International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **13(1)**(2025), 65–92
Available Online: http://ijmaa.in

Construction of Two Factor Asymmetrical Factorial Designs that are Characterized by Balance with Orthogonal Factorial Structure (BAFDs)

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Abstract

The construction of two factor BAFD's with block size equal to the levels levels of the first factor or block size equal the levels of the second factor has been pointed out. It should be noted that the construction methods involved utilizes balanced arrays, orthogonal arrays and transitive arrays. Several theorems that enable generation of resolvable balanced incomplete block designs have been proved. These theorems involve galois fields and methods of step cycles. Using resolvable balanced incomplete block designs and balanced arrays we have also pointed out the construction of two factor BAFD's with block size which is equal to a common multiple of the levels of the two factors. It should also be noted that the designs constructed are such that the main effects of each of the two factors are estimated with full/maximum efficiency. The designs are balanced with orthogonal factorial structure (OFS).

Keywords: Block size; balanced arrays; orthogonal arrays; transitive arrays; resolvable balanced incomplete block designs; main effects; full efficiency; maximum efficiency; step cycles; Galois fields; balance; orthogonal factorial structure.

1. Introduction

In many situations there arise scenarios when an experimenter has to use factors at different levels. The problem of obtaining confounded plans for such cases has received a good deal of attention. To this extent, [39], by trial and hit methods obtained confounded plans of the type $3^m \times 2^n$, where m and n are any positive integers. Using orthogonal arrays of strength 2 [28] gave methods for constructing Extended Group Design (EGD) designs for $s_1 \times s_2$ experiments in blocks of size $s_1 < s_2$. Thomson [35] starting from a basic $s_1 \times s_2$ design in blocks of size s_2 ($s_2 < s_1$, s_1 being a prime number or power of prime) obtained three factor designs. Rao [30] constructed some series of designs from

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orthogonal latin squares for $s_1 \times s_2$ experiments in block of size s_1 and $s_2 - 1$ replications. Tharthare [34] gave a cass of balanced designs with OFS. Muller [26] considered the use of balanced incomplete block designs for the construction of $s_1 \times s_2$ balanced factorial with OFS when $s_1 > s_2$. Informative accounts and subsequent developments have been done by [16,22,33]. Sreenath [31] proposed a general method of obtaining block design for asymmetrical confounded factorial designs using block designs for asymmetrical factorial experiments.

Rajarathinam [29] used the methods that are used in construction of variance balanced designs in order to construct variance balanced block designs that are highly efficient. Ghosh [15] also extended the work of [29] in the construction of variance balanced block designs using methods used in construction of variance balanced designs.

Gupta [17] describes a general method of construction of supersaturated designs for asymmetric factorials obtained by exploiting the concept of resolvable orthogonal arrays and Hadamard matrices. El-Helbawy [18] considered three forms of a general null hypothesis Ho on the factorial parameters of a general asymmetrical factorial paired comparison experiment in order to determine optimal or efficient designs. Kumar [21] constructed designs by using confounding through equation methods. Construction of confounded asymmetrical factorial experiments in row-column settings and efficiency factor of confounded effects was worked out. Agarwal [1] attempted to construct asymmetrical factorial type switch over designs having strip type arrangement of combination of the levels. To start with two factors at different levels were been considered. Jyoti Divecha [13] described a method of constructing cost-efficient response surface designs (RSDs) as compared to the replicated central composite designs (RCCDs), that are useful for modelling and optimization of the experiments asymmetric in some qualitative, quantitative factors with at least two unrestricted quantitative factors while the remaining take two or three levels. Voss [36] identified a Kronecker product structure for a particular class of asymmetric factorial designs in blocks, including the classes of designs generated by several of the generalizations of the classical method in the literature. Das [11] focuses on the construction and analysis of an extra ordinary type of asymmetrical factorial experiment which corresponds to fraction of a symmetrical factorial experiment as indicated by [10]. Gupta [7] establishes a lower bound for $tr[(M_d)^2]$ with respect to a main effects model, where d is an $s_1 \times s_2 \times, \cdots, \times s_m$ levels of asymmetric orthogonal array of strength at least 1. Nonisomorphic asymmetrical MS-optimal orthogonal arrays of strength 1 with $N = 6$, 8 and 12 runs are also presented, where A design d is called D-optimal if it maximizes $det(M_d)$ and is called MS-optimal if it maximizes $tr(M_d)$ and minimizes $tr[(M_d)^2]$ among those which maximize $tr(M_d)$, where M_d stands for the information matrix produced from d under a given model.

Mainardi [24] conceptualized the fundamental aspects of the Complete, Fractional, Central Composite Rotational and Asymmetrical factorial designs. Recent applications of these powerful tools were described. Bahl [4] developed a method for the construction of $p \times 3 \times 2$ asymmetrical factorial experiments with $(p-1)$ replications. Suen [32] proposed A general method of obtaining block designs for asymmetrical confounded factorial experiments using the block designs for symmetrical factorial experiments. Xue Min Zi [40] asymmetrical factorial designs containing clear effects. Metrika [25] explained how to choose an optimal $(s^2)s^n$ design for the practical need, where s is any prime or prime power accordingly considers the clear effects criterion for selecting good designs. Murthy [27] deals with situations where there was a need for designing an asymmetrical factorial experiment involving interactions. Failing to get a satisfactory answer to this problem from the literature, the authors have developed an adhoc method of constructing a design. It is transparent from that the design provides efficient estimates for all the required main effects and interactions. The later part of this paper deals with the issues of how this method is extended to more general situations and how this adhoc method is translated into a systematic approach. [2] developed The R package DoE.base which can be used for creating full factorial designs and general factorial experiments based on orthogonal arrays. Besides design creation, some analysis functionality is also available, particularly (augmented) half-normal effects plots.

Jalil [20] Alludes that monograph is an outcome of the research works on the construction of factorial experiments (symmetrical and asymmetrical). In this booklet, construction frameworks have been described for factorial experiments. The construction frameworks include general construction method of p^n factorial experiments, construction methods with confounded effects and detection method of confounded effects in a confounded plan. The concepts of combinatorial, matrix operations and linear equation technique have been deployed to develop the methods. Dipa Rani Das [9] discussed an Alternative Method of Construction and Analysis of Asymmetrical Factorial Experiment of the type 6 in Blocks of Size 12. Dipa Rani Das [11] focuses on the construction and analysis of an extra ordinary type of asymmetrical factorial experiment which corresponds to fraction of a symmetrical factorial experiment as indicated by [10]. For constructing this design, we have used 3 choices and for each choice we have used 5 different cases. Finding the block contents for each case we have seen that there are mainly two different cases for each choice. In case of analysis of variance, we have seen that, for the case where the highest order interaction effect is confounded in 4 replications, the loss of information is same for all the choices.

Klaus Hinkelmann [19] in his book chapter he discuss different methods of constructing systems of confounding for asymmetrical factorial designs, including: Combining symmetrical systems of confounding via the Kronecker product method, use of pseudo-factors, the method of generalized cyclic designs, method of finite rings (this method is also used to extend the Kempthorne parameterization from symmetrical to asymmetrical factorials), and the method of balanced factorial designs. We show the equivalence of balanced factorial designs and extended group divisible partially balanced incomplete block designs, establishing again a close link between incomplete block designs and confounding in factorial designs.

Angela Dean [12] in her book chapter discusses confounding in single replicate experiments in which at least one factor has more than two levels. First, the case of three-levelled factors is considered and the techniques are then adapted to handle m-levelled factors, where m is a prime number. Next, pseudo factors are introduced to facilitate confounding for factors with non-prime numbers of levels. Asymmetrical experiments involving factors or pseudo factors at both two and three levels are also considered, as well as more complicated situations where the treatment factors have a mixture of 2, 3, 4, and 6 levels. Analysis of an experiment with partial confounding is illustrated using the SAS and R software packages.

Gachii [14] alludes that Asymmetrical single replicate factorial designs in blocks are constructed using the deletion technique. Results are given that are useful in simplifying expressions for calculating loss of information on main effects and interactions, due to confounding with blocks. Designs for estimating main effects and low order interactions are also given. 1. Introduction Consider a single replicate factorial experiment.

Conto Lopez [23] in his work presents the results of a systematic literature review (SLR) and a taxonomical classification of studies about run orders for factorial designs published between 1952 and 2021. The objective here is to describe the findings and main and future research directions in this field. The main components considered in each study and the methodologies they used to obtain run sequences are also highlighted, allowing professionals to select an appropriate ordering for their problem. This review shows that obtaining orderings with good properties for an experimental design with any number of factors and levels is still an unresolved issue.

Rahul Mukerjee [5] in his present book gives, for the first time in book form, a comprehensive and up-to-date account of this modern theory. Many major classes of designs are covered in the book. While maintaining a high level of mathematical rigor, it also provides extensive design tables for research and practical purposes. Sunanda Bagchi [3] in his work discusses the construction of 'inter-class orthogonal' main effect plans (MEPs) for asymmetrical experiments. In such a plan, the factors are partitioned into classes so that any two factors from different classes are orthogonal. The researcher also defined the concept of "partial orthogonality" between a pair of factors. In many of our plans, partial orthogonality has been achieved when (total) orthogonality is not possible due to divisibility or any other restriction. We present a method of obtaining inter-class orthogonal MEPs. Using this method and also a method of 'cut and paste' we have obtained several series of inter-class orthogonal MEPs. One of them happens to be a series of orthogonal MEP (OMEPs) [see Theorem 3.6], which includes an OMEP for a 330 experiment on 64 runs.

Ching-Shui Cheng [8] book provides a rigorous, systematic, and up-to-date treatment of the theoretical aspects of factorial design. To prepare readers for a general theory, the author first presents a unified treatment of several simple designs, including completely randomized designs, block designs, and row-column designs. As such, the book is accessible to readers with minimal exposure to experimental design. Liuping Hu [6] Lee discrepancy has wide applications in design of experiments, which can be used to measure the uniformity of fractional factorials. An improved lower bound of Lee discrepancy for asymmetrical factorials with mixed two-, three- and four-level is presented. The new lower bound is more accurate for a lot of designs than other existing lower bound, which is a useful complement to the lower bounds of Lee discrepancy and can be served as a benchmark to search uniform designs with mixed levels in terms of Lee discrepancy.

