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An Alternative Proof of a Result of Bhayo and Sándor

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Abstract

In 2021, the second author of this paper pointed out that the method of applying some known results to prove Theorem 1.6 in the paper, 'On certain old and new trigonometric and hyperbolic inequalities' by B. A. Bhayo and J. Sándor contains a mistake and corrected this mistake by providing another simple proof. In this note, we aim to give an alternative simple proof by using series expansions of hyperbolic sine and hyperbolic cosine functions. Moreover, we propose sharp lower bounds for hyperbolic tangent function.

Keywords: Trigonometric and hyperbolic inequalities; l'Hôpital's rule of monotonicity; increasing and decreasing functions.
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1. Introduction

In 2015, Bhayo and Sándor [4] published a proof for the following double sided inequality.

Theorem 1.1 ([4])**.** *If $x > 0$ then*

$$\exp\left[\frac{1}{2}\left(\frac{x}{\tanh x} - 1\right)\right] < \frac{\sinh x}{x} < \exp\left[\frac{x}{\tanh x} - 1\right]. \tag{1}$$

The inequalities (1) give very sharp bounds for the hyperbolic sinc function, viz. $(\sinh x)/x$. Due to the importance of hyperbolic sinc function in many branches of science and mathematics, it deserves to study the bounds for approximation of this function. To prove the inequalities (1) the authors of [4] used the following lemmas.

Lemma 1.2 ([1])**.** *Let $f, g : [a, b] \to \mathbb{R}$ be continuous. Moreover, let f, g be differentiable on (a, b) and $g'(x) \neq 0$, on (a, b). Let,*

$$A(x) = \frac{f(x) - f(a)}{g(x) - g(a)},\ B(x) = \frac{f(x) - f(b)}{g(x) - g(b)},\ x \in (a, b).$$

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(i) $A(x)$ and $B(x)$ are increasing(strictly increasing) on (a,b) if $f'(\cdot)/g'(\cdot)$ is increasing(strictly increasing) on (a,b).

(ii) $A(x)$ and $B(x)$ are decreasing(strictly decreasing) on (a,b) if $f'(\cdot)/g'(\cdot)$ is decreasing(strictly decreasing) on (a,b).

The Lemma 1.2 is known in the literature as l'Hôpital's rule of monotonicity. For Lemma 1.3 we refer to [7].

Lemma 1.3. *Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < R$, where a_n and b_n are real numbers for $n = 0, 1, 2, \cdots$ such that $b_n > 0$. If the sequence a_n/b_n is strictly increasing(or decreasing), then the function $A(x)/B(x)$ is also strictly increasing(or decreasing) on $(0, R)$.*

The above lemmas are being widely used for ample number of applications. The second author [2] pointed out that the method employed to obtain double inequality (1) in [4] is not correct and it contains a flaw. Further, using l'Hôpital's rule of monotonicity the new proof of inequality (1) is given in [2]. In this note, we present an alternative simple proof by utilizing series expansions of hyperbolic sine and hyperbolic cosine functions. In addition to this, we obtain sharp lower bound for hyperbolic tangent by using the left inequality of (1) and one of the results of paper [5].

2. An Alternative Simple Proof

Consider the function

$$f(x) = \frac{\log\left(\frac{\sinh x}{x}\right)}{\frac{x}{\tanh x} - 1} = \frac{f_1(x)}{f_2(x)}$$

where $f_1(x) = \log((\sinh x)/x)$ and $f_2(x) = x/\tanh x - 1$ with $f_1(0+) = 0 = f_2(0+)$. After differentiating we get

$$\begin{aligned}\frac{f_1'(x)}{f_2'(x)} &= \frac{x\cosh x - \sinh x}{x \sinh x} \cdot \frac{\tanh^2 x}{\tanh x - x\,\mathrm{sech}^2 x} \\ &= \frac{1 - x\coth x}{\left(\frac{x}{\sinh x}\right)^2 - x\coth x} \\ &= \frac{\sinh^2 x - x\sinh x\cosh x}{x^2 - x\sinh x\cosh x} \\ &= \frac{\cosh 2x - 1 - x\sinh 2x}{2x^2 - x\sinh 2x} = \frac{A(x)}{B(x)}.\end{aligned}$$

Using series expansions [6]

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \text{ and } \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

we write

$$\begin{aligned}\frac{A(x)}{B(x)} &= \frac{\sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} x^{2n} - 1 - x \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+1}}{2x^2 - x \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+1}} \\ &= \frac{\sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} x^{2n} - \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+2}}{2x^2 - \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} x^{2n+2}} \\ &= \frac{\sum_{n=2}^{\infty} \left[\frac{2^{2n-1}}{(2n-1)!} - \frac{2^{2n}}{(2n)!} \right] x^{2n}}{\sum_{n=2}^{\infty} \frac{2^{2n-1}}{(2n-1)!} x^{2n}} = \frac{\sum_{n=2}^{\infty} a_n x^{2n}}{\sum_{n=2}^{\infty} b_n x^{2n}}\end{aligned}$$

where

$$a_n = \frac{2^{2n-1}}{(2n-1)!} - \frac{2^{2n}}{(2n)!} \text{ and } b_n = \frac{2^{2n-1}}{(2n-1)!}.$$

