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On Order of Entire Functions Sharing One or Two Points Relatively

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Abstract

Using the idea of relative sharing of values of meromorphic functions given by Banerjee, Dutta [1], we prove some results on order of entire functions on the basis of some previous papers which concerned with the unicity of entire and meromorphic functions sharing zero-one sets.

Keywords: Entire functions; Meromorphic functions; Value sharing; Relative sharing; Order.
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1. Introduction and Definitions

For a non-constant meromorphic function f, the order of f is denoted by ρ_f and is defined by [2]

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \text{ where } T(r,f) \text{ is the Nevanlinna's characteristic function of } f.$$

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$ and let $a \in \mathbb{C} \cup \{\infty\}$. If $f-a$ and $g-a$ have the same zeros CM (counting multiplicities) and IM (ignoring multiplicities) then we say that f and g share the value 'a' CM and IM respectively. Similarly f, g share '∞' CM or IM means that $\frac{1}{f}$, $\frac{1}{g}$ share '0' CM or IM respectively. For the standard definitions and notations on value distribution theory we refer [2]. In 1980, Ueda [5] introduced the following definition.

Definition 1.1 ([5]). *If k is a positive integer or ∞, then $E(a,k,f) = \{ z \in \mathbb{C}$: z is a zero of $f-a$ of order $\leq k \}$, where $\mathbb{C}$ is the complex plane.*

In 2007, Banerjee and Dutta [1] introduced the idea of relative sharing of values of two meromorphic functions with respect to another meromorphic function.

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Definition 1.2 ([1]). *Let f and g be two non-constant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. We say that f, g share 'a' CM (IM) relatively with respect to a meromorphic function h, provided the functions F and G share 'a' CM (IM), where $F = \frac{f}{h}$ and $G = \frac{g}{h}$.*

The purpose to introduce this definition of relative sharing of values of two meromorphic functions f and g is to study some properties of f and g by that of F and G constructed with the help of a suitably chosen meromorphic function h.

Ozawa [3], Ueda [4, 5, 6] have proved some unicity theorems for entire functions. Their main interest lies in the problem: How does the distribution of zero-one sets affect the unicity in the case of entire functions ? After that Yi [7] have proved a unicity theorem for meromorphic functions which share the same set. In this paper, we shall be concerned with the same problem using the idea of relative sharing.

2. Known Results

In 1976, Ozawa [3] proved the following theorems.

Theorem 2.1. *Let f, g be two entire functions of finite order. If f, g share 0, 1 CM and $2\delta(0, f) > 1$ then $fg \equiv 1$ unless $f \equiv g$.*

Theorem 2.2. *Let f, g be two entire functions of finite order and share 0, 1 CM with $\delta(0, f) > 0$. Then $\delta(0, f) = \frac{1}{p}$ with a positive integer p. If $\delta(0, f) \neq \frac{1}{p}$ with an integer $p \geq 2$, 0 is lacunary for f and $fg \equiv 1$.*

Theorem 2.3. *Let f and g be two entire functions. Suppose that f, g share 1 CM and that $\delta(0, f) > 0$ and 0 is lacunary for g. Then $fg \equiv 1$, $f = e^H$ (H is entire) unless $f \equiv g$.*

In 1980, Ueda [4] proved the follwing results.

Theorem 2.4. *Let f and g be entire functions. Assume that $\rho_f = \rho_g = \infty$ and f, g share 0, 1 CM with $\delta(0, f) > \frac{5}{6}$. Then $fg \equiv 1$ unless $f \equiv g$.*

Theorem 2.5. *Let f and g be entire functions. Assume that $\rho_f = \rho_g = \infty$ and f, g share 0, 1 CM. Further assume that all zero-points excepting atmost finite number have multiplicities ≥ 7. Then $fg \equiv 1$ unless $f \equiv g$.*

In 1980, Ueda [5] also proved the following theorem.