The purpose of this paper is to use the methods used in construction of variance balanced design in order to construct variance balanced asymmetrical factorial designs that posses additional property known as orthogonal factorial structure (OFS). Specifically we shall in this paper construct asymmetrical factorial designs with OFS where two factors are involved and in which main effects are estimated with full efficiencies while two order interactions are estimated with maximum efficiency. In this regard, [38] explained the definitions given below in a detailed context.

2. Definitions

Definition 2.1. *Let $\Psi(j_1, \ldots, j_m)$ denote the treatment effect corresponding to a treatment combination $j_1 \ldots j_m$. These treatment effects are unknown parameters in the context of a factorial experiment; a linear parametric function*

$$\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_m=0}^{s_m-1} \ell(j_1 \cdots j_m)\Psi(j_1 \cdots j_m) \tag{1}$$

where $\ell(j_1 \cdots j_m)$ are real numbers, not all zero, such that

$$\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_m=0}^{s_m-1} \ell(j_1 \cdots j_m) = 0 \tag{2}$$

is called a treatment contrast.

Definition 2.2. *A design is called connected if rank$(C) = v - 1$. C is the design matrix(incidence matrix) and v is the total number of treatment combinations.*

Definition 2.3. *A treatment contrast is of the form $\underline{\ell}'\underline{\Psi}$, where the $v \times 1$ coefficient vector $\underline{\ell}$ is non-null and the sum of elements of $\underline{\ell}$ equal to zero. Such a treatment contrast will be said to be normalised if $\underline{\ell}'\underline{\ell} = 1$. Two treatment contrasts $\underline{\ell}_1'\underline{\Psi}$ and $\underline{\ell}_2'\underline{\Psi}$ will be called mutually orthogonal if $\underline{\ell}_1'\underline{\ell}_2 = 0$. A set of treatment contrasts will be called orthonormal if the contrasts in the set are all normalised and mutually orthogonal.*

Definition 2.4. *A treatment contrast $\underline{\ell}'\underline{\Psi}$ is estimable if $\underline{\ell}' \in \mathcal{R}(C)$, where for any matrix A, $\mathcal{R}(A)$ stands for it's row space.*

Definition 2.5. *Clearly, for an estimable treatment contrast $\underline{\ell}'\underline{\Psi}$, there exists a $v \times 1$ vector $\underline{\ell}^*$ such that $\underline{\ell}' = \underline{\ell}^{*'}C$. The best linear unbiased estimator (BLUE) of $\underline{\ell}'\underline{\Psi}$ is given by $\underline{\ell}'\hat{\underline{\Psi}} = \underline{\ell}^{*'}\underline{Q}$. All treatment contrasts are estimable if and only if the design is connected.*

Definition 2.6. *A factorial design will be said to have orthogonal factorial structure (OFS) if the BLUEs of estimable treatment contrasts belonging to distinct interactions are mutually orthogonal, i.e. uncorrelated.*

Definition 2.7. *In a factorial design, an interaction F^y, $y \in \Omega$, will be said to be balanced if either*

(a). all treatment contrasts belonging to F^y are estimable and the BLUEs of all normalised contrasts belonging to F^y have the same variance or;

(b). No contrast belonging to F^y is estimable.

Definition 2.8. *An experiment involving $m \geq 2$ factors $F_1, F_2, \ldots, F_m$ that appear at $s_1, \ldots, s_m (\geq 2)$ levels is called an $s_1 \times \cdots \times s_m$ factorial experiment (or an $s_1 \times \cdots \times s_m$ factorial for brevity).*

The purpose of this paper is to give simplified methods of constructing two factor asymmetrical factorial designs that are characterized by balance with orthogonal factorial structure.
In this paper we shall involve well known arrangements of arrays such as Difference Schemes, Orthogonal Arrays, Balanced Arrays, Transitive Arrays, and Hadarmard Matrices whose definitions are given below.

Definition 2.9. *An $r \times c$ array D with entries from $\mathcal{A}$ is called a difference scheme based on $(\mathcal{A}, +)$ if it has the property that for all i and j with $1 \leq i, j \leq c$, the vector difference between the i^{th} and j^{th} columns contains every element of $\mathcal{A}$ equally often if $i \neq j$*

Definition 2.10. *A $k \times b$ array A with entries from a set of v symbols is called an orthogonal array of strength t if each $t \times b$ subarray of A contains all possible v^t column vectors with the same frequency $\lambda = \frac{b}{v^t}$. It is denoted $OA(b, k, v, t; \lambda)$; the number λ is called the index of the array. The numbers b and k are known as the number of assemblies and constraints of the orthogonal array respectively.*

0	0	0	0	0	0	0	0	0
0	1	2	0	1	2	0	1	2
0	2	1	0	2	1	0	2	1
0	0	0	1	1	1	2	2	2
0	1	2	1	2	0	2	0	1
0	2	1	1	0	2	2	1	0
0	0	0	2	2	2	1	1	1
0	1	2	2	0	1	1	2	0
0	2	1	2	1	0	1	0	2

Table 1: This difference scheme is derived from $(GF(3), +)$

Example 2.11.

$$\begin{array}{cccccccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}$$
$$OA(8, 4, 2, 3; 1)$$

Definition 2.12. *Let A be a $k \times b$ array with entries from a set of v symbols. Consider the v^t ordered t-tuples $(x_1, \ldots, x_t)$ that can be formed from a t-rowed subarray of A, and let there be associated a non-negative integer $\lambda(x_1, \ldots, x_t)$ that is invariant under permutations of $x_1, \ldots, x_t$. If for any t-rowed subarray of A the v^t ordered t-tuples $(x_1, \ldots, x_t)$, each occur $\lambda(x_1, \ldots, x_t)$ times as a column, then A is said to ba a balanced array of strength t. It is denoted by $BA(b, k, v, t)$ and the numbers $\lambda(x_1, \ldots, x_t)$ are called the index parameters of the array.*

Clearly a BA(b,k,v,t) with $\lambda(x_1,\ldots,x_t)=\lambda$ for all t-tuples $(x_1,\ldots,x_t)$ is simply an orthogonal array $OA(b,k,v,t;\lambda)$.

Example 2.13.

$$\begin{array}{cccccccccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$
$$BA(10,5,2,2)$$

Definition 2.14. *A transitive array $TA(b,k,v,t;\lambda)$ is a $k\times b$ array of v symbols such that for any choice of t rows, the $\frac{v!}{(v-t)!}$ ordered t-tuples of distinct symbols each occur λ times as a column.*

Example 2.15.

$$\begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 \\ 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 \end{array}$$
$$TA(12,4,4,2;1)$$

Definition 2.16. *A hadamard matrix of order n is an $n\times n$ matrix H_n of $+1's$ and $-1's$ whose rows are orthogonal, that is, which satisfies*

$$H_n H_n^T = nI_n \tag{3}$$

For example, here are hadamard matrices of order $1,2$ and 4.

$$H_1=[1], H_2=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, H_4=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{4}$$

Theorem 2.17. *For all k and s,there always exists a $BA[T][k,s,\lambda]$ for some λ, where $BA[T][k,s,\lambda]$ is a $BA[[T][k,s,\lambda]]$ with parameters $\lambda(x,y)=(k-1)$ or $k\lambda$ accordingly as $x=y$ or Not.*

Proof. For all k and s, there exists a $TA[(ks-1)ksn,ks,ks,2]$ for some n. Let the symbols of the transitive array be denoted by $[0,1,...,ks-1]$. If we replace each symbol in the transitive array by $x(mod k)$. Then the transitive array becomes a $BA[(ks-1)ksn,ks,s,2]$ with parameters $\lambda(x,y)=(k-1)kn$ or k^2n according as $x=y$ or not, which is a $BA[T][ks,s,kn]$. The method of construction in Theorem 2.17 does not usually provide balanced arrays with a small number of assemblies as we desire. □

Definition 2.18. *An orthogonal array $OA[N,k,s,2]$ is said to be a-resolvable if it is statistically equivalent to the juxtaposition of $\frac{N}{as}$ arrays such that each factor occurs in each of these arrays a times at each level. A 1-resolvable orthogonal array is also called completely resolvable, otherwise it is called Partly resolvable.*

Let Ω^* be the set of all $m-$ component binary vectors, that is $\Omega^* = \Omega \cup \{(0,0,\ldots,0)\}$ where Ω is the set of all none null binary component vectors for $y = (y_1, y_2, \ldots, y_m) \in \Omega$ let

$$Z^y = \otimes_{i=1}^{m} Z_i^{y_i} \tag{5}$$

where for $1 \leq i \leq m$,

$$Z_i^{y_i} = I_i \quad \text{if} \quad y_i = 1 \tag{6}$$

$$= J_i \quad \text{if} \quad y_i = 0 \tag{7}$$

where I is an identity matrix and J is matrix of $1's$ both of order $m \times m$ From (5) and (6) we can ultimately obtain

$$C = r(\otimes_{i=1}^{m} I_i) - k^{-1}NN' \tag{8}$$

where C is the design matrix and N is the incidence matrix of a BAFD. (8) shows that the design has property A see [38]. For connected equireplicate designs with property A and a common replication number r the interaction efficiencies are given by

$$E(y) = 1 - \frac{1}{rk} g(y) \text{and E(y) = 1 if and only if g(y)=0} \tag{9}$$

[38] proved the following Corollary 2.20

Definition 2.19. *A $BA(T)(k,s,1)$ is a balanced array which is obtained by deleting S assemblies of the form $(i,i,\cdots,i)^T$ for $i = 0,1,2,\cdots,s-1$*

Corollary 2.20. *If s is a prime power, then there exists a $TA[s(s-1),s,s,2]$.*

While the following corollaries and theorem are well proved and suitable examples given in [37].

