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Common Fixed Point Theorem for Lipschitz Type Mapping

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Abstract

In this paper, we prove some common fixed point theorem for Lipschitz type mapping with condition $\mathcal{C}$ and $\mathcal{C}'$.

Keywords: Common fixed point; Gornicki mappings; Lipschitz type mapping.
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1. Introduction

It is well known that in the setting of metric space, strict contractive condition do not ensure the existence of common fixed point unless the space is assumed compact or the strict conditions are replaced by stronger conditions as in [2]-[3]. In 1986, Jungck [5] introduced the notion of compatible maps. This concept was frequently used to prove existence theorems in common fixed point theory. However, the study of common fixed points of non compatible mappings is also very interesting. Work along these lines has recently been initiated by Pant [4]. In 2002, Aamri [1] introduced a new property which generalize the concept of non compatible mappings, and give some common fixed point theorems under strict contractive conditions. In 2021, Popescu [17] introduced a new class of Picard operator, namely, the Gornicki mappings, which extends the notion of enriched contractions [11]. Aamri [1] proved new common fixed point theorem using E.A. properties. Currently Ravindra K. Bisht [12] proved fixed point theorem for Lipschitz type mapping satisfying condition $\mathcal{C}$. Our aim of this paper to prove common fixed point theorem for this Lipschitz type mapping using property E.A.

2. Preliminaries

Definition 2.1 ([1]). *Let S and T be two self mappings of a metric space (X, d). We say that T and S satisfy the property (E.A) if there exists a sequence x_n such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} = Sx_n = t$ for some $t \in X$.*

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Theorem 2.2 ([1]). *Let S and T be two weakly compatible self mappings of a metric space (X,d) such that*

(i) T and S satisfy the property (E.A),

(ii) $d(Tx,Ty) < max\ d(Sx,Sy), \frac{d(Tx,Sx)+d(Ty,Sy)}{2}, \frac{d(Ty,Sx)+d(Tx,Sy)}{2}$ *for all* $x \neq y \in X$.

(iii) $TX \subset SX$.

Then S and T have unique common fixed point.

Definition 2.3 ([17]). *Let (X,d) be a complete metric space and let $T : X \to X$ be a self mapping. We say that T is a Gornicki mapping if T satisfies $d(Tx,Ty) \leq M[d(x,y)+d(x,Tx)+d(y,Ty)]$ and there exist non negative real constants $a,\ b$ with $a < 1$ such that for arbitrary $x \in X$ there exists $u \in X$ with $d(u,Tu) \leq ad(x,Tx)$ and $d(u,x) \leq bd(x,Tx)$.*

Definition 2.4 ([12]). *Suppose $\alpha \in A(f)$, $\kappa \in K(f)$ and $l \in L(f)$, such that for each $x \in X$, there exists $y \in X$ satisfying*

(i) $d(y,Ty) \leq \alpha(x,y)d(x,Tx)$;

(ii) $d(x,y) \leq \kappa(x,y)[d(x,Tx)]^{l(x,y)}$,

where A denote the class of function $\alpha : X \times X \to [0,\infty)$, satisfying for any $x_n \subset X$, if $f(x_n)$ converges, then $\lim_{n\to\infty} s\alpha(x_n,x_{n+1}) < 1$ *Then T has a unique fixed point.*

Definition 2.5 ([12]). *A mapping χ is defined by $\chi(t) < t$ for all $t > 0$,where $\chi : [0,\infty) \to [0,\infty]$ be a continuous mapping.*

Definition 2.6 ([13]). *Let f and g be R-weakly commuting self-mappings of type A_f or of type A_g of a complete metric space (X,d) such that $fX \subseteq gX$ and $d(fx,fy) \leq hd(gx,gy)$, $0 \leq h < 1$. Then f and g have a common fixed point if and only if f and g are (f,g)-orbitally continuous.*

Theorem 2.7 ([12]). *Suppose that (X,d) is a complete metric space and $T : X \to X$ is a mapping satisfying $d(Tx,Ty) \leq M[d(x,y)+d(x,Tx)+d(y,Ty)]$, where $M \in [0,1)$ and the following condition: $\mathcal{C}$ Assume that there exist real constants a,b with $a \in [0,1)$ and $b > 0$ such that for arbitrary $x \in X$ there exists $u \in X$ satisfying*

(i) $d(u,Tu) \leq ad(x,Tx)$;

(ii) $d(u,x) \leq bd(x,Tx)$.

Then T has a fixed point.

Theorem 2.8. *Suppose that (X,d) is a complete metric space and $T : X \to X$ is an orbitally continuous mapping satisfying $\mathcal{C}$. Then T has a fixed point.*

3. Main Results

Theorem 3.1. *Suppose (X, d) is a complete metric space and $T, S : X \to X$ is an arbitally continuous mapping and one-one mapping such that;*

(1) T and S satisfy condition $\mathcal{C}$;

(2) $d(Tx, Ty) < \max\left\{d(Sx, Sy), \frac{[d(Tx,Sx)+d(Ty,Sy)]}{2}, \frac{[d(Ty,Sx)+d(Tx,Sy)]}{2}\right\}$ for all$x \neq y \in X$; Then T and S have a unique common fixed point.

