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On a Class of Perturbed Variational-Like Inequalities

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Abstract

In this work, we present and examine perturbed mixed quasi variational-like inequalities, a novel class of variational-like inequalities. We prove the existence and the uniqueness of the solution, using the auxiliary principle technique. In addition, we examine a few iterative techniques for solving quasi variational-like perturbed mixed inequality problems.

Keywords: perturbed mixed quasi variational-like inequalities; auxiliary principle technique; iterative methods.

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1. Introduction

Variational inequality theory is a significant area of mathematics with several applications in the pure and applied sciences. This theory offers a comprehensive framework for addressing a wide variety of linear and nonlinear problems, see, for example [2,7,8,10,16]. The concept of convexity has been expanded in multiple directions, finding applications in areas like nonlinear optimization, network equilibrium, and economics, see, for example [4,17]. The concepts of invex and preinvex functions represent a significant generalization of convex functions, see, for example [3,9,12]. Parida demonstrated that the minimum of preinvex functions can be described using a type of variational inequalities called variational-like inequalities, see [15]. Recently many extensions and generalizations of the variational-like inequality theory have been considered and studied, see, for example [5,11,13,14]. In this work, we present and examine a new category of perturbed variational-like inequalities. We use the auxiliary principle technique to prove the existence and the uniqueness of the solution of perturbed mixed quasi variational-like inequality. Additionally, we propose and evaluate several iterative methods for solving these types of variational-like inequalities.

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2. Preliminaries

Let H be a real Hilbert space, with its norm denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let K be a nonempty subset of H and $\eta\colon K\times K\to H$ be a continuous function. We now recall some basic definitions and concepts.

Definition 2.1 ([17]). *A set K is said to be an invex set, if there exists a function $\eta(\cdot,\cdot)$, such that*

$$u+t\eta(v,u)\in K,\quad \forall u,v\in K,\ t\in[0,1].$$

From now on, the set K is an invex set in H, unless otherwise specified.

Definition 2.2 ([17]). *A function $F\colon K\to\mathbb{R}$ is said to be a preinvex function, if there exists a function $\eta(\cdot,\cdot)$, such that*

$$F(u+t\eta(v,u))\leq(1-t)F(u)+tF(v),\quad \forall u,v\in K,\ t\in[0,1].$$

It follows that a minimum of a differentiable preinvex function F on the invex set K can be characterized by the variational-like inequality

$$\langle F'(u),\eta(v,u)\rangle\geq 0,\quad \forall v\in K.$$

Definition 2.3 ([12]). *A function $F\colon K\to\mathbb{R}$ is said to be a strongly preinvex function, if there exists a function $\eta(\cdot,\cdot)$ and a constant $\mu>0$ such that*

$$F(u+t\eta(v,u))\leq(1-t)F(u)+tF(v)-\mu(1-t)t\left\|\eta(v,u)\right\|^2,\quad \forall u,v\in K,\ t\in[0,1].$$

We note that the differentiable strongly preinvex function is a strongly invex function, i.e.,

$$F(v)-F(u)\geq\langle F'(u),\eta(v,u)\rangle+\mu\left\|\eta(v,u)\right\|^2.$$

It is clear that, for $\eta(v,u)=v-u$, the invex set K is a convex set, the preinvex functions are convex functions and strongly preinvex functions reduce to strongly convex functions. Let K be a invex set in H. Given an operator $A\colon H\to H$ and $\eta\colon K\times K\to H$, we consider the problem of finding $u\in K$ and constant $\rho>0$ such that

$$\langle Au,\eta(v,u)\rangle+\varphi(u,v)-\varphi(u,u)+\rho\left\|\eta(v,u)\right\|^2\geq 0,\quad \text{for all } v\in K, \tag{1}$$

where the form $\varphi\colon H\times H\to\mathbb{R}$, is non-differentiable and satisfies the following properties:

(i) $\varphi(u,v)$ is linear in the first argument;

(ii) $\varphi(u,v)$ is bounded, that is, there exists a constant $\gamma > 0$ such that,

$$\varphi(u,v) \leq \gamma \|u\| \|v\|, \quad \text{for all } u,v \in H$$

(iii) $\varphi(u,v) - \varphi(u,w) \leq \varphi(u,v-w), \quad \text{for all } u,v,w \in H.$

The problem (1) is called perturbed mixed quasi variational-like inequality.