Theorem 2.6. *Let f and g be non-constant entire functions such that f and g share 0, 1 CM. Further assume that there exists a complex number 'a'($\neq 0, 1$) satisfying $E(a, k, f) = E(a, k, g)$, where k is a positive integer (≥ 2) or ∞. Then f and g must satisfy one of the following four relations.*

(i) $f \equiv g$;

(ii) $\left(f - \frac{1}{2}\right)\left(g - \frac{1}{2}\right) \equiv \frac{1}{4}$ $\left(\text{This occurs only for } a = \frac{1}{2}\right)$;

(iii) $fg \equiv 1$ *(This occurs only for* $a = -1$*);*

(iv) $(f-1)(g-1) \equiv 1$ *(This occurs only for* $a = 2$*).*

In 1983, Ueda [6] further show that the order restriction of f and g in Theorem 2.1 can be removed perfectly.

Theorem 2.7. *Let f and g be entire functions. Assume that f and g share* 0, 1 *CM and* $\delta(0, f) > \frac{1}{2}$. *Then* $fg \equiv 1$ *unless* $f \equiv g$.

In 1987, Yi [7] proved the following result which is an extension of Theorem 2.3 by Ozawa [3].

Theorem 2.8. *Let f and g be meromorphic functions such that f and g share* 1 *CM. If* $\delta(0, f) + \delta(0, g) > 1$ *and* $\delta(\infty, f) = \delta(\infty, g) = 1$, *then* $f \equiv g$ *or* $fg \equiv 1$.

3. Main Results

Our main results are the following theorems.

Theorem 3.1. *Let f, g be two entire functions of finite order. If there is an entire function h such that* $F = \dfrac{f}{h}$ *and* $G = \dfrac{g}{h}$ *become two entire functions with* $T(r, h) = o(T(r, f))$ *and* $T(r, h) = o(T(r, g))$ *and F, G share* 0, 1 *CM with* $2\delta(0; F) > 1$ *then* $\rho_f = \rho_g$.

Proof. Given

$$T(r,h) = o(T(r, f)) \text{ and } T(r,h) = o(T(r, g)). \tag{1}$$

Then h is of finite order and so are F and G. Now by Theorem 2.1

$$F \equiv G \text{ or } FG \equiv 1.$$

If $F \equiv G$, then obviously $\rho_f = \rho_g$.
If $FG \equiv 1$, then

$$F = \frac{1}{G} \text{ and } G = \frac{1}{F}.$$

$$\text{So, } \rho_F = \rho_G. \tag{2}$$

Now $F = \dfrac{f}{h}$ gives

$$\begin{aligned} T(r, F) &\leq T(r, f) + T(r, h) + O(1) \\ &\leq T(r, f)(1 + o(1)) + O(1), \text{ using (1).} \end{aligned}$$

$$\text{So, } \rho_F \leq \rho_f. \tag{3}$$

Again from $f = hF$ we obtain

$$\rho_f \leq \rho_F. \tag{4}$$

Therefore (3) and (4) gives

$$\rho_f = \rho_F. \tag{5}$$

Similarly from the relation $G = \frac{g}{h}$ we obtain

$$\rho_g = \rho_G. \tag{6}$$

From (2), (5) and (6) we have

$$\rho_f = \rho_g.$$

Hence the proof. □

Example 3.2. *Let*

$$f(z) = (1 - e^z)\left(1 - e^{-z}\right),\ g(z) = e^{-2z}\left(1 - e^{-z}\right)^2 \text{ and } h(z) = e^{-z}\left(1 - e^{-z}\right).$$

Here f, g are entire functions of finite order and h is also an entire function. So, $F = \frac{f}{h}$ and $G = \frac{g}{h}$ are two entire functions and share 0 and 1 CM. Now

$$T(r, f) \leq 2\log 2 + \frac{2r}{\pi} \text{ and } T(r, g) \leq 2\log 2 + \frac{4r}{\pi}.$$