Proof. Suppose (X, d) is a complete metric space and $T, S : X \to X$ is an arbitally continuous mapping satisfying (1), (2) by Theorem (2.7), T has a fixed point, i.e, $T(z_1) = z_1$. S has a fixed point, i.e, $S(z_2) = z_2$, since, (X, d) is a complete metric space. Therefore, there exists $z_1 \in X$ such that $x_n \to z_1$ as $n \to \infty$ but T is arbital continuous. So, $\lim_{n\to\infty} Tx_n = Tz_1 = z_1$. Similarly, $\lim_{n\to\infty} Sx_n = sz_2 = z_2$ from condition (2),

$$d(Tx_n, Tz_2) < \max\left\{d(Sx_n, Sz_2), \frac{[d(Tx_n, Sx_n) + d(Tz_2, Sz_2)]}{2}, \frac{[d(Tz_2, Sx_n) + d(Tx_n, Sz_2)]}{2}\right\}$$

letting $n \to \infty$,

$$\begin{aligned} d(Tz_1, Tz_2) &< \max\left\{d(Sz_2, Sz_2), \frac{[d(Tz_1, Sz_2) + d(Tz_2, Sz_2)]}{2}, \frac{[d(Tz_2, Sz_2) + d(Tz_1, Sz_2)]}{2}\right\} \\ &< max\left\{0, \frac{d(Tz_1, Tz_2)}{2}, \frac{d(Tz_1, Tz_2)}{2}\right\} \\ &< \frac{d(Tz_1, Tz_2)}{2} \end{aligned}$$

which is contradiction, so, $Tz_1 = Tz_2$ implies $z_1 = z_2$, since T is one-one, using (3) and (4), $\lim_{n\to\infty} Tx_n = Tz_1 = z_1 = z_2 = Sz_2 = \lim_{n\to\infty} Sx_n$. Clearly T and S satisfy the property E.A. So, By Theorem 1, T and S have a unique common fixed point. □

Example 3.2. *Let $X = [2, \infty]$ with usual metric $d(x, y) = |x - y|$ define $T, S : X \to X$ by $T(x) = 2x + 3$, $S(x) = x^2$ for all $x \in X$. Then*

(1) T and S is one-one mapping and satisfy condition C.

(2) $d(Tx, Ty) = |2x + 3 - x^2| < \max\left\{d(Sx, Sy), \frac{[d(Tx,Sx)+d(Ty,Sy)]}{2}, \frac{[d(Ty,Sx)+d(Tx,Sy)]}{2}\right\}$

Then T and S have a common fixed point $T(3) = 2.3 + 3 = 9 = S(3) = 3^2 = 9$, 3 is a fixed point for T and S.

Theorem 3.3. *Suppose (X, d) is a complete metric space and $A, B : X \to X$ is an orbitally continuous and one-one mapping such that,*

(1) A and B satisfy condition C'

(2) $d(Ax, Ay) \leq \chi\{[d(x, y) + d(x, Ax) + d(y, Ay)]\}$

Then A and B have common fixed point.

Proof. Let (X, d) is a complete metric space and $A, B : X \to X$ is an orbitally continuous and one-one mapping satisfying condition (1) and (2), then by Theorem (3.1), A has a fixed point, i.e $Ay_1 = y_1$ and B has a fixed point, i.e $By_2 = y_2$. Since (X, d) is complete. Therefore there exists $y_1 \in X$ such that $x_n \to y_1$ as $n \to \infty$, but A is orbital continuous, so, $\lim_{n\to\infty} Ax_n = Ay_1 = y_1$. Similarly, $\lim_{n\to\infty} Bx_n = By_2 = y_2$ from condition (2),

$$
\begin{aligned}
d(Ax_n, Ay_2) &\leq \chi[d(x_n, y_2) + d(x_n, Ax_n) + d(y_2, Ay_2)] \\
\textit{letting } \ n \to \infty & \\
d(Ay_1, Ay_2) &\leq \chi[d(y_1, y_2) + d(y_1, Ay_1) + d(y_2, Ay_2)] \\
&\leq [d(Ay_1, Ay_2), d(y_2, Ay_2)] \\
&\leq \chi[d(Ay_1, Ay_2)] \\
&< d(Ay_1, Ay_2), \ \textit{since } \chi(t) < t
\end{aligned}
$$

which is contradiction., so, $Ay_1 = Ay_2$ implies $y_1 = y_2$, since A is one-one, using (3) and (4), $\lim_{n\to\infty} Ax_n = Ay_1 = y_1 = y_2 = By_2 = \lim_{n\to\infty} Bx_n$. Clearly A and B satisfy the property E.A. So, By Theorem (1), A and B have a unique common fixed point. □

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