We now discuss some special cases of the problem (1):

(I) For $\rho = 0$, problem (1) reduces to

$$\langle Au, \eta(v,u)\rangle + \varphi(u,v) - \varphi(u,u) \geq 0, \quad \text{for all } v \in K, \tag{2}$$

which is known as the mixed quasi variational-like inequality, studied by Noor [13]. Moreover, if $\eta(v,u) = v - u$, then the problem (1) is equivalent to finding $u \in K$ such that

$$\langle Au, v-u\rangle + \varphi(u,v) - \varphi(u,u) \geq 0, \quad \text{for all } v \in K, \tag{3}$$

which is known as mixed quasi variational inequality studied by Kikuchi and Oden [7].

(II) If $\varphi(u,v) = j(v)$ where j is a convex, lower semi-continuous, proper and non-differentiable functional, then the problem (1) is equivalent to finding $u \in K$ such that

$$\langle Au, \eta(v,u)\rangle + j(v) - j(u) + \rho \|\eta(v,u)\|^2 \geq 0, \quad \text{for all } v \in K, \tag{4}$$

which is known as perturbed mixed variational-like inequality, see Noor [14].

(III) For $\rho = 0$ and $\eta(v,u) = v - u$, then the problem (1) reduces to the problem of finding $u \in K$ such that

$$\langle Au, v-u\rangle + j(v) - j(u) \geq 0, \quad \text{for all } v \in K, \tag{5}$$

which is known as mixed variational inequality, introduced and studied by Lions and Stampacchia [10].

(IV) If $\varphi(u,v) = 0$ and $\rho = 0$, then the problem (1) reduces to the problem of finding $u \in K$ such that

$$\langle Au, v-u\rangle \geq 0, \quad \text{for all } v \in K, \tag{6}$$

which is well known original variational inequality, introduced by Stampacchia [16].

Definition 2.4 ([1]). *The function $\varphi\colon H \times H \to \mathbb{R}$ is called skew-symmetric, if*

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) \geq 0, \quad \forall u,v \in H.$$

Definition 2.5. *An operator $A\colon H \to H$ is said to be:*

(a) strongly monotone, if there exists a constant $\alpha > 0$, such that

$$\langle Au - Av, u - v\rangle \geq \alpha \left\| u - v \right\|^2, \quad \forall u, v \in H.$$

(b) Lipschitz continuous, if there exists a constant $\beta > 0$, such that

$$\left\| Au - Av \right\| \leq \beta \left\| u - v \right\|, \quad \forall u, v \in H.$$

Definition 2.6 ([11])**.** *The function $\eta\colon H \times H \to H$ is said to be:*

(a) strongly monotone, if there exists a constant $\sigma > 0$, such that

$$\langle \eta(u,v), u - v\rangle \geq \sigma \left\| u - v \right\|^2, \quad \forall u, v \in H.$$

(b) Lipschitz continuous, if there exists a constant $\delta > 0$, such that

$$\left\| \eta(u,v) \right\| \leq \delta \left\| u - v \right\|, \quad \forall u, v \in H.$$

If $\eta(u,v) = Au - Av$ for $A\colon H \to H$, then the definition 2.6 reduces to the strongly monotonicity and Lipschitz continuity of the nonlinear operator A, definition 2.5.

Assumption 2.7. *The function $\eta\colon H \times H \to H$ satisfies the condition:*

$$\eta(u,v) = \eta(u,w) + \eta(w,v), \quad \forall u, v, w \in H.$$

3. Main results

In this section, we use the auxiliary principle technique of Glowinski, Lions, and Tremolieres [6], to prove the existence and uniqueness of the solution of perturbed mixed quasi variational-like inequality (1).