Clearly both the series $A(x)$ and $B(x)$ are convergent in $(0, R)$ by ratio test for any $R > 0$ and $b_n > 0$ for $n \geqslant 2$. Moreover, $a_n/b_n = 1 - 1/n$ implies that the sequence $\{a_n/b_n\}$ is increasing. By Lemma 1.3, $A(x)/B(x)$ is increasing on $(0, R)$ for every $R > 0$. Hence we conclude that $A(x)/B(x)$ is increasing on $(0, \infty)$. Applying Lemma 1.2, $f(x)$ is also increasing on $(0, \infty)$. Consequently $f(0^+) < f(x) < f(\infty^-)$. Lastly, the limits $f(0^+) = \frac{1}{2}$ and $f(\infty^-) = 1$ prove the result.

3. Sharp Lower Bound for $\tanh x$

The inequality

$$\tanh x < \frac{2x}{\sqrt{4x^2 + 9} - 1}, \quad x > 0 \tag{2}$$

is due to Bhayo et. al. [5]. This inequality (2) can be written as

$$\frac{\sqrt{4x^2 + 9} - 1}{2} < \frac{x}{\tanh x},$$

from which we obtain

$$\frac{\sqrt{4x^2 + 9} - 3}{4} < \frac{1}{2} \left(\frac{x}{\tanh x} - 1 \right)$$

or

$$e^{(\sqrt{4x^2+9}-3)/4} < e^{(x/\tanh x - 1)/2}. \tag{3}$$

Now the inequalities (1) and (3) yield

$$e^{(\sqrt{4x^2+9}-3)/4} < \frac{\sinh x}{x}, \quad x > 0.$$

Due to the known result $(\sinh x)/x < \cosh x$, we write $e^{(\sqrt{4x^2+9}-3)/4} < \cosh x$, which is equivalent to $e^{(\sqrt{4x^2+9}-3)/2} < \cosh^2 x$, i.e., $e^{(3-\sqrt{4x^2+9})/2} > \operatorname{sech}^2 x$, or $e^{(3-\sqrt{4x^2+9})/2} > 1 - \tanh^2 x$. This gives us

$$\sqrt{1 - e^{\frac{3-\sqrt{4x^2+9}}{2}}} < \tanh x, \;\; x > 0. \tag{4}$$

Motivated by the inequality (4), we propose the following:

Theorem 3.1. *For $x > 0$, we have*

$$\sqrt{1 - e^{\frac{3-\sqrt{12x^2+9}}{2}}} < \tanh x. \tag{5}$$

Proof. Let us suppose,

$$g(x) = \frac{3-\sqrt{12x^2+9}}{2} - \ln(\operatorname{sech}^2 x), \;\; x > 0.$$

Differentiation with respect to x gives

$$g'(x) = \frac{-6x}{\sqrt{12x^2+9}} + 2\tanh x = 2x\left[\frac{\tanh x}{x} - \frac{1}{\sqrt{\frac{4}{3}x^2+1}}\right] > 0,$$

due to $\frac{1}{\sqrt{\frac{4}{3}x^2+1}} < \frac{1}{\sqrt{x^2+1}}$ and the inequality [3, (3.4)]. Hence $g(x)$ is strictly increasing for $x > 0$. This yields $g(x) > g(0) = 0$ giving us the desired inequality (5). □

Now by combining inequalities (2) and (4), we get a double inequality as follows:

$$\sqrt{1 - e^{\frac{3-\sqrt{12x^2+9}}{2}}} < \tanh x < \frac{2x}{\sqrt{4x^2+9}-1}, \;\; x > 0.$$

The sharpness of the inequality (5) can be seen from the following graph of difference function

$$F(x) = \tanh x - \sqrt{1 - e^{\frac{3-\sqrt{12x^2+9}}{2}}}.$$

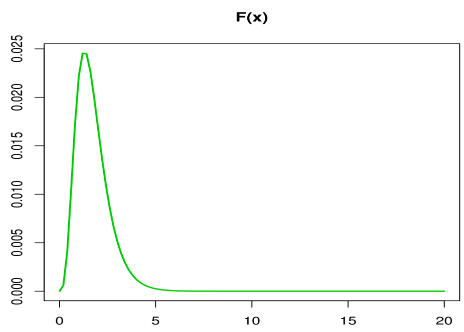

Figure 1: Graph of the curve $F(x)$ for $x \in (0, 20)$.

Since, $\lim_{x\to\infty} F(x) = 0$, it is obvious that the lower bound gets more closer to $\tanh x$ as $x \to \infty$.

Thus we provided an alternative simple proof of Theorem 1.1 and established a sharp lower bound for hyperbolic tangent.

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