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Wavelets and its Application in Signal and Image Processing

Research Article

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Abstract: This paper deals with a brief introduction to wavelet transforms. Special attention is given to continuous wavelet transforms, inverse wavelet transform, discrete wavelet transforms. Recent applications of wavelet transform analysis like image processing are discussed. Many recent references are included in the end to stimulate further study and research on wavelet transforms and their diverse applications.

Keywords: Wavelets, Continuous Wavelet Transform, Inverse Wavelet Transform, Discrete Wavelet Transform, image processing.

1. Introduction

Since the days of Joseph Fourier, his analysis has been used in all branches of engineering and science. Simply stated, the Fourier method is the most powerful technique for signal analysis. It transforms the signal from one domain to another domain in which many important characteristics of the signal are revealed. One usually refers to this transform domain as the spectral or frequency domain, while the domain of the original signal is usually the time or spatial domain. The Fourier analysis includes both the Fourier transform and the Fourier series. The Fourier transform is applicable to functions that are defined on the real line, while the Fourier series is used to analyze functions that are periodic.

One of the areas that significantly benefited from the Fourier analysis is digital signal processing. But, the Fourier transform has its limitations. For example, this transformation cannot be applied to non-stationary signals. These signals, e.g., speech and image, have different characteristics at different time or space. Although the modified version of the Fourier transform, referred to as short-time Fourier transform can resolve some of the problems associated with non-stationary signals, but does not address all issues of concern. The short-time Fourier transform is extensively used in speech signal processing but rarely, if ever, used in image processing. The ideal tool for studying stationary signals is the Fourier transform. In other words, stationary signals decompose canonically into linear combination of waves (sines and cosines). In the same way, signals that are not stationary decompose into linear combination of wavelets. The wavelet transform is successfully applied to non-stationary signals for analysis and processing and provides an alternative to the short-time Fourier transform (STFT). In contrast to STFT, which uses a single analysis window, the wavelet transform uses short windows at high frequencies and long windows at low frequencies.

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For some applications it is desirable to see the wavelet transform as signal decomposition onto a set of basis functions. In fact, basis functions called wavelets always underlie the wavelet analysis. They are obtained from a single prototype wavelet by dilations and contractions (scalings) as well as shifts. Therefore, in a wavelet transform, the notion of the scale is introduced as an alternative to frequency, leading to a so called time-scale representation. This means that a signal is mapped into a time-scale plane (the equivalent of the time-frequency plane used in the STFT). There are several types of wavelet transforms, and, depending on the application, one may be preferred to the others. For a continuous input signal, the time and scale parameters can be continuous, leading to the continuous wavelet transform. They may as well be discrete, leading to a wavelet series expansion. Also, the wavelet transform can be defined for discrete-time signals, leading to a Discrete Wavelet Transform.

2. Continuous Wavelet Transform

The short time Fourier transform (STFT) provides one of many ways to generate a time-frequency analysis of signals. But fixed time-frequency resolution of the STFT poses a serious constraint in many applications. It is observed that the radii Δt and Δf of the window function for STFT do not depend upon location in the t-w plane. Once the window function is chosen, the time-frequency resolution is fixed throughout the processing. Our objective is to devise a method that can give good time-frequency resolution at an arbitrary location in the t-w plane. In other words, we must have a window function whose radius increases in time (reduces in frequency) while resolving the low frequency contents, and decreases in time (increases in frequency) while resolving the high-frequency contents of a signal. This objective leads us to the development of wavelet functions $\psi(t)$. The integral wavelet transform of a function $f(t) \in L^2$ with respect to some analyzing wavelet ψ is defined as

$$W_\psi f(a,b) = \int_{-\infty}^{\infty} f(t)\overline{\psi_{a,b}(t)}dt \tag{1}$$

where

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right), \quad a > 0, b \in R$$

The parameters a and b are called dilation and translation parameters, respectively. The normalization factor $a^{-1/2}$ is included so that $\|\psi_{a,b}\| = \|\psi\|$. For ψ to be a window function, we start with a function $\psi(t)$ of the real variable t. This function is called a "mother wavelet" provided it is well localized and oscillating (By oscillating it resembles a wave, but by being localized it is a wavelet). The localization condition is expressed as decreasing rapidly to zero when $|t|$ tends to infinity. The second condition suggests that $\psi(t)$ vibrates like a wave. Here we require that the integral of $\psi(t)$ be zero i.e.

$$\int_{-\infty}^{\infty} \psi(t)dt = 0 \tag{2}$$

In addition to this, a wavelet is constructed so that it has a higher order of vanishing moments. A wavelet is said to have vanishing moments of order m if

$$0 = \int_{-\infty}^{\infty} \psi(t)dt = \int_{-\infty}^{\infty} t\psi(t)dt = \cdots = \int_{-\infty}^{\infty} t^{m-1}\psi(t)dt$$

The "mother wavelet", $\psi(t)$, generates the other wavelets, $\psi_{a,b}(t), a > 0, b \in R$, of the family by change of scale (the scale of $\psi(t)$ is conventionally 1, and that of $\psi_{a,b}(t)$ is $a > 0$) and translation in time (the function $\psi(t)$ is conventionally centered around 0, and $\psi_{a,b}(t)$ is then centered around b). Thus

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right), \quad a > 0, b \in R \tag{3}$$