Theorem 3.1. *Let H be a real Hilbert space and K be a invex set of H. Let A be strongly monotone and Lipschitz continuous with constants $\alpha > 0$, $\beta > 0$, respectively and the function η is strongly monotone and Lipschitz continuous with constants $\sigma > 0$, $\delta > 0$, respectively. If Assumption 2.7 holds, and for a constant $\lambda > 0$, such that*

$$0 < \lambda < 2\frac{\alpha - (\omega + 2\rho\delta^2)}{\beta^2 - (\omega + 2\rho\delta^2)^2}, \quad \lambda < \frac{1}{\omega + 2\rho\delta^2}, \quad \alpha > \omega + 2\rho\delta^2, \tag{7}$$

where

$$\omega = \beta\sqrt{1 - 2\sigma + \delta^2} + \gamma.$$

then, there exists a unique solution $u \in K$ of the problem (1).

Proof. Existence: We now use the auxiliary principle technique to prove the existence of a solution for (1). For a given $u \in K$, we consider the auxiliary problem of finding $w \in K$ satisfying the variational inequality

$$\langle w, v - w\rangle \geq \langle u, v - w\rangle - \lambda\langle Au, \eta(v, w)\rangle - \lambda\varphi(u, v) + \lambda\varphi(u, w) - \lambda\rho\left\|\eta(v, w)\right\|^2, \quad (8)$$

for all $v \in K$, where $\lambda > 0$ is a constant.

Let w_1, w_2 be to solution of (1) related to $u_1, u_2 \in K$ respectively. It is enough to show that the mapping $u \mapsto w(u)$ has a fixed point belonging to H satisfying (1). We show that

$$\|w_1 - w_2\| \leq k\|u_1 - u_2\|,$$

with $0 < k < 1$, where k is independent of u_1 and u_2. Since w_1, w_2 are both solution of (1) related to $u_1, u_2 \in K$, we have

$$\langle w_1, v - w_1\rangle \geq \langle u_1, v - w_1\rangle - \lambda\langle Au_1, \eta(v, w_1)\rangle - \lambda\varphi(u_1, v) + \lambda\varphi(u_1, w_1) - \lambda\rho\left\|\eta(v, w_1)\right\|^2, \quad (9)$$

and

$$\langle w_2, v - w_2\rangle \geq \langle u_2, v - w_2\rangle - \lambda\langle Au_2, \eta(v, w_2)\rangle - \lambda\varphi(u_2, v) + \lambda\varphi(u_2, w_2) - \lambda\rho\left\|\eta(v, w_2)\right\|^2, \quad (10)$$

for all $v \in H$. Taking $v = w_2$ in (9) and $v = w_1$ in (10), we have

$$\langle w_1, w_2 - w_1\rangle \geq \langle u_1, w_2 - w_1\rangle - \lambda\langle Au_1, \eta(w_2, w_1)\rangle - \lambda\varphi(u_1, w_2) + \lambda\varphi(u_1, w_1) - \lambda\rho\left\|\eta(w_2, w_1)\right\|^2$$

and

$$\langle w_2, w_1 - w_2\rangle \geq \langle u_2, w_1 - w_2\rangle - \lambda\langle Au_2, \eta(w_1, w_2)\rangle - \lambda\varphi(u_2, w_1) + \lambda\varphi(u_2, w_2) - \lambda\rho\left\|\eta(w_1, w_2)\right\|^2,$$

Adding this inequalities and using the Assumption 2.7 and the properties (i), (iii), we have

$$\begin{aligned}\langle w_1 - w_2, w_1 - w_2\rangle &\leq \langle u_1 - u_2, w_1 - w_2\rangle - \lambda\langle Au_1 - Au_2, \eta(w_1, w_2)\rangle \\ &\quad + \lambda\varphi(u_1 - u_2, w_2 - w_1) + 2\lambda\rho\left\|\eta(w_1, w_2)\right\|^2, \\ &= \langle u_1 - u_2 - \lambda(Au_1 - Au_2), w_2 - w_1\rangle + \lambda\varphi(u_1 - u_2, w_2 - w_1) \\ &\quad + \lambda\langle Au_1 - Au_2, w_1 - w_2 - \eta(w_1, w_2)\rangle + 2\lambda\rho\left\|\eta(w_1, w_2)\right\|^2,\end{aligned}$$