Again we know that

$$T(r, f) \geq \frac{1}{3}\log^+ M\left(\frac{r}{2}, f\right).$$

Since $h = e^{-z} - e^{-2z}$ and at $z = -r$, $|h| = |e^{2r} - e^r|$, so we have for large r

$$M(r, h) \geq e^{2r} - e^r.$$

Similarly we get $M(r, F) \geq e^{2r} - e^r$, for sufficiently large r. Therefore for large r we have

$$T(r, h) \geq \frac{1}{3}\log\left(e^r - e^{\frac{r}{2}}\right) \text{ and } T(r, F) \geq \frac{1}{3}\log\left(e^r - e^{\frac{r}{2}}\right).$$

So it is clear that $\frac{T(r,h)}{T(r,f)} \geq \frac{\pi}{6}$ and $\frac{T(r,h)}{T(r,g)} \geq \frac{\pi}{12}$ for large r. Further

$$\delta(0, F) = 1 - \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{F}\right)}{T(r, F)}.$$

Now modulus of poles of $\frac{1}{F}$ being $0, 2\pi, 4\pi, \cdots$ we have, $N\left(r, \frac{1}{F}\right) = O(\log r)$ and so $\delta(0; F) = 1$. Hence the condition $2\delta(0; F) > 1$ is satisfied. Here $\rho_f = \rho_g$ although $T(r, h) = o(T(r, f))$ and $T(r, h) = o(T(r, g))$ are not satisfied.

Theorem 3.3. *Let f, g be two entire functions of finite order and let there is an entire function h such that $F = \frac{f}{h}$ and $G = \frac{g}{h}$ become two entire functions with $T(r, h) = o(T(r, f))$ and $T(r, h) = o(T(r, g))$. If F, G share 0, 1 CM, $\delta(0; F) > 0$ and $\delta(0, F) \neq \frac{1}{p}$ with an integer $p \geq 2$, then 0 is lacunary for f iff 0 is lacunary for h and $\rho_f = \rho_g$.*

Proof. By the similar argument as Theorem 3.1, we obtain F, G satisfy all conditions of Theorem 2.2. So we get $FG = 1$ and 0 is lacunary for F. Now as Theorem 3.1, from $FG = 1$, we can easily obtain $\rho_f = \rho_g$. Again 0 is lacunary for F gives $F = e^P$, where P is an entire function and so $f = he^P$. Thus 0 is lacunary for f iff 0 is lacunary for h. Hence the theorem. □

Example 3.4. *The functions in Example 3.2 satisfy all the conditions of Theorem 3.3 except $T(r, h) = o(T(r, f))$ and $T(r, h) = o(T(r, g))$ and $\rho_f = \rho_g$. Moreover 0 is neither lacunary for f nor for h.*

Theorem 3.5. *Let f, g be two entire functions of finite order and let there is an entire function h such that $F = \frac{f}{h}$ and $G = \frac{g}{h}$ become two entire functions with $T(r, h) = o(T(r, f))$ and $T(r, h) = o(T(r, g))$. If F, G share 1 CM and that $\delta(0, F) > 0$ and 0 is lacunary for F then $\rho_f = \rho_g$ and 0 is lacunary for f iff 0 is lacunary for h.*

Proof. Using Theorem 2.3 and exactly proceeding like Theorem 3.3 we obtain the result. □

Example 3.6. *Take $f(z) = e^z$, $g(z) = e^{-3z}$ and $h(z) = e^{-z}$. Here $T(r, f) = \frac{r}{\pi}$, $T(r, g) = \frac{3r}{\pi}$, $T(r, h) = \frac{r}{\pi}$ and $T(r, F) = \frac{2r}{\pi}$. So $T(r, h) = O(T(r, f))$, $T(r, h) = O(T(r, g))$, $\delta(0, F) = 1$ and 0 is lacunary for F. Here also $\rho_f = \rho_g$ although $T(r, h) = o(T(r, f))$ and $T(r, h) = o(T(r, g))$ are not satisfied. Moreover 0 is lacunary for both f and h.*