Alex Grossmann and Jean Morlet have shown that, if $\psi(t)$ is real-valued, this collection can be used as if it were an orthonormal basis. This means that any signal of finite energy can be represented as a linear combination of wavelets $\psi_{a,b}(t)$ and that the co-efficient of this combination are, upto a normalizing factor, the scalar products, $\int\limits_{-\infty}^{\infty} f(t)\psi_{a,b}(t)dt$. These scalar products measure, in a certain sense, the fluctuations of the signal $f(t)$ around the point b, at the scale given by $a > 0$. Strictly speaking, wavelet transform provides time-scale analysis and not time-frequency analysis. However, by proper scale-to-frequency transformation, one can get an analysis that is very close to time-frequency analysis. Observe that in (3), by reducing a, the support of $\psi_{a,b}$ is reduced in time and hence covers a larger frequency range and vice-versa. Therefore $\frac{1}{a}$ is a measure of frequency. The parameter b, on the other hand, indicates the location of the wavelet window along the time axis. Thus, by changing (a, b), $W_\psi f$ can be computed on the entire time-frequency plane. Furthermore because of the condition (2), we conclude that all wavelets must oscillate, giving them the nature of small waves and hence the name wavelets.

3. Inverse Wavelet Transform

The purpose of the inverse transform is to reconstruct the original function from its integral wavelet transform. It involves a two-dimensional integration over the scale parameter a and the translation parameter b. The expression for the inverse wavelet transform is

$$f(t) = \frac{1}{c_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{a^2} \left[W_\psi f(a,b)\right] \psi_{a,b}(t)\, da\, db \tag{4}$$

where C_ψ is a constant that depends on the choice of wavelet is given by

$$C_\psi = \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega < \infty \tag{5}$$

where $\hat{\psi}(\omega)$ is the Fourier transform of $\psi(t)$. The above condition (5), known as admissibility condition, restricts the class of functions that can be wavelets. Equation (4) is essentially a superposition integral. Integration with respect to a, sums all the contributions of the wavelet components at location b, while the integral with respect to b includes all locations along the b-axis. Since computation of the inverse wavelet transform is used only for synthesizing the original signal, it is not used as frequently as the integral wavelet transform for the analysis of signals. In subsequent sections where the discrete wavelet transform (DWT) is introduced, the inverse of the DWT is very useful in data communication and signal processing.

4. Discrete Wavelet Transform

In signal analysis, it is sometimes necessary to partition the (positive) frequency axis into disjoint frequency bands. For computational efficiency and convenience in discussions, we will only consider "binary partitions". Here we take the discrete values of the scale parameter a and the translation parameter b. We take $a = 2^{-j}$ and $b = k2^{-j}$, $j, k \in$ Z. With these values of a and b, the integral of (1) becomes

$$W_\psi f(2^{-j}, k2^{-j}) = 2^{j/2} \int_{-\infty}^{\infty} f(t)\psi(2^j t - k)dt$$

We now discretize the function $f(t)$. For simplicity, assume the sampling rate to be 1. In that case, the above integral can be written as

$$W_\psi f(2^{-j}, k2^{-j}) \approx 2^{j/2} \sum_n f(n)\psi(2^j n - k)$$

To compute the wavelet transform of a function at some point in the time-scale plane, we do not need to know the function values for the entire time axis. All we need is the function at those values of time at which the wavelet is non-zero.

5. Image Processing

Digital image processing is a rapidly evolving field with growing applications in science and engineering. It has become an essential part of contemporary scientific and technological activity. Image processing is used in telecommunications, in the transmission and analysis of satellite images and in medical imaging (echography, topography and nuclear magnetic resonance). Image processing holds the possibility of developing the ultimate machine that could perform the visual functions of all living beings. Many theoretical as well as technological break throughs are required before we could build such a machine. In the meantime, there is an abundance of image processing applications that can serve mankind with the available and anticipated technology in the near future.

Image processing is done on the numerical representation of the image. For a black and white image, the numerical representation of it is created by replacing the x and y coordinates of an image point with those of the closest point on a sufficiently fine grid. The value $f(x, y)$ of the "gray scale" is then assigned to the corresponding grid point. The image thus becomes a large, typically square matrix. Image processing is done on this matrix. These matrices are enormous, and as soon as one deals with a sequence of images, the volume of numerical data that must be processed becomes immense. Is it possible to reduce this volume by considering the "hidden laws" or correlations that exist among the different pieces of numerical information representing the image? This question leads us to define the goals of the scientific discipline called "image processing". The major topics of digital image processing are representation, processing techniques and communication. Image representation includes tasks ranging from acquisition, digitization and display to mathematical characterization of images for subsequent processing. A proper representation is a prerequisite to an efficient processing technique such as enhancement, filtering and restoration, analysis, reconstruction from projections and image communication. The attention of the signal processing community was caught when Daubechies and Mallat, in addition to their contribution to the theory of wavelets, established connections to discrete signal processing results. Since then, a number of theoretical, as well as practical contributions have been made on various aspects of Wavelet Transform's. This transformation provides a general technique that is applicable to many tasks in signal processing.

6. Conclusion

Mathematical aspects of wavelets play a very significant role in differing the results of engineering applications. In this paper, we reviewed wavelet theory, an important mathematical tool for signal and image processing. Compared to other tools, such as the Fourier transform, wavelet transforms often provide a better spatial domain localization property, critical to many image applications. The application of wavelets in image processing is only a decade old. Wavelets have demonstrated their importance in almost all areas of signal processing and image processing. In many areas, techniques based on wavelet transforms represent the best of the available solutions. Theoretical research inspired by wavelets has led to new techniques that are more promising in certain situations. Explorations of new frontiers are likely to bring us more successful applications in image denoising, image enhancement, image restoration, etc.

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