From Cauchy-Schwarz inequality and the property (ii), we obtain

$$\begin{aligned} \|w_1 - w_2\|^2 &\leq \|u_1 - u_2 - \lambda(Au_1 - Au_2)\| \|w_1 - w_2\| + \lambda\gamma \|u_1 - u_2\| \|w_1 - w_2\| \\ &\quad + \lambda \|Au_1 - Au_2\| \|w_1 - w_2 - \eta(w_1, w_2)\| + 2\lambda\rho\delta^2 \|w_1 - w_2\|^2, \end{aligned} \tag{11}$$

Since A, η are both strongly monotone and Lipschitz continuous, we have

$$\begin{aligned} \|u_1 - u_2 - \lambda(Au_1 - Au_2)\|^2 &= \|u_1 - u_2\|^2 - 2\lambda\langle Au_1 - Au_2, u_1 - u_2\rangle + \lambda^2 \|Au_1 - Au_2\|^2 \\ &\leq (1 - 2\alpha\lambda + \beta^2\lambda^2) \|u_1 - u_2\|^2, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \|w_1 - w_2 - \eta(w_1, w_2)\|^2 &= \|w_1 - w_2\|^2 - 2\langle w_1 - w_2, \eta(w_1, w_2)\rangle + \|\eta(w_1, w_2)\|^2 \\ &\leq (1 - 2\sigma + \delta^2) \|w_1 - w_2\|^2, \end{aligned} \tag{13}$$

Combining (11), (12) and (13), we have

$$\begin{aligned} \|w_1 - w_2\| &\leq \left(\sqrt{1 - 2\alpha\lambda + \beta^2\lambda^2} + \lambda\gamma + \lambda\beta\sqrt{1 - 2\sigma + \delta^2}\right) \|u_1 - u_2\| + 2\lambda\rho\delta^2 \|w_1 - w_2\|, \\ &= \big(\theta(\lambda) + \lambda\omega\big) \|u_1 - u_2\| + 2\lambda\rho\delta^2 \|w_1 - w_2\|, \end{aligned}$$

where

$$\theta(\lambda) = \sqrt{1 - 2\alpha\lambda + \beta^2\lambda^2},$$

It follows that

$$\|w_1 - w_2\| \leq k \|u_1 - u_2\|,$$

where

$$k = \frac{\theta(\lambda) + \lambda\omega}{1 - 2\lambda\rho\delta^2}.$$

From (7) it follows that $k < 1$, the mapping $u \mapsto w(u)$ is a contraction, so has a fixed point, which is the solution of the problem (1).

Uniqueness: Let $u_1, u_2 \in K$, $u_1 \neq u_2$ be two solution of the problem (1), that is

$$\langle Au_1, \eta(v, u_1)\rangle + \varphi(u_1, v) - \varphi(u_1, u_1) + \rho \|\eta(v, u_1)\|^2 \geq 0, \quad \text{for all } v \in K, \tag{14}$$

and

$$\langle Au_2, \eta(v, u_2)\rangle + \varphi(u_2, v) - \varphi(u_2, u_2) + \rho \|\eta(v, u_2)\|^2 \geq 0, \quad \text{for all } v \in K, \tag{15}$$

Taking $v = u_2$ in (14) and $v = u_1$ in (15), we have

$$\langle Au_1, \eta(u_2, u_1)\rangle + \varphi(u_1, u_2) - \varphi(u_1, u_1) + \rho \|\eta(u_2, u_1)\|^2 \geq 0,$$

and

$$\langle Au_2, \eta(u_1, u_2)\rangle + \varphi(u_2, u_1) - \varphi(u_2, u_2) + \rho \|\eta(u_1, u_2)\|^2 \geq 0,$$

Using the Assumption 2.7 the properties (i), (iii) and adding the above inequalities, we have

$$\langle Au_1 - Au_2, \eta(u_1, u_2)\rangle \leq \varphi(u_1 - u_2, u_2 - u_1) + 2\rho \|\eta(u_1, u_2)\|^2, \tag{16}$$

which can be written as

$$\langle Au_1 - Au_2, u_1 - u_2\rangle \leq \langle Au_1 - Au_2, u_1 - u_2 - \eta(u_1, u_2)\rangle + \varphi(u_1 - u_2, u_2 - u_1) + 2\rho \|\eta(u_1, u_2)\|^2.$$