Theorem 3.7. *Let f, g be two non-constant entire functions and let there is an entire function h such that $F = \frac{f}{h}$ and $G = \frac{g}{h}$ become two non-constant entire functions with $T(r, h) = o(T(r, f))$ and $T(r, h) = o(T(r, g))$. If F and G share 0, 1 CM and satisfy $\rho_f = \rho_g = \infty$, $\delta(0, f) > \frac{5}{6}$, then $\rho_f = \rho_g$.*

Proof. Using Theorem 2.4 we can easily get the result. □

Theorem 3.8. *Let f, g be two non-constant entire functions and let there is an entire function h such that $F = \frac{f}{h}$ and $G = \frac{g}{h}$ become two non-constant entire functions with $T(r, h) = o(T(r, f))$ and $T(r, h) = o(T(r, g))$. Let F and G share 0, 1 CM. Further assume that all zero-points of F, G excepting atmost finite number have multiplicities ≥ 7. Then $\rho_f = \rho_g$.*

Proof. The proof can be done using Theorem 2.5 and proceeding as Theorem 3.1. □

Theorem 3.9. *Let f, g be two non-constant entire functions and let there is an entire function h such that $F = \frac{f}{h}$ and $G = \frac{g}{h}$ become two non-constant entire functions with $T(r, h) = o(T(r, f))$ and $T(r, h) = o(T(r, g))$. If F and G share 0, 1 CM and there exists a complex number a $(\neq 0, 1)$ satisfying $E(a, k, F) = E(a, k, G)$, where k is a positive integer (≥ 2) or ∞, then $\rho_f = \rho_g$.*

Proof. F and G satisfy all the conditions of Theorem 2.6. So four cases will arise.

Case 1: $F \equiv G$. In this case obviously $\rho_f = \rho_g$.

Case 2: $\left(F - \frac{1}{2}\right)\left(G - \frac{1}{2}\right) \equiv \frac{1}{4}$. Then

$$\frac{1}{F} = 2 - \frac{1}{G} \text{ and } \frac{1}{G} = 2 - \frac{1}{F} \text{ and so } \rho_F = \rho_G.$$

Case 3: $FG \equiv 1$. Then

$$F = \frac{1}{G} \text{ and } G = \frac{1}{F} \text{ and so } \rho_F = \rho_G.$$

Case 4: $(F-1)(G-1) \equiv 1$. Then

$$\frac{1}{F} = 1 - \frac{1}{G} \text{ and } \frac{1}{G} = 1 - \frac{1}{F} \text{ and so } \rho_F = \rho_G.$$

Now proceeding exactly as Theorem 3.1 we have from Case 2, Case 3 and Case 4

$$\rho_f = \rho_g.$$

Hence the proof. □

Using Theorem 2.7 we will show that, the order restriction of f and g in Theorem 3.1 can be removed perfectly.

Theorem 3.10. *Let f, g be two entire functions and let there is an entire function h such that $F = \frac{f}{h}$ and $G = \frac{g}{h}$ become two entire functions with $T(r,h) = o(T(r,f))$ and $T(r,h) = o(T(r,g))$. If F, G share 0, 1 CM and $2\delta(0;F) > 1$ then $\rho_f = \rho_g$.*

Proof. Proof is analogues to Theorem 3.1. □

The following Theorem is an extension of Theorem 3.5.

Theorem 3.11. *Let f, g be two entire functions and let there is an entire function h such that $F = \frac{f}{h}$ and $G = \frac{g}{h}$ become two entire functions with $T(r,h) = o(T(r,f))$ and $T(r,h) = o(T(r,g))$. If F, G share 1 CM and $\delta(0,F) + \delta(0,G) > 1$ and $\delta(\infty,F) = \delta(\infty,G) = 1$, then $\rho_f = \rho_g$.*

Proof. Proof can be done using Theorem 2.8. □

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