International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **12(4)**(2024), 55–63
Available Online: http://ijmaa.in

An Orientability of a Graph

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Abstract

In 1962 Ore posed the problem, characterize graphs which are cover graphs; i.e., characterize those graphs which are orientable as an ordered set. In 1985 O. Pretzel have studied the orientability of graph in terms of girth and chromatic numbers of a graph. In 1995 Nešetřil and Rödl have studied the graphs of arbitrary large girth that are not covering graphs. In 2021 Bhavale and Waphare introduced the concept of a poset dismantlable by doubly irreducibles and proved that a graph is orientable as a poset dismantlable by doubly irreducible if and only if it is (non-trivial) adjunct of ears. In this paper, we survey an orientability of several classes of graphs in particular, covering graphs of posets. We will also discuss in detail about the various types of orientability of graphs. In the end, we provide a list of some open problems related to the orientability of graphs.

Keywords: Graph; Poset; Cover graph; Orientable graph.

2020 Mathematics Subject Classification: 18A10, 06A06, 05C60.

1. Introduction

In 1962 Ore [28] posed the problem, characterize graphs which are cover graphs; i.e., characterize those graphs which are orientable as an ordered set. The problem is still open. In 1985 Pretzel [29] have studied the orientability of a graph in terms of girth and chromatic number of graph. Pretzel [29] has studied some equivalent and necessary conditions for a finite graph to be the covering graph of a partially ordered set. For each $k \geq 1$, Aigner and Prins [1] have introduced a notion of a vertex colouring, here called k-good colouring, such that a 1-good colouring is the usual concept, and graphs that have a 2-good colouring are precisely covering graphs. Aigner and Prins [1] presented some inequalities for the corresponding chromatic numbers χ_k, especially for χ_2. that is there exist graphs that satisfy these inequalities for $k = 2$ but are not covering graphs. Aigner and Prins [1] also showed that χ_2 cannot be bounded by a function of $\chi = \chi_1$. A construction of Nešetřil and Rödl [26] is used to show that χ_2 is not bounded by a function of the girth. In 2014 Bhavale and Waphare have given

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a partial solution to the open problem of orientability by characterizing covering graphs of posets dismantlable by doubly irreducibles.

2. Preliminaries

A *graph* $G = (V, E)$ is a mathematical structure consisting of two sets V and E. The elements of V are called vertices, and the elements of E are called edges. Two graphs G_1 and G_2 are *isomorphic*, if there is one-one correspondence between the vertices of G_1 and those of G_2 such that the number of edges joining any two vertices of G_1 equals the number of edges joining the corresponding vertices of G_2. The *girth* of a graph G with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth. It is denoted by $g(G)$. A *planar graph* is a graph that can be drawn without crossing. A *directed graph* $D = (V, A)$ consists of two finite sets V and A. The elements of V are called *vertices*, and the elements of A are called *arcs*. A *chromatic number* of a graph G is the smallest number of colours $\chi(G)$ needed to colour the vertices of G so that no two adjacent vertices have same colour. It is denoted by $\chi(G)$. Orientability of graphs is already studied by Pretzel [29] in terms of girth and the chromatic number of a graph. An *ear* of a loopless connected graph G is a subgraph of G such that it is a maximal path in which all internal vertices are of degree two in G or it is a cycle in which all but one vertex have degree two in G. If G is a cycle itself then that cycle is the only ear of G. *A partially ordered set* (in short, *a poset*) is a set P of elements together with a binary relation $\leq$ on P which is reflexive, antisymmetric and transitive. An element $x \in P$ is an *upper bound* for a subset $S \subset P$ if $s \leq x$ for all $s \in S$. An *upper cone of* S denoted by S^u is defined as $S^u = \{x \in P \mid s \leq x, \ \forall s \in S\}$. The least element of S^u is called *join* of S, denoted by $\vee S$. An *lower cone of* S denoted by S^l is defined as $S^l = \{x \in P \mid x \leq s, \ \forall s \in S\}$. The greatest element of S^l is called *meet* of S, denoted by $\wedge S$. In particular, $\vee\{a, b\}$ and $\wedge\{a, b\}$ are respectively denoted by $a \vee b$ and $a \wedge b$. A *lattice* is a poset in which every pair of elements has the meet and the join. An element x in a lattice L is called *join-reducible (meet-reducible)* in L, if there exist $y, z \in L$ both distinct from x such that $y \vee z = x (y \wedge z = x)$. A lattice L is called *distributive lattice* if for any elements a, b and c of L, it satisfies following distributive properties:

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

2. $a \wedge (b \vee c) = (a \vee b) \wedge (a \vee c)$.

A lattice L is *modular* if for every $x, y, z \in L$ with $x \leq z$, $x \vee (y \wedge z) = (x \vee y) \wedge z$. An element x in a lattice L is called *join-irreducible (meet-irreducible)* if it is not join-reducible (meet-reducible). An element x in a lattice L is called *doubly irreducible* if it is both join-irreducible and meet-irreducible. An element a of a poset P is called *doubly irreducible* in P if a has at most one upper cover and at most one lower cover in P.

Definition 2.1 ([9]). *A finite lattice L of order n is called dismantlable if there exists a chain $L_1 \subset L_2 \subset \cdots \subset L_n (= L)$ of sublattices of L such that $|L_i| = i$ for all i.*

Thus a dismantlable lattice is a lattice which can be completely dismantled by removing one element at each stage. An element a of a poset P is called *irreducible* in P if a is an isolated element or a has precisely one upper cover or precisely one lower cover in P.

Definition 2.2. *An n-element poset P is called poset dismantlable by irreducibles if there exists a chain* $P_1 \subset P_2 \subset \cdots \subset P_n(= P)$ *of subposets of P such that* P_1 *has one element and* $P_{i-1} = P_i \setminus \{x\}$*, where x is an irreducible element in* P_i*, for all i.*

Bhavale and Waphare (see [4]) introduced the concept of poset dismantlable by doubly irreducibles.

Definition 2.3. *An n-element poset P is called poset dismantlable by doubly irreducibles if there exists a chain* $P_1 \subset P_2 \subset \cdots \subset P_n(= P)$ *of subposets of P such that* P_1 *has one element and* $P_{i-1} = P_i \setminus \{x\}$*, where x is a doubly irreducible element in* P_i*, for all i.*