Since A, η are both strongly monotone and Lipschitz continuous, and using (ii), we get

$$\begin{aligned} \alpha \|u_1 - u_2\|^2 &\leq \|Au_1 - Au_2\| \|u_1 - u_2 - \eta(u_1, u_2)\| + \gamma \|u_1 - u_2\|^2 + 2\rho\delta^2 \|u_1 - u_2\|^2 \\ &\leq \left(\beta\sqrt{1 - 2\sigma + \delta^2} + \gamma + 2\rho\delta^2\right) \|u_1 - u_2\|^2 \\ &= (\omega + 2\rho\delta^2) \|u_1 - u_2\|^2. \end{aligned} \tag{17}$$

Thus, it follows that

$$(\alpha - \omega - 2\rho\delta^2) \|u_1 - u_2\|^2 \leq 0. \tag{18}$$

which is implies $u_1 = u_2$, the uniqueness of the solution of (1), since $\alpha > \omega + 2\rho\delta^2$. □

We use the auxiliary principle technique to propose and analyze several iterative algorithms for solving perturbed mixed quasi variational-like inequalities. (1). Now, we consider the auxiliary variational-like inequality associated with the problem (1). For a given $u \in K$, consider the problem of finding $w \in K$, satisfying the auxiliary variational-like inequality

$$\langle \lambda Aw + \Phi'(w) - \Phi'(u), \eta(v, w)\rangle \geq \lambda\varphi(w, w) - \lambda\varphi(w, v) - \lambda\rho \|\eta(v, w)\|^2, \quad \forall v \in K, \tag{19}$$

where $\lambda > 0$ is a constant and $\Phi'(u)$ is the differential of a strongly preinvex function $\Phi(u)$. We remark, if $w = u$, then w is a solution of (1). Next, we propose and analyze an iterative algorithm for solving the problem (1).

Algorithm 3.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme:*

$$\begin{aligned} \langle \lambda Au_{n+1} + \Phi'(u_{n+1}) - \Phi'(u_n), \eta(v, u_{n+1})\rangle &\geq \lambda\varphi(u_{n+1}, u_{n+1}) - \lambda\varphi(u_{n+1}, v) \\ &\quad -\lambda\rho \|\eta(v, u_{n+1})\|^2, \quad \forall v \in K. \end{aligned} \tag{20}$$

which is known as the proximal point algorithm for solving perturbed mixed quasi variational-like inequalities. If $\rho = 0$, Algorithm 3.2 reduces to:

Algorithm 3.3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme:*

$$\langle \lambda A u_{n+1} + \Phi'(u_{n+1}) - \Phi'(u_n), \eta(v, u_{n+1}) \rangle \geq \lambda\varphi(u_{n+1}, u_{n+1}) - \lambda\varphi(u_{n+1}, v), \quad \forall v \in K. \tag{21}$$

which is known as the proximal point algorithm for solving mixed quasi variational-like inequality (2). In particular, if $\eta(v, u) = v - u$, then Algorithm 3.3 reduces to:

Algorithm 3.4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme:*

$$\langle \lambda A u_{n+1} + \Phi'(u_{n+1}) - \Phi'(u_n), v - u_{n+1} \rangle \geq \lambda\varphi(u_{n+1}, u_{n+1}) - \lambda\varphi(u_{n+1}, v), \quad \forall v \in K. \tag{22}$$

which is known as the proximal method for solving mixed quasi variational inequality (3).

Theorem 3.5. *Let A be strongly η-monotone with constant $\alpha > 0$ and Φ be differentiable strongly preinvex function with modulus μ. If Assumption 2.7 holds and the continuous function φ is skew-symmetric, then the solution $\{u_n\}$ generated by Algorithm 4.1 converges to the solution u of the problem (1) for $\alpha > 2\rho$.*

Proof. Let $u \in K$ be a solution of (1). Then, for $v = u_{n+1}$, we have

$$\langle Au, \eta(u_{n+1}, u) \rangle + \varphi(u, u_{n+1}) - \varphi(u, u) + \rho \left\| \eta(u_{n+1}, u) \right\|^2 \geq 0, \tag{23}$$

