n-2}\left(a_2a_3\cdots a_n + a_1a_3\cdots a_n + \cdots + a_1a_2\cdots a_{n-1}\right) \leq \left(a_1+\cdots+a_n\right)^{n-1} .$$

We may multiply the right-hand-side out and organize both sides with the $T[\cdots]$ notations.

$$\begin{aligned} c_0T[1,1,\cdots,0] \leq c_1T[(n-1),0,\cdots,0] + c_2T[(n-2),1,0,\cdots,0] + c_3T[(n-3),2,0,\cdots,0] \\ + c_4T[(n-3),1,1,0,\cdots,0] + \cdots + c_kT[1,1,\cdots,1,0]. \end{aligned}$$

Notice that the sum of coefficients of all terms at the left-hand-side is equal to the sum of the right-hand-side. According to the Muirhead's Inequality, the above inequality is true because all the $T[\cdots]$ terms at the right-hand-side are greater or equal to the $T[\cdots]$ terms at the left-hand-side. □

The next two examples will show us another pattern in the inequalities that involve quadratic means, arithmetic means and geometric means.

Example 2.4. *For positive real numbers a and b, we know that*

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Let $Q = \sqrt{\frac{a^2+b^2}{2}}$ *denote the quadratic mean,* $A = \frac{a+b}{2}$ *denote the arithmetic mean, and* $G = \sqrt{ab}$ *denote the geometric mean. The above identity is equivalent to:*

$$(2A)^2 = 2Q^2 + 2G^2.$$

Since $2Q^2 + 2G^2 \geq (Q+G)^2$ *according to the QM-AM Inequality, we have*

$$2A \geq Q + G.$$

Example 2.5. *For positive real numbers a, b and c, we know that*

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca) \geq a^2 + b^2 + c^2 + 6\,(abc)^{\frac{2}{3}}$$

according to the AM-GM Inequality. Let Q denote the quadratic mean, A denote the arithmetic mean, and G denote the geometric mean. The above inequality is equivalent to:

$$(3A)^2 \geq 3Q^2 + 6G^2.$$

Since $3Q^2 + 6G^2 \geq Q^2 + 4QG + 4G^2 = (Q+2G)^2$ *according to the AM-GM Inequality, we have*

$$3A \geq Q + 2G.$$

In these two examples we notice that, when the number of the sequence increases, the coefficients of the arithmetic mean and the geometric mean also increase accordingly with certain pattern. We therefore generalize the pattern into the next result.

Theorem 2.6. *For any n positive real numbers, with Q, A,and G being their quadratic mean, arithmetic mean and geometric mean respectively, we have*

$$nA \geq Q + (n-1)G.$$

Proof. Let $a_1, a_2, \cdots, a_n$ be positive real numbers. According to AM-GM inequality we first notice that

$$\sum_{1 \leq i \leq j \leq n} \left(a_i a_j\right) \geq \frac{n(n-1)}{2} \left(a_1^{n-1} \cdot a_2^{n-1} \cdots a_n^{n-1}\right)^{\frac{2}{n(n-1)}} = \frac{n(n-1)}{2} G^2.$$

Therefore,

$$\begin{aligned}(nA)^2 &= (a_1 + a_2 + \cdots + a_n)^2 \\ &= a_1^2 + a_2^2 + \cdots + a_n^2 + 2 \sum_{1 \le i \le j \le n} (a_i a_j) \\ &\ge nQ^2 + n(n-1)G^2.\end{aligned}$$

We want to prove that

$$nQ^2 + n(n-1)G^2 \ge (Q + (n-1)G)^2 .$$

Since

$$(Q + (n-1)G)^2 = Q^2 + 2(n-1)QG + (n-1)^2G^2$$

and

$$\left(nQ^2 + n(n-1)G^2\right) - \left(Q^2 + 2(n-1)QG + (n-1)^2G^2\right) = (n-1)(Q-G)^2 \ge 0,$$

we have

$$(nA)^2 \ge nQ^2 + n(n-1)G^2 \ge (Q + (n-1)G)^2 .$$

The claimed inequality is then proved by taking the square root at both sides. □

As we know that quadratic mean, arithmetic mean, geometric mean, and harmonic mean are special cases of power means with degree 2, 1, 0, and -1 respectively. Our first theorem introduces the relation between power means of degree 1, 0, and -1. Our second theorem introduces the relation between power means of degree 2, 1, and 0. We also tried power means with different degrees. However, we have not found any other patterns.

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