Corollary 2.21. *If a hadarmard matrix of order $4k$ exists, then a $BA(T)[k,2,1]$ exists, and can always be constructed.*

Corollary 2.22. *If k and s are both powers of the same prime p a $BA(T)[k,s,1]$ can always be constructed.*

Corollary 2.23. *If $s = p^n, k = 2s^l$ where p is an odd prime, $n \geq 1$ and $l \geq 0$, then a $BA(T)[k,s,1]$ can always be constructed.*

Theorem 2.24. *The existence of a partly resolvable (Definition 2.18) $OA[ks^2,ks,s,2]$ is equivalent to the existence of a $BA[T][k,s,1]$.*

Example 2.25. *Table 2 shows a difference scheme $D(6,6,3)$ constructed in a similar way from $GF(3)$.*

0	0	0	0	0	0
0	1	2	1	2	0
0	2	1	1	0	2
0	2	2	0	1	1
0	0	1	2	2	1
0	1	0	2	1	2

Table 2: A difference Scheme $D(6,6,3)$

Example 2.26. *Suppose $k = 2$ and that $s = 5$, we can construct $OA[50,10,5,2]$ by developing a difference scheme $D(10,10,5)$ and also a $BA[45,10,5,2]$. The Difference Scheme $D(10,10,5)$ is*

0	0	0	0	0	0	0	0	0	0
0	4	3	1	2	1	0	4	2	3
0	3	1	2	4	4	2	0	1	3
0	1	2	4	3	1	2	3	0	4
0	2	4	3	1	4	1	3	2	0
0	2	3	2	3	0	4	1	4	1
0	1	1	3	0	2	4	4	3	2
0	0	4	4	2	3	3	1	1	2
0	3	0	1	1	2	3	2	4	4
0	4	2	0	4	3	1	2	3	1

Table 3: Table $D(10,10,5)$

Example 2.27. *For $k = 3$ and $s = 3$ we can construct a $BA(T)[3,3,1]$ by first constructing a completely resolvable $OA[27,9,3,2]$. Applying theorem 2.24, we obtain $BA(T)[3,3,1]$.*

Example 2.28. *Let $M = [0,1,2]$. Among the Differences of corresponding elements of any two rows of the following array,* 0 *occurs* 6 *times, where* 1 *and* 2 *each occur* 8 *times.*

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	2	0	1	0	0	2	1	2	1	2	1	1	0	2	0	0	1	2	1	2
1	1	2	2	0	1	0	0	2	1	2	2	2	1	1	0	2	0	0	1	2	1
2	1	1	2	2	0	1	0	0	2	1	1	2	2	1	1	0	2	0	0	1	2
1	2	1	1	2	2	0	1	0	0	2	2	1	2	2	1	1	0	2	0	0	1
2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2	0	0
0	2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2	0
0	0	2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2
1	0	0	2	1	2	1	1	2	2	0	2	0	0	1	2	1	2	2	1	1	0
0	1	0	0	2	1	2	1	1	2	2	0	2	0	0	1	2	1	2	2	1	1
2	0	1	0	0	2	1	2	1	1	2	1	0	2	0	0	1	2	1	2	2	1
2	2	0	1	0	0	2	1	2	1	1	1	1	0	2	0	0	1	2	1	2	2

hence we can construct a $BA[66,12,3,2]$ with parameters $\lambda(x,y) = 6$ or 8 according as $x = y$ or not. i.e $BA(T)[4,3,2]$.

Example 2.29. *Let $M = [0,1,2,3]$. Among the differences of the corresponding elements of any two rows of the following array,* 0 *occurs* 4 *times, where* 1, 2 *and* 3 *occur* 6 *times each.*

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	2	0	2	1	1	3	2	3	1	0	3	2	0	2	3	3	1	2	1
3	3	0	1	2	0	2	1	1	3	2	1	1	0	3	2	0	2	3	3	1	2
2	3	3	0	1	2	0	2	1	1	3	2	1	1	0	3	2	0	2	3	3	1
3	2	3	3	0	1	2	0	2	1	1	1	2	1	1	0	3	2	0	2	3	3
1	3	2	3	3	0	1	2	0	2	1	3	1	2	1	1	0	3	2	0	2	3
1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2	0	2
2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2	0
0	2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2
2	0	2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3
1	2	0	2	1	1	3	2	3	3	0	3	2	0	2	3	3	1	2	1	1	0
0	1	2	0	2	1	1	3	2	3	3	0	3	2	0	2	3	3	1	2	1	1

hence we can construct a $BA[88,12,4,2]$ with parameters $\lambda(x,y) = 4$ or 6 according as $x = y$ or not,i.e $BA(T)[3,4,2]$

Example 2.30. *For $k = 3$ and $s = 3$ we can construct a $BA(T)[3,3,1]$ by first constructing a completely resolvable $OA[27,9,3,2]$. Applying theorem 2.24, we obtain $BA(T)[3,3,1]$.*

Example 2.31. *Let $M = [0,1,2]$. Among the Differences of corresponding elements of any two rows of the following array,0 occurs 6 times, where 1 and 2 each occur 8 times.*

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	2	0	1	0	0	2	1	2	1	2	1	1	0	2	0	0	1	2	1	2
1	1	2	2	0	1	0	0	2	1	2	2	2	1	1	0	2	0	0	1	2	1
2	1	1	2	2	0	1	0	0	2	1	1	2	2	1	1	0	2	0	0	1	2
1	2	1	1	2	2	0	1	0	0	2	2	1	2	2	1	1	0	2	0	0	1
2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2	0	0
0	2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2	0
0	0	2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2
1	0	0	2	1	2	1	1	2	2	0	2	0	0	1	2	1	2	2	1	1	0
0	1	0	0	2	1	2	1	1	2	2	0	2	0	0	1	2	1	2	2	1	1
2	0	1	0	0	2	1	2	1	1	2	1	0	2	0	0	1	2	1	2	2	1
2	2	0	1	0	0	2	1	2	1	1	1	1	0	2	0	0	1	2	1	2	2

hence we can construct a $BA[66,12,3,2]$ with parameters $\lambda(x,y) = 6$ or 8 according as $x = y$ or not. i.e $BA(T)[4,3,2]$.

Example 2.32. *Let $M = [0,1,2,3]$. Among the differences of the corresponding elements of any two rows of the following array,0 occurs 4 times, where 1, 2 and 3 occur 6 times each.*

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	2	0	2	1	1	3	2	3	1	0	3	2	0	2	3	3	1	2	1
3	3	0	1	2	0	2	1	1	3	2	1	1	0	3	2	0	2	3	3	1	2
2	3	3	0	1	2	0	2	1	1	3	2	1	1	0	3	2	0	2	3	3	1
3	2	3	3	0	1	2	0	2	1	1	1	2	1	1	0	3	2	0	2	3	3
1	3	2	3	3	0	1	2	0	2	1	3	1	2	1	1	0	3	2	0	2	3
1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2	0	2
2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2	0
0	2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2
2	0	2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3
1	2	0	2	1	1	3	2	3	3	0	3	2	0	2	3	3	1	2	1	1	0
0	1	2	0	2	1	1	3	2	3	3	0	3	2	0	2	3	3	1	2	1	1

hence we can construct a $BA[88,12,4,2]$ with parameters $\lambda(x,y) = 4$ or 6 according as $x = y$ or not,i.e $BA(T)[3,4,2]$.

3. $s_1 \times s_2$ BAFD'S with Block Size $s_1 (s_1 \leq s_2)$

We shall discuss the construction of two factor BAFD's. Construction of more than two factor BAFD's using two factor BAFD's will be discussed in another paper. We are only interested in BAFD's in which the main effects are estimated with high efficiencies. These designs can usually be constructed using arrays discussed in the previous chapter. Let F_1 and F_2 be the two factors in a BAFD at s_1 and s_2 levels respectively. We assume that $s_1 \leq s_2$ without loss of generality. Let N denote the incidence matrix of BAFD. By equations (5) and (6), the eigenvalues of NN^T are:

$$g(1,0) = r + (s_2 - 1)\lambda_{01} - \lambda_{10} - (s_2 - 1)\lambda_{11} \tag{10}$$

$$g(0,1) = r - \lambda_{01} + (s_1 - 1)\lambda_{10} - (s_1 - 1)\lambda_{11} \tag{11}$$

$$g(1,1) = r - \lambda_{01} - \lambda_{10} + \lambda_{11} \tag{12}$$

using the equality due to [28]

$$\sum(\theta(k_1) \cdot \theta(k_2) \cdots \theta(k_m) \cdot \lambda_{k_1 k_2 \cdots k_m}) = r(k-1) \tag{13}$$

and (9), where $\theta(k_t) = 1$ or $(s_t - 1)$ according as $k_t = 0$ or 1 and $\sum$ denotes the summation over $2^m - 1$ terms, we obtain

$$r(k-1) = (s_1 - 1)\lambda_{10} + (s_2 - 1)\lambda_{01} + (s_1 - 1)(s_2 - 1)\lambda_{11} \tag{14}$$

and so we can derive the efficiencies of the main effects as follows;-

$$\begin{aligned}
E[0,1] &= 1 - \frac{1}{rk} g[0,1] \\
&= 1 - \left[\frac{r - \lambda_{01} + (s_1-1)\lambda_{10} - (s_1-1)\lambda_{11}}{rk}\right] \\
&= \frac{kr - r + \lambda_{01} - (s_1-1)\lambda_{10} + (s_1-1)\lambda_{11}}{kr} \\
&= \frac{\begin{array}{c}(s_1-1)\lambda_{10} + (s_2-1)\lambda_{01} + (s_1-1)(s_2-1)\lambda_{11} \\ +\lambda_{01} - (s_1-1)\lambda_{10} + (s_1-1)\lambda_{11}\end{array}}{kr} \\
&= \frac{(s_2-1)\lambda_{01} + (s_1-1)(s_2-1)\lambda_{11} + \lambda_{01} + (s_1-1)\lambda_{11}}{kr} \\
&= \frac{[s_2 - 1 + 1]\lambda_{01} + [s_1-1]\lambda_{11}[s_2 - 1 + 1]}{kr} \\
&= \frac{s_2[\lambda_{01} + (s_1-1)\lambda_{11}]}{kr} \\
&= \frac{s_2[\lambda_{01} + (s_1-1)\lambda_{11}]}{kr[k-1]}[k-1] \\
&= \frac{s_2[\lambda_{01} + (s_1-1)\lambda_{11}]}{r[k-1]} \frac{[k-1]}{k} \\
&= \frac{[k-1]s_2}{k} \frac{[\lambda_{01} + (s_1-1)\lambda_{11}]}{r[k-1]} \\
&= \frac{[k-1]s_2}{k} \frac{[\lambda_{01} + (s_1-1)\lambda_{11}]}{[s_1-1]\lambda_{10} + [s_2-1]\lambda_{01} + [s_1-1][s_2-1]\lambda_{11}}
\end{aligned} \tag{15}$$

using equation (14). Similarly

$$\begin{aligned}
E[1,0] &= 1 - \frac{1}{rk} g[1,0] \\
&= \frac{[k-1]s_1}{k} \frac{[\lambda_{10} + (s_2-1)\lambda_{11}]}{[s_2-1]\lambda_{01} + [s_1-1]\lambda_{10} + [s_1-1][s_2-1]\lambda_{11}}
\end{aligned} \tag{16}$$

If the main effect of F_1 are estimated with full efficiency i.e $E[1,0] = 1$ then the block size k must be a multiple of s_1. We shall assume that $k = s_1$ throughout this section. For $k = s_1$ equation (16) becomes

$$E[1,0] = (s_1 - 1)\frac{\lambda_{10} + (s_2-1)\lambda_{11}}{(s_2-1)\lambda_{01} + (s_1-1)\lambda_{10} + (s_1-1)(s_2-1)\lambda_{11}} \tag{17}$$

$E[1,0] = 1$ if and only if $\lambda_{01} = 0$; that is two treatments at the same level of F_1 never occur together in the same block.

Theorem 3.1. *In an $s_1 \times s_2$ BAFD with block size s_1, the main effects are estimated with full efficiency if and only if $\lambda_{01} = 0$. This design is equivalent to a $BA[\lambda_{10}s_2 + \lambda_{11}s_2(s_2-1), s_1, s_2, 2]$ with parameters $\lambda(x,y) = \lambda_{10}$ or*

λ_{11} *according as* $x = y$ *or not.*

Proof. The first part of the theorem has been shown: we only need to prove the latter part. Suppose such a balanced array exists: if we identify columns, rows and symbols with blocks, the levels of F_1, and the levels of F_2 respectively then it is the specified BAFD. In proving Theorem 3.1, we did not use the condition $s_1 \leq s_2$; hence the theorem is true for all s_1 and s_2. For $k = s_1$ and $\lambda_{01} = 0$ in equation 15, we have

$$\begin{aligned} E[0,1] &= \frac{[s_1-1]s_2[0+(s_1-1)\lambda_{11}]}{s_1\left\{(s_1-1)\lambda_{10}+0+(s_1-1)(s_2-1)\lambda_{11}\right\}} \\ &= \frac{(s_1-1)^2 s_2\lambda_{11}}{s_1\left\{(s_1-1)\lambda_{10}+(s_1-1)(s_2-1)\lambda_{11}\right\}} \\ &= \frac{(s_1-1)^2 s_2\lambda_{11}}{s_1(s_1-1)\left\{\lambda_{10}+(s_2-1)\lambda_{11}\right\}} \\ &= \frac{(s_1-1)s_2\lambda_{11}}{s_1\left\{(s_2-1)\lambda_{11}+\lambda_{10}\right\}} \end{aligned} \tag{18}$$

$E[0,1]$ has the maximum value of $\frac{(s_1-1)s_2}{s_1(s_2-1)}$ when $\lambda_{10} = 0$. □

Theorem 3.2. *In an* $s_1 \times s_2$ *BAFD with block size* $s_1 (s_1 \leq s_2)$, *if the main effects of* F_1 *are estimated with full efficiency and the main effects* F_2 *are estimated with maximum efficiency* $\frac{(s_1-1)s_2}{s_1(s_2-1)}$ *then the BAFD has parameters* $\lambda_{10} = \lambda_{01} = 0$ *and* $\lambda_{11} \neq 0$. *This design is equivalent to a* $TA[\lambda_{11}s_2(s_2-1), s_1, s_2, 2]$.

Since $\lambda_{10} = 0$ means that two treatments at the same level of F_2 do not occur together in the same block, which implies $s_2 \geq k = s_1$ we do not need $s_1 \leq s_2$ in the construction of the designs in theorem 3.2. The construction of $TA[s_2(s_2-1)\lambda_{11}, s_1, s_2, 2]$ has been discussed in [37]. Deleting any $(s_2 - s_1)$ constraints from a $TA[s_2(s_2-1)\lambda_{11}, s_2, s_2, 2]$ we obtain a $TA[s_2(s_2-1)\lambda_{11}, s_1, s_2, 2]$. If we restrict $\lambda_{11} = 1$ then the existence of a $TA[s_2(s_2-1), s_1, s_2, 2]$ is equivalent to the existence of $s_1 - 1$ mutually orthogonal latin squares of order s_2 or $s_1 - 2$ mutually orthogonal latin squares of order s_2 with different elements in the diagonal.

Example 3.3. *A* 3×4 *BAFD with* $b = 12$, $k = 3$, $r = 3$, $\lambda_{01} = \lambda_{10} = 0$ *and* $\lambda_{11} = 1$ *can be constructed from a* $TA[12, 3, 4, 2]$

Blocks	1	2	3	4	5	6	7	8	9	10	11	12
levels of F_1	Levels	of	F_2									
0	0	1	2	3	0	1	2	3	0	1	2	3
1	1	0	3	2	2	3	0	1	3	2	1	0
2	2	3	0	1	3	2	1	0	1	0	3	2

Table 4: Table of a 3×4 BAFD

In this design, $E[1,0] = 1$, $E[0,1] = \frac{8}{9}$ and $E[1,1] = \frac{5}{9}$.

Example 3.4. *A* 3×6 *BAFD with* $b = 30, k = 3, r = 5$ $\lambda_{01} = \lambda_{10} = 0$ *and* $\lambda_{11} = 1$ *can be constructed from a* $TA[30, 3, 6, 2]$. *The efficiencies are* $E[1, 0] = 1.0$, $E[0, 1] = \frac{4}{5}$ *and* $E[1, 1] = \frac{3}{5}$.

Example 3.5. *A* 7×20 *BAFD with* $b = 80$, $k = 7$, $r = 4$, $\lambda_{01} = \lambda_{10} = 0$, $\lambda_{11} = 1$, *can be constructed from a* $TA[80, 7, 20, 2]$. *The efficiencies are* $E[1, 0] = 1.0$, $E[0, 1] = \frac{120}{133}$ *and* $E[1, 1] = \frac{23}{28}$.

Example 3.6. *A* 12×15 *BAFD with* $b = 630, k = 12, r = 42, \lambda_{01} = \lambda_{10} = 0$, $\lambda_{11} = 3$, *can be constructed from a* $TA[630, 12, 15, 2]$. *The efficiencies are* $E[1, 0] = 1.0$, $E[0, 1] = \frac{165}{168}$ *and* $E[1, 1] = \frac{51}{56}$.

Corollary 3.7. *In an* s^2 *symmetrical factorial design* (FD) *with block size* s *and if all the main effects are estimated with full efficiency then the FD has parameters* $\lambda_1 = 0$ *and* $\lambda_2 \neq 0$.*This design is equivalent to a* $TA[\lambda_2 s(s-1), s, s, 2]$.

Example 3.8. *If* s *is a prime power, then there exists a* $TA[s(s-1), s, s, 2]$ *by corollary 2.20. Hence we can always construct an* s^2 *symmetrical FD with* $r = s - 1, b = s(s-1), k = s, \lambda_1 = 0, E_1 = 1$ *and* $E_2 = \frac{s-2}{s-1}$ *assuming that* $\lambda_2 = 1$.

Example 3.9. *A* 6^2 *symmetrical FD with* $r = 10, b = 60, k = 6, \lambda_1 = 0, \lambda_2 = 2$ *can be constructed from a* $TA[60, 6, 6, 2]$. *The efficiencies are:* $E_1 = 1$ *and* $E_2 = \frac{4}{5}$.

Example 3.10. *A* 21×21 *FD can be constructed from a* $TA[2, 100, 21, 21, 2]$ *with* $\lambda_1 = 0, \lambda_2 = 5, b = 2, 100, k = 21$ *and* $r = 100$. *The efficiencies are* $E_1 = 1$ *and* $E_2 = \frac{19}{20}$. *Similarly we can construct* $10^2, 12^2, 14^2$ *BAFD'S by using* $TA[360, 10, 10, 2]$, $TA[660, 12, 12, 2]$ *and* $TA[1092, 14, 14, 2]$ *respectively.*

4. $s_1 \times s_2$ BAFD's with Block Size $s_2 (s_1 < s_2)$

If the main effects of F_2 are estimated with full efficiency then the block size k must be a multiple of s_2. Assume that $k = s_2$ throughout this section. By Theorem 3.1, $E[0, 1] = 1$ if and only if $\lambda_{10} = 0$. Furthermore the design is equivalent to a $BA[\lambda_{01} s_1 + \lambda_{11} s_1 (s_1 - 1), s_2, s_1, 2]$ with parameters $\lambda(x, y) = \lambda_{01}$ or λ_{11} according as $x = y$ or not, if we identify the columns, rows, and symbols of the balanced array with blocks, the levels of F_2 and the levels of F_1 of the design.

Example 4.1. *A* 2×3 *BAFD with* $b = 4, k = 3, r = 2, \lambda_{10} = 0$ *and* $\lambda_{01} = \lambda_{11} = 1$ *can be constructed from the* $OA[4, 3, 2, 2]$

Blocks	1	2	3	4
Levels of F_2	levels	of	F_1	
0	0	0	1	1
1	0	1	0	1
2	0	1	1	0

Table 5: Table of a 2×3 BAFD

In this design, the efficiencies are: $E[0, 1] = 1$ and $E[1, 0] = E[1, 1] = \frac{2}{3}$

Example 4.2. *A* 11×12 *BAFD with* $b = 121, k = 12, r = 11, \lambda_{10} = 0$ *and* $\lambda_{01} = \lambda_{11} = 1$ *can be constructed from an* $OA[121, 12, 11, 2]$*. The efficiencies are* $E[0,1] = 1$, $E[1,0] = \frac{11}{12}$ *and* $E(1,1) = \frac{11}{12}$.

Example 4.3. *A* 3×13 *BAFD with* $b = 27, k = 13, r = 9, \lambda_{10} = 0, \lambda_{01} = \lambda_{11} = 3$ *can be constructed from an* $OA[27, 13, 3, 2]$, $E[0,1] = 1.0$, $E[1,0] = \frac{12}{13}$, $E[1,1] = \frac{12}{13}$.

Example 4.4. *A* 4×9 *BAFD with* $b = 32, k = 9, r = 8, \lambda_{10} = 0, \lambda_{01} = \lambda_{11} = 2$ *can be constructed from an* $OA[32, 9, 4, 2]$, $E[0,1] = 1.0$, $E[1,0] = \frac{8}{9}$, $E[1,1] = \frac{8}{9}$.

Example 4.5. *A* 5×11 *BAFD with* $b = 50, k = 11, r = 10, \lambda_{10} = 0, \lambda_{01} = \lambda_{11} = 2$ *can be constructed from an* $OA[50, 11, 5, 2]$, $E[0,1] = 1.0$, $E[1,0] = \frac{10}{11}$, $E[1,1] = \frac{10}{11}$.

Example 4.6. *A* 2×8 *BAFD with* $b = 16, k = 8, r = 8, \lambda_{10} = 0, \lambda_{01} = \lambda_{11} = 4$ *can be constructed from an* $OA[16, 8, 2, 2]$, $E[0,1] = 1.0$, $E[1,0] = \frac{7}{8}$, $E[1,1] = \frac{7}{8}$.

For $\lambda_{10} = 0$ and $k = s_2$ equation (16) becomes

$$\begin{aligned} E[1,0] &= \frac{(k-1)s_1}{k} \frac{\lambda_{10} + (s_2-1)\lambda_{11}}{(s_2-1)\lambda_{01} + (s_1-1)\lambda_{10} + (s_1-1)(s_2-1)\lambda_{11}} \\ &= \frac{(s_2-1)s_1}{s_2} \frac{(s_2-1)\lambda_{11}}{(s_2-1)\lambda_{01} + (s_1-1)(s_2-1)\lambda_{11}} \\ &= \frac{s_1\lambda_{11}(s_2-1)^2}{s_2(s_2-1)[\lambda_{01} + (s_1-1)\lambda_{11}]} \\ &= \frac{s_1\lambda_{11}(s_2-1)}{s_2[\lambda_{01} + (s_1-1)\lambda_{11}]} \\ &= \frac{(s_2-1)s_1\lambda_{11}}{s_2[\lambda_{01} + (s_1-1)\lambda_{11}]} \\ &= \frac{(s_2-1)s_1}{s_2[(s_1-1) + \frac{\lambda_{01}}{\lambda_{11}}]} \\ &= \frac{(s_2-1)s_1}{s_2} \frac{1}{(s_1-1) + \frac{\lambda_{01}}{\lambda_{11}}} \end{aligned} \tag{19}$$

Note that $\lambda_{01} \neq 0$, since $k = s_2 > s_1$ implies that at least two treatments in a given block have the same level of F_1. To maximize $E(1,0)$, it is required that $\frac{\lambda_{01}}{\lambda_{11}}$ be as small as possible.

Theorem 4.7. *In an* $s_1 \times s_2 (s_1 \leq s_2)$ *BAFD with block size* s_2 *and* $\lambda_{10} = 0$ *the following inequality holds:*

$$\frac{\lambda_{01}}{\lambda_{11}} \geq \frac{s_2 - s_1}{s_2} \tag{20}$$

when the equality holds, $E(1,0) = 1.0$ *and* $E(1,1) = \frac{s_2-2}{s_2-1}$.

Proof. $g(0,1) = 0$ in this BAFD since the main effect of F_2 is estimated with full efficiency. By equation (11) we have

$$r = \lambda_{01} + (s_1 - 1)\lambda_{11} \tag{21}$$

substituting r in equation (10) in equation (21), we have

$$\begin{aligned} g(1,0) &= \lambda_{01} + (s_1 - 1)\lambda_{11} + (s_2 - 1)\lambda_{01} - (s_2 - 1)\lambda_{11} \\ &= s_2\lambda_{01} + [s_1 - 1 - s_2 + 1]\lambda_{11} \\ &= s_2\lambda_{01} - (s_2 - s_1)\lambda_{11} \end{aligned} \tag{22}$$

but $g(1,0) \geq 0$, since $g(1,0)$ is an eigenvalue of the none negative definite matrix NN'. Therefore we have equation (20) and the equality holds if and only if $g(1,0) = 0$. i.e $E(1,0) = 1$, i.e.

$$\begin{aligned} g(1,1) &= r - \lambda_{01} - \lambda_{10} + \lambda_{11} \\ &= \lambda_{01} + (s_1 - 1)\lambda_{11} - \lambda_{01} - 0 + \lambda_{11} \\ &= (s_1 - 1)\lambda_{11} + \lambda_{11} \end{aligned}$$

but

$$\begin{aligned} E[1,1] &= 1 - \frac{1}{rk}[g(1,1)] \\ &= 1 - \frac{[s_1 - 1]s_2 + s_2}{s_2[\lambda_{01} + (s_1 - 1)\lambda_{11}]} \end{aligned}$$

if $\frac{\lambda_{01}}{\lambda_{11}} = \frac{s_2 - s_1}{s_2}$ and using equations (14) and (21)

$$\begin{aligned} &= 1 - \frac{(s_1 - 1)s_2 + s_2}{s_2[s_2 - s_1 + (s_1 - 1)s_2]} \\ &= \frac{s_2[s_2 - s_1 + (s_1 - 1)s_2] + -[(s_1 - 1)s_2 + s_2]}{s_2[s_2 - s_1 + (s_1 - 1)s_2]} \\ &= \frac{s_2[s_1 s_2 - s_1] - s_1 s_2}{s_2[s_1 s_2 - s_1]} \\ &= \frac{s_2 s_1 s_2 - s_1 s_2 - s_1 s_2}{s_2[s_1 s_2 - s_1]} \\ &= \frac{s_2 - 2}{s_2 - 1} \quad \text{as} \quad \text{required.} \end{aligned}$$

Since a necessary condition for $E[1,0] = 1$ is that block size k must be a multiple of s_1 we must assume that $s_2 = ms_1 (= k)$ for some integer m in order to construct a BAFD such that all main effects are estimated with full efficiency. When $s_2 = ms_1$ equation (20) becomes

$$\frac{\lambda_{01}}{\lambda_{11}} \geq \frac{m - 1}{m} \tag{23}$$

□

Corollary 4.8. *In an $s_1 \times s_2$ BAFD with block size $s_2 (> s_1)$ the main effects of F_1 and F_2 are estimated with full efficiency if and only if $s_2 = ms_1, \lambda_{10} = 0$ and $\frac{\lambda_{01}}{\lambda_{11}} = \frac{m-1}{m}$ for some m. The design is equivalent to a*

$BA[(ms_1-1)s_1\lambda, ms_1, s_1, 2]$ with parameters $\lambda(x,y) = (m-1)\lambda$ or $m\lambda$ according as $x = y$ or not, i.e. a $BA(T)(m, s_1, \lambda)$.

By theorem 2.17 for any given m and s_1 we can always construct a $BA(T)(m, s_1, \lambda)$ for some λ. Thus we can always construct an $ms_1 \times s_1$ BAFD such that all main effects are estimated with full efficiency, but a large replication may be needed. The construction of a $BA(T)(m, s_1, 1)$ for some m and s_1 are discussed in [37]. In Example 2.31 and 2.32, we can also construct a $BA(T)[4, 3, 2]$ and a $BA(T)(3, 4, 2)$.

Example 4.9. *A 2×4 BAFD with $b = 6, k = 4, r = 3, \lambda_{10} = 0$, can be constructed from a $BA(T)(2, 2, 1) = BA[6, 4, 2, 2]$ with $\lambda(x,y) = 1$ or 2 according as $x = y$ or not*

Blocks	1	2	3	4	5	6
Levels of F_2	Levels	of	F_1			
0	1	0	1	0	1	0
1	0	1	1	0	0	1
2	1	0	0	1	0	1
3	0	1	0	1	1	0

Table 6: Table of a 2×4 BAFD

In this design, the efficiencies are: $E[0,1] = 1$, $E[1,0] = 1$ and $E[1,1] = \frac{2}{3}$.

Example 4.10. *A 7×42 BAFD with $b = 287, k = 42, r = 41, \lambda_{10} = 0, \lambda_{01} = 5, \lambda_{11} = 6$ can be constructed from a $BA(T)[6, 7, 1] = BA[287, 42, 7, 2]$ with $\lambda(x,y) = 5$ or 6 according as $x = y$ or Not. The efficiencies of this designs are: $E[0,1] = 1.0$, $E[1,0] = 1.0$ and $E[1,1] = \frac{40}{41}$.*

Example 4.11. *A 17×238 BAFD with $b = 4029, k = 238, r = 237, \lambda_{10} = 0, \lambda_{01} = 13, \lambda_{11} = 14$, can be constructed from a $BA(T)[14, 17, 1] = BA[4029, 238, 17, 2]$ with $\lambda(x,y) = 13$ or 14 according as $x = y$ or Not. The efficiencies of this design are: $E[0,1] = 1.0$, $E[1,0] = 1.0$ and $E[1,1] = \frac{236}{237}$.*

Example 4.12. *A 23×391 BAFD with $b = 62790, k = 391, r = 2730, \lambda_{10} = 0, \lambda_{01} = 112, \lambda_{11} = 119$ can be constructed from a $BA(T)[17, 23, 7] = BA[62, 790, 391, 23, 2]$ with $\lambda(x,y) = 112$ or 119 according as $x = y$ or Not. This design has efficiencies: $E[0,1] = 1.0$, $E[1,0] = 1.0$ and $E[1,1] = \frac{389}{390}$.*

5. $s_1 \times s_2$ BAFD's with Block Size a Common Multiple of s_1 and s_2

In an $s_1 \times s_2$ BAFD with block size s_2, if s_2 is not a multiple of s_1, then the main effect of F_1 cannot be estimated with full efficiency. To estimate all main effects with full efficiency, the block size k must be a common multiple of s_1 and s_2. Let $s_1 = ps$ and $s_2 = qs$ where $s > 1$. A method is given below to construct $s_1 \times s_2$ BAFD with block size pqs such that all the main effects are estimated with full efficiency.

Theorem 5.1. *If there exists a resolvable BIBD with qs treatments and block size q, then there exists a $ps \times qs$ BAFD with block size pqs such that all main effects are estimated with full efficiency.*

Proof. Construct a $BA(T)(p,s,n)$ for some integer n by Theorem 2.17. In the resolvable BIBD, there being s blocks in each replication , we can number the block in each replication by $0,1,\cdots,s-1$. Replacing each symbol in the balanced array by a group of symbols which represents blocks in the BIBD for each replication, we obtain a $pqs \times [ps-1]snr^{'}$ matrix, where $r^{'}$ is the number of replications in the BIBD. Assign i^{th} level of F_1 to the rows from the $(i_q+1)^{th}$ to the $(i+1)^{th}$, where $i=0,1,\cdots,ps-1$.Identifying columns and symbols with blocks and the levels of F_2, we get a $ps \times qs$ design with block size pqs.

We shall show that all the main effects of the design constructed above are estimated with full efficiency.Let $\lambda^{'}$ be the number of blocks in which two treatments occur together in the BIBD, then $(qs-1)\lambda^{'}=(q-1)r^{'}$. Assume that $r^{'}=(qs-1)m$ and $\lambda^{'}=(q-1)m$, where m need not be an integer. Let $\lambda_{01},\lambda_{10},\lambda_{11}$ denote the parameters and r denote the number of replications in the $ps \times qs$ design, then through inspection we have

$$\begin{aligned} \lambda(x,y) &= (ps-1)^{x+1}(qs-1)^{y+1}(p-1)^{x}(q-1)^{y}mn+(xy)(pq)(s-1)^{xy}mn \\ x,y &= 0 \quad \text{or} \quad 1 \quad \text{in} \quad \text{mod } 2 \end{aligned} \tag{24}$$

so

$$\left\{ \begin{array}{c} \lambda_{01}=(ps-1)(q-1)mn \\ \lambda_{10}=(qs-1)(p-1)mn \\ \lambda_{11}=(p-1)(q-1)mn+pq(s-1)mn \\ \lambda_{00}=r=(ps-1)(qs-1)mn \end{array} \right\} \tag{25}$$

substituting the parameters of the equations (10), (11) and (12) in equations (25) and (9) we get

$$E[0,1]=E[1,0]=1 \quad \text{and} \quad E[1,1]=-\frac{s-1}{(ps-1)(qs-1)}+1$$

Given any q and s, there always exists a resolvable BIBD with qs treatments and block size q if the number of replications is allowed to be large.

Example 5.2. *The irreducible BIBD of qs treatments with block size q in which each of the $\binom{qs}{q}$ possible $q-$ element combinations form a block is resolvable with parameters*

$$v=qs, \quad b=\binom{qs}{q}, \quad r=\binom{qs-1}{q-1}, \quad k=q, \quad \lambda=\binom{qs-2}{q-2} \tag{26}$$

□

Definition 5.3. *Suppose $(\mathcal{F},\mathcal{A})$ is a (v,k,λ)-BIBD, a parallel class in $(\mathcal{F},\mathcal{A})$ is a subset of disjoint blocks from $\mathcal{A}$ whose union is $\mathcal{F}$. A partition of $\mathcal{A}$ into r parallel classes is called a resolution; and $(\mathcal{F},\mathcal{A})$ is said to be a resolvable BIBD if $\mathcal{A}$ has at least one resolution. We say that $\mathcal{F}$ is a finite set of points called treatments, where*

$\mathcal{F} = \Big\{0, 1, 2, \cdots, v-1\Big\}.$

Several methods of constructing a resolvable Resolvable Incomplete Block Designs which can be used to construct $s_1 \times s_2$ BAFD's whose block designs are common multiples of s_1 and s_2 are illustrated below. To construct a BIBD with block size $k = 3$ and a finite number of symbols $V = v$ one can use the methods of one step cycles, two step cycles or three steps cycles. The method of one step cycles is applicable when

$$v = 2y + 1 = 24m + 3 \text{ or when}$$
$$v = 2y + 1 = 24m + 9 \qquad (27)$$

we may denote the element 0 by k and the others by $1, 2, 3, \cdots, 2y$ place k at the center of the circle and the other elements $1, 2, 3, \cdots, 2y$ at equidistant intervals on their circumference. The companions of k are to be different on each parallel class. If we suppose that on the first parallel class they are 1 and $y + 1$ on the second 2 and $2 + y$ and so on, then the diameters through k will give for each parallel class a triplet in which k appears. On each parallel class we have to find $\frac{2(y-1)}{3}$ other triplets satisfying the conditions of the problem. Every triplet formed from the remaining $2y - 2$ elements will be represented by an inscribed triangle joining the points representing these elements. The sides of the triangles are the chords joining these $2y - 2$ points. The sides of the triangles so represented are denoted by the letters p, q, r. The term, triad or grouping denotes any of p, q, r which determines the dimensions of an inscribed triangle. If p, q, r are proportional to the smaller arcs subtended then $p + q = r$ or $p + q + r = 2y$. If $\frac{(y-1)}{3}$ scalene triangles can be inscribed in the circle so that to each triangle corresponds an equal complimentary triangle having its equal sides parallel to those of the first and its vertices at free points then the system of $\frac{2(y-1)}{3}$ triangles with the corresponding diameter will give an arrangement for one parallel class. If the system is permuted cyclically$(y - 1)$ times we get arrangements for the other $(y - 1)$ parallel classes.

The method of two step cycles is applicable when $v = 12m + 3$. When v is of this form and m is odd we cannot get sets of complimentary triangles as required. Hence to apply a similar method we have to find $\frac{2(y-1)}{3}$ different dissimilar inscribed triangles having no vertex in common and satisfying the conditions $p + q = r$ or $p + q + r = 2y$. These solutions are also central. p, q, r are proportional to the smaller arcs subtended. In the first part of this solution $\frac{v}{3}$ of these triangles must be selected to form an arrangement of the first parallel class. By rotating this arrangement two steps at a time we obtain triples for $\frac{v}{3}$ parallel classes in all. The method of three step cycles applies if

$$v = 18m + 3 \text{ or } v = 18m + 9 \text{ or } v = 18m + 15 \qquad (28)$$

It gives a solution for every value of v except $v = 15$. In this, method we may with equal propriety represent all the elements by symbols placed at equidistant intervals round the circumference of a circle. Such solutions are termed as none central. The symbols may be $1, 2, 3, \cdots, v$, or letters $a_1, b_1, c_1, a_2, b_2, c_2, \cdots$ any triplet will be represented by a circle whose sides are chords of a circle. The arrangement of any parallel class is to include all the elements and therefore the triangles representing the triplets for a given parallel class are $\frac{v}{3}$ in number, and so each element appears in only one triplet, thus no two triangles can have a common vertex. The complete three steps solution will require the determination of a system of $\frac{(n-1)}{2}$ inscribed triangles. In the first part of the solution, $\frac{v}{3}$ of these triangles must be selected to form an arrangement for the first parallel class, so that by rotating this arrangement three steps at a time we obtain triplets for $\frac{v}{3}$ parallel classes in all. If $q = 4t - 1$ is a prime power, then there exists a resolvable balanced incomplete block design with block size $k = 2t$ and number of symbols $v = 4t$ and $\lambda = 2t - 1$. As for the point set we take $\mathcal{F} = GF(q) \bigcup \left\{\infty\right\}$. Developing the parallel classes.

$$H_0^2 \cup \left\{0\right\} \quad \text{and} \quad H_1^2 \cup \left\{\infty\right\} \tag{29}$$

over $GF(q)$ produces the required blocks and resolution. The multiplicative cosets $H_0^e, H_1^e, H_2^e, \cdots, H_{e-1}^e$ are defined by $H_m^e = \left\{x^t : t \equiv m \ \ mod \ \ e\right\}$, where x denotes a primitive element of $GF(q)$. If $q = 4t + 1$ is a prime power, then there exists a resolvable balanced incomplete block design with block size $k = 2t + 1$ and number of symbols $v = 4t + 2$ and $\lambda = 4t$. As for the point set we take $\mathcal{F} = GF(q) \cup \left\{\infty\right\}$. Developing the parallel classes.

$$\begin{matrix} H_0^2 \cup \left\{0\right\} \\ H_1^2 \cup \left\{\infty\right\} \end{matrix} \text{and} \begin{matrix} H_0^2 \cup \left\{\infty\right\} \\ H_1^2 \cup \left\{0\right\} \end{matrix} \tag{30}$$

over $GF(q)$ produces the required blocks and resolution. Let $\lambda \leq k - 1$. Suppose there is a difference family $DF_\lambda(k, v) \left\{A_0, A_1, A_2, A_3, \cdots, A_{s-1}\right\}$ over a ring R whose base blocks are mutually disjoint. If there is a set of k distinct units $\left\{u_0, u_1, u_2, \cdots, u_{k-1}\right\}$ whose differences are all units of R, then there exists a Resolvable balanced incomplete block design with block size $K = k$ and number of symbols $V = kv$, where s represents the number of base blocks and that

$$B_j^j = A_j \times \left\{i\right\} = \left\{(a_1{}^j, i), (a_2{}^j, i), \cdots, (a_k{}^j, i)\right\}; \ \ i \in I_k, \ \ j \in I_s \tag{31}$$