We consider the generalized Bregman function,

$$B(u, v) = \Phi(u) - \Phi(v) - \langle \Phi'(v), \eta(u, v) \rangle \geq \mu \left\| \eta(u, v) \right\|^2, \quad \forall\, v \in K, \tag{24}$$

associated with the differentiable strongly preinvex function Φ. Now, combining (20), (23), (24) and using the Assumption 2.7, it follows that

$$\begin{aligned}
B(u, u_n) - B(u, u_{n+1}) &= \Phi(u_{n+1}) - \Phi(u_n) - \langle \Phi'(u_n), \eta(u, u_n) \rangle + \langle \Phi'(u_{n+1}), \eta(u, u_{n+1}) \rangle, \\
&= \Phi(u_{n+1}) - \Phi(u_n) - \langle \Phi'(u_n), \eta(u_{n+1}, u_n) \rangle \\
&\quad + \langle \Phi'(u_{n+1}) - \Phi'(u_n), \eta(u, u_{n+1}) \rangle, \\
&\geq \mu \left\| \eta(u_{n+1}, u_n) \right\|^2 + \langle \Phi'(u_{n+1}) - \Phi'(u_n), \eta(u, u_{n+1}) \rangle, \\
&\geq \mu \left\| \eta(u_{n+1}, u_n) \right\|^2 + \langle \lambda A u_{n+1}, \eta(u_{n+1}, u) \rangle \\
&\quad + \lambda\varphi(u_{n+1}, u_{n+1}) - \lambda\varphi(u_{n+1}, u) - \lambda\rho \left\| \eta(u, u_{n+1}) \right\|^2, \\
&= \mu \left\| \eta(u_{n+1}, u_n) \right\|^2 + \lambda\langle A u_{n+1} - Au, \eta(u_{n+1}, u) \rangle + \lambda\langle Au, \eta(u_{n+1}, u) \rangle \\
&\quad + \lambda\varphi(u_{n+1}, u_{n+1}) - \lambda\varphi(u_{n+1}, u) - \lambda\rho \left\| \eta(u, u_{n+1}) \right\|^2, \\
&= \mu \left\| \eta(u_{n+1}, u_n) \right\|^2 + Q.
\end{aligned}$$

Since A is strongly η - monotone and φ is skew-symmetric, we have

$$\begin{aligned} Q &= \lambda\langle Au_{n+1} - Au, \eta(u_{n+1}, u)\rangle + \lambda\langle Au, \eta(u_{n+1}, u)\rangle \\ &\quad + \lambda\varphi(u_{n+1}, u_{n+1}) - \lambda\varphi(u_{n+1}, u) - \lambda\rho \left\|\eta(u, u_{n+1})\right\|^2, \\ &\geq \lambda\alpha \left\|\eta(u_{n+1}, u)\right\|^2 + \lambda\varphi(u, u) - \lambda\varphi(u, u_{n+1}) - \lambda\varphi(u_{n+1}, u) \\ &\quad + \lambda\varphi(u_{n+1}, u_{n+1}) - 2\lambda\rho \left\|\eta(u_{n+1}, u)\right\|^2, \\ &= \lambda(\alpha - 2\rho) \left\|\eta(u_{n+1}, u)\right\|^2. \end{aligned}$$

Therefore,

$$B(u, u_n) - B(u, u_{n+1}) \geq \mu \left\|\eta(u_{n+1}, u_n)\right\|^2 + \lambda(\alpha - 2\rho) \left\|\eta(u_{n+1}, u)\right\|^2.$$

If $u_{n+1} = u_n$ it is clear that u_n is a solution of the problem (1). Otherwise, the assumption $\alpha > 2\rho$ implies that the sequence $B(u, u_n) - B(u, u_{n+1})$ is nonnegative, and we must have $\lim_{n\to\infty} \|\eta(u_{n+1}, u_n)\| = 0$. It follows that the sequence $\{u_n\}$ is bounded. Let $\widetilde{u} \in K$ be a cluster point of the sequence $\{u_n\}$. Taking the limit in (20), we conclude that $\widetilde{u}$ is a solution of the variational-like inequality (1). □

4. Conclusions

In this work, we have introduced and analyzed a class of perturbed mixed quasi variational-like inequalities. We proved the existence and uniqueness of the solution of this class of variational-like inequalities. Additionally, we propose and analyze iterative methods for solving this problem, using auxiliary variational-like inequality associated with perturbed mixed quasi variational-like inequality.

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