In order to get further blocks we put

$$C_x = \left\{(u_0, 0), (u_1, 1), (u_2, 2), \cdots, (u_{k-1}, k - 1)\right\} \cdot x \qquad x \in R \tag{32}$$

where $(u, i) \cdot x$ means (ux, i). We must partition the blocks into $r = \frac{\lambda(kv-1)}{(k-1)}$ parallel classes. The first parallel class P_0 will take all the blocks $u_i B_j{}^i$ where $i \in I_k$, $j \in I_s$ and the blocks C_x, where x is distinct from all $a^j{}_i$, $i \in I_k$, $j \in I_s$. Other classes are given by $P_g = \mathcal{T}_g P_0$, where $\mathcal{T}_g : (x, i) \mapsto (x + g, i), g \in \mathbb{R}$, that is

$$P_g = \left\{\mathcal{T}_g(B) : B \in P_0\right\} \tag{33}$$

We can still construct more parallel classes. Let $Q_x = \left\{\mathcal{T}_g C_x : g \in R\right\}$ with

$$x \in \bigcup_{0\leq j\leq s-1} A_j \ \ and \ \ R_x = \left\{\mathcal{T}_g C_x : g \in R\right\} with \ \ x \in R\backslash \bigcup_{o\leq j\leq s-1} A_j \tag{34}$$

Both Q_x and R_x are parallel classes. We take each parallel class Q_x λ times and each class R_x $\lambda - 1$ times. If v is even and $v \geq 4$ a resolvable balanced incomplete block design with block size equal to $k = 2$ and number of symbols $V = v$ can be constructed as follows: We take the point set $\mathcal{F}$ to be $\mathcal{F} = \mathbb{Z}_{v-1} \cup \left\{\infty\right\}$. For $j \in \mathbb{Z}_{v-1}$, define

$$\pi_j = \left\{\left\{\infty, j\right\}\right\} \bigcup \left\{\left\{i + j \ \ mod(v-1), j - i \ \ mod(v-1)\right\}\right\} : \ \ 1 \leq i \leq \frac{v-2}{2} \tag{35}$$

π_j is a parallel class and each pair of points occurs in exactly one π_j. To construct a resolvable incomplete block design with block size $k = q$ and number of symbols $V = q^2$ where q is a prime number, we can construct an affine plane of order q. This is done by defining $P = \mathbb{F}_q \times \mathbb{F}_q$. For any $a, b \in \mathbb{F}_q$, we define a block

$$L_{a,b} = \left\{(x, y) \in P : y = ax + b\right\} \tag{36}$$

and for any $c \in \mathbb{F}_q$, we define

$$L_{\infty,c} = \left\{(c, y) : y \in \mathbb{F}_q\right\} \tag{37}$$

Finally we define

$$L = \left\{L_{a,b} : a, b \in \mathbb{F}_q\right\} \bigcup \left\{L_{\infty,c} : c \in \mathbb{F}_q\right\} \tag{38}$$

Then (P, L) is the required affine plane order q and hence the required balanced incomplete block design which resolvable.

Definition 5.4. *For $0 \leq d \leq m$, we define the Gaussian Coefficient $\begin{bmatrix} m \\ d \end{bmatrix}_q$ as follows*

$$\begin{bmatrix} m \\ d \end{bmatrix}_q = \left\{\begin{matrix} \frac{(q^m-1)(q^{m-1}-1)\cdots(q^{m-d+1}-1)}{(q^d-1)(q^{d-1}-1)\cdots(q-1)} & if \quad d \neq 0 \\ 1 & if \quad d = 0 \end{matrix}\right\} \tag{39}$$

To construct a resolvable balanced incomplete block design $(q^m, b, r, q^d, \lambda)$, where $m \geq 2, 1 \leq d \leq m - 1$

$$b = \begin{bmatrix} m \\ d \end{bmatrix}_q^{q^{m-d}}, \quad r = \begin{bmatrix} m \\ d \end{bmatrix}_q, \quad and \quad \lambda = \begin{bmatrix} m-1 \\ d-1 \end{bmatrix}_q \tag{40}$$

we use equations (36) and equation (37) and (38). However, in some cases it might not be possible to construct resolvable balanced incomplete block designs with the given properties in (40). Like for example there exists $(8, 4, 3)-$BIBDs that are not resolvable. To construct a Resolvable balanced incomplete block design (q^d, q^{d-1}, λ), where q is a prime power and $m \geq 2$ with $\lambda = \frac{q^{d-1}-1}{q-1}$. We first construct a symmetrical BIBD

$$\left(\frac{q^{d+1}-1}{q-1}, \frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1}\right) \tag{41}$$

From which we obtain a quasiresidual BIBD from the symmetrical BIBD above. The quasiresidual BIBD is an affine resolvable BIBD with parameters

$$v = q^d, \quad k = q^{d-1}, \quad \lambda = \frac{q^{d-1}-1}{q-1} \tag{42}$$

Example 5.5. *A* 4×6 *BAFD with block using* 12 *can be constructed using* $BA[10,6,2,2]$ *and a resolvable BIBD with 4 treatments and block size* 2 *as shown below:*

Consider the following BIBD with 4 *treatments and block size* 2 *where* $X_0, X_1, Y_0, Y_1, Z_0, Z_1$ *represents the blocks.*

X_0	X_1	Y_0	Y_1	Z_0	Z_1
0	2	0	1	0	1
1	3	2	3	3	2

Table 7: Table of BIBD[4,6,2]

Also consider the $BA(T)(3,2,1)$ given below

0	0	0	0	0	1	1	1	1	1
0	0	1	1	1	0	0	0	1	1
1	0	0	1	1	1	1	0	0	0
0	1	1	0	1	0	1	1	0	0
1	1	1	0	0	1	0	0	0	1
1	1	0	1	0	0	0	1	1	0

Table 8: Table of BA(T)[3,2,1]

By theorem 5.1 we can construct a 4×6 BAFD with $k = 12, r = \lambda_{00} = 15, b = 30, \lambda_{10} = 5, \lambda_{01} = 6, \lambda_{11} = 8$ with $E[1,0] = 1, E[0,1] = 1, E[1,1] = \frac{14}{15}$.

Blocks	1	2	3	4	5	6	7	8	9	10
levels of F_2	Levels	of	F_1							
0	X_0	X_0	X_0	X_0	X_0	X_1	X_1	X_1	X_1	X_1
1	X_0	X_0	X_1	X_1	X_1	X_0	X_0	X_0	X_1	X_1
2	X_1	X_0	X_0	X_1	X_1	X_1	X_1	X_0	X_0	X_0
3	X_0	X_1	X_1	X_0	X_1	X_0	X_1	X_1	X_0	X_0
4	X_1	X_1	X_1	X_0	X_0	X_1	X_0	X_0	X_0	X_1
5	X_1	X_1	X_0	X_1	X_0	X_0	X_0	X_1	X_1	X_0

Blocks	11	12	13	14	15	16	17	18	19	20
levels of F_2										
0	Y_0	Y_0	Y_0	Y_0	Y_0	Y_1	Y_1	Y_1	Y_1	Y_1
1	Y_0	Y_0	Y_1	Y_1	Y_1	Y_0	Y_0	Y_0	Y_1	Y_1
2	Y_1	Y_0	Y_0	Y_1	Y_1	Y_1	Y_1	Y_0	Y_0	Y_0
3	Y_0	Y_1	Y_1	Y_0	Y_1	Y_0	Y_1	Y_1	Y_0	Y_0
4	Y_1	Y_1	Y_1	Y_0	Y_0	Y_1	Y_0	Y_0	Y_0	Y_1
5	Y_1	Y_1	Y_0	Y_1	Y_0	Y_0	Y_0	Y_1	Y_1	Y_0

Blocks	21	22	23	24	25	26	27	28	29	30
levels of F_2										
0	Z_0	Z_0	Z_0	Z_0	Z_0	Z_1	Z_1	Z_1	Z_1	Z_1
1	Z_0	Z_0	Z_1	Z_1	Z_1	Z_0	Z_0	Z_0	Z_1	Z_1
2	Z_1	Z_0	Z_0	Z_1	Z_1	Z_1	Z_1	Z_0	Z_0	Z_0
3	Z_0	Z_1	Z_1	Z_0	Z_1	Z_0	Z_1	Z_1	Z_0	Z_0
4	Z_1	Z_1	Z_1	Z_0	Z_0	Z_1	Z_0	Z_0	Z_0	Z_1
5	Z_1	Z_1	Z_0	Z_1	Z_0	Z_0	Z_0	Z_1	Z_1	Z_0

Table 9: Table of a 4×6 BAFD

Example 5.6. *A* 6×8 *BAFD with parameters* $k = 24$, $r = \lambda_{00} = 35$, $b = 70$, $\lambda_{01} = 15$, $\lambda_{10} = 14$, $\lambda_{11} = 18$ *and efficiencies* $E[0,1] = 1$, $E[1,0] = 1$, $E[1,1] = \frac{34}{35}$ *can be constructed by using both the* $BA(T)(3,2,1)$ *given in example 5.5 and a resolvable BIBD with 8 treatments and block of size* 4 *given below that has been constructed by developing parallel classes as in equation* (29). *The base blocks will be* ${H_0}^2 \cup \left\{0\right\} = \left\{1,2,4,0\right\}$

and $H_1{}^2 \cup \left\{\infty\right\} = \left\{3,5,6,\infty\right\}$ *The resolvable BIBD will be given as follows after replacing* ∞ *with* 7.

1	3	2	4	3	5	4	6	5	0	6	1	0	2
2	5	3	6	4	0	5	1	6	2	0	3	1	4
4	6	5	0	6	1	0	2	1	3	2	4	3	5
0	7	1	7	2	7	3	7	4	7	5	7	6	7

Table 10: Table of BIBD[8,14,4]

Example 5.7. *A* 6×9 *BAFD with parameters* $k = 18$, $r = \lambda_{00} = 20$, $b = 60$, $\lambda_{01} = 5$, $\lambda_{10} = 4$, $\lambda_{11} = 7$ *and efficiencies* $E[0,1] = 1$, $E[1,0] = 1$, $E[1,1] = \frac{19}{20}$ *can be constructed by using both the* $BA(T)(2,3,1)$ *and the resolvable BIBD with* 9 *treatments and block size* 3 *given below. This has been constructed as in equations* (36), (37) *and* (38). *That is, by constructing an affine plane of order* 3.

0	1	2	0	1	2	0	1	2	0	3	6
3	4	5	4	5	3	5	3	4	1	4	7
6	7	8	8	6	7	7	8	6	2	5	8

Table 11: Table of a BIBD[9,12,3]

Example 5.8. *A* 10×15 *BAFD with parameters* $k = 30$, $r = \lambda_{00} = 63$, $b = 315$, $\lambda_{01} = 9$, $\lambda_{10} = 7$, $\lambda_{11} = 13$ *and efficiencies* $E[0,1] = 1$, $E[1,0] = 1$, $E[1,1] = \frac{61}{63}$ *can be constructed by using both the* $BA(T)[2,5,1]$ *and a resolvable BIBD with* 15 *treatments and block size* 3 *given below which was constructed by using the method of two step cycles where the first parallel class gives a set of triplets*

$$(k \cdot 1 \cdot 2), (3 \cdot 7 \cdot 10), (4 \cdot 5 \cdot 13), (6 \cdot 9 \cdot 11), (8, \cdot 12, \cdot 14).$$

From which by a cyclical two step permutation we get solution

0	3	4	6	8	0	5	6	8	10
1	7	5	9	12	3	9	7	11	14
2	10	13	11	14	4	12	1	13	2
0	7	8	10	12	0	9	10	12	14
5	11	9	13	2	7	13	11	1	4
6	14	3	1	4	8	2	5	3	6
0	13	14	2	4	0	1	2	4	6
11	3	1	5	8	13	5	3	7	10
12	6	9	7	10	14	8	11	9	12
0	11	12	14	2					
9	1	13	3	6					
10	4	7	5	8					

Table 12: Table of BIBD[15,35,3]

Example 5.9. *A* 6×9 *BAFD with parameters* $k = 27$, $r = \lambda_{00} = 20$, $b = 96$, $\lambda_{01} = 5$, $\lambda_{10} = 4$, $\lambda_{11} = 7$ *and efficiencies* $E[0,1] = 1$, $E[1,0] = 1$, $E[1,1] = \frac{29}{30}$ *can be constructed by using both the* $BA(T)[3,3,1]$ *and a* *resolvable BIBD with* 9 *treatments and block size* 3 *given below which was constructed by using the method of one step cycles where the first parallel class gives a set of triplets*

$$(k \cdot 1 \cdot 5), (3 \cdot 4 \cdot 6), (7 \cdot 8 \cdot 2)$$

from which by a one cyclical one step permutation we get the solution

0	3	7	0	4	8	0	5	1	0	6	2
1	4	8	2	5	1	3	6	2	4	7	3
5	6	2	6	7	3	7	8	4	8	1	5

Table 13: Table of BIBD[9,12,3]

Example 5.10. *An* 8×14 *BAFD with parameters* $k = 28$*,* $r = \lambda_{00} = 91$*,* $b = 312$*,* $\lambda_{01} = 42$*,* $\lambda_{10} = 39$*,* $\lambda_{11} = 46$ *and efficiencies* $E[1,0] = 1.0$*,* $E[0,1] = 1.0$*,* $E[1,1] = \frac{89}{91}$ *can be constructed by using both the* $BA(T)[4,2,2]$ *which can be constructed by using Theorem 2.17 and a resolvable BIBD with* 14 *treatments and block size* 7 *given by developing parallel classes as in Example 5.10. The base blocks will be* ${H_0}^2 \cup \left\{0\right\} = \left\{1,4,3,12,9,10,0\right\}, {H_1}^2 \cup \left\{\infty\right\} = \left\{2,8,6,11,5,7,\infty\right\}$ *and* ${H_0}^2 \cup \left\{\infty\right\} = \left\{1,4,3,12,9,10,\infty\right\}, {H_1}^2 \cup \left\{0\right\} = \left\{2,8,6,11,5,7,0\right\}$ *The resolvable BIBD is given as follows after replacing* ∞ *with* 13.

1	2	2	3	3	4	4	5	5	6	6	7
4	8	5	9	6	10	7	11	8	12	9	0
3	6	4	7	5	8	6	9	7	10	8	11
12	11	0	12	1	0	2	1	3	2	4	3
9	5	10	6	11	7	12	8	0	9	1	10
10	7	11	8	12	9	0	10	1	11	2	12
0	13	1	13	2	13	3	13	4	13	5	13
7	8	8	9	9	10	10	11	11	12	12	0
10	1	11	2	12	3	0	4	1	5	2	6
9	12	10	0	11	1	12	2	0	3	1	4
5	4	6	5	7	6	8	7	9	8	10	9
2	11	3	12	4	0	5	1	6	2	7	3
3	0	4	1	5	2	6	3	7	4	8	5
6	13	7	13	8	13	9	13	10	13	11	13
0	1	1	2	2	3	3	4	4	5	5	6
3	7	4	8	5	9	6	10	7	11	8	12
2	5	3	6	4	7	5	8	6	9	7	10
11	10	12	11	0	12	1	0	2	1	3	2
8	4	9	5	10	6	11	7	12	8	0	9
9	6	10	7	11	8	12	9	0	10	1	11
12	13	13	0	13	1	13	2	13	3	13	4
6	7	7	8	8	9	9	10	10	11	11	12
9	0	10	1	11	2	12	3	0	4	1	5
8	11	9	12	10	0	11	1	12	2	0	3
4	3	5	4	6	5	7	6	8	7	9	8
1	10	2	11	3	12	4	0	5	1	6	2
2	12	3	0	4	1	5	2	6	3	7	4
13	5	13	6	13	7	13	8	13	9	13	10
12	0	0	1								
2	6	3	7								
1	4	2	5								
10	9	11	10								
7	3	8	4								
8	5	9	6								
13	11	13	12								

Table 14: Table of a BIBD(14,52,7)

Example 5.11. *A* 22×55 *BAFD with parameters* $k = 110, r = \lambda_{00} = 567, b = 6,237$ *,*$\lambda_{01} = 42, \lambda_{10} = 27, \lambda_{11} = 52$ *and efficiencies* $E[0,1] = 1$*,* $E[1,0] = 1$*,* $E[1,1] = \frac{562}{567}$ *can be constructed by using both the* $BA(T)[2,11,1]$ *which can be constructed by using theorem 2.24 and a resolvable BIBD with* 55 *treatments and block size* 5 *given below which was constructed by developing parallel classes as in equations* (32), (33), (34). *Thus we develop parallel classes* P_g*,* Q_x*,* R_x*.* $R = \left\{0,1,2,\cdots,10\right\}$*. To construct a resolvable balanced incomplete* *block design with block size* 5 *and number of treatments* 55, $k = 5$, *suppose* $\lambda = 2 \leq k - 1$ *then* $s = \frac{\lambda(v-1)}{k(k-1)} = 1$ *base block hence* P_g *parallel classes:* $P_0, P_1, P_2, \cdots P_{10}$ *[11 blocks].* Q_x *parallel classes:* Q_0, Q_1, Q_2, Q_3, Q_4 *each of which is taken* $\lambda = 2$ *times.* R_x *parallel classes:* $R_0, R_1, R_2, R_3, R_4, R_5$ *each of which is taken* $\lambda - 1 = 2 - 1 = 1$ *times. Thus total number of parallel classes* $= 11 + 2 \times 5 + 6 \times 1 = 27$. *As an example the parallel classes* P_0, Q_1, R_0 *are given by*

P_0

5	11	17	23	29	10	21	32	43	54	0
15	31	47	8	24	30	6	37	13	44	1
20	41	7	28	49	35	16	52	33	14	2
25	51	22	48	19	40	26	12	53	39	3
45	36	27	18	9	50	46	42	38	34	4

Q_1

5	10	15	20	25	30	35	40	45	50	0
11	16	21	26	31	36	41	46	51	1	6
17	22	27	32	37	42	47	52	2	7	12
23	28	33	38	43	48	53	3	8	13	18
29	34	39	44	49	54	4	9	14	19	24

R_0

0	5	10	15	20	25	30	35	40	45	50
1	6	11	16	21	26	31	36	41	46	51
2	7	12	17	22	27	32	37	42	47	52
3	8	13	18	23	28	33	38	43	48	53
4	9	14	19	24	29	34	39	44	49	54

Table 15: Table of Parallel classes P_o, Q_1, R_o

6. Conclusion

The results presented in this paper relate to connected factorial designs. The disconnected case poses some special problems. As a matter of fact, the results in this paper, at least in their present forms do not remain valid in the disconnected case. The following example illustrates the point.

Example 6.1. *Consider a disconnected* 2^3 *design in two blocks as shown below.*

$BLOCKI:$ 000, 100, 010, 001

$BLOCKII:$ 110, 101, 011, 111

Clearly, each interaction is represented by a single contrast.It may be seen from the elementary considerations that the contrasts belonging to interactions $F(1,1,0), F(1,0,1), F(0,1,1)$ are estimable while those belonging to $F(1,0,0)$,

$F(0,1,0), F(0,0,1), F(1,1,1)$ are not estimable. Moreover, the BLUE's of the contrasts belonging to $F(1,1,0), F(1,0,1), F(0,1,1)$ may be seen to be mutually orthogonal, i.e Uncorrelated. Hence the design has OFS. Also trivially, the design is balanced since each interaction is represented by a single contrast. Thus the design is balanced and has OFS. However, the C-matrix is not of the form

$$C = \sum_{y \in \Omega} \rho(y) M^y \tag{43}$$

note that if the C-matrix be of the form above, one must have $M^y C = CM^y$ for every $y \in \Omega$. For this design, explicit computation shows that, in particular $M(0,0,1) = M^{001}$ does not commute with C. Thus the above example shows the form (43) does not hold. This calls for suitable modifications of these results to make them applicable to the disconnected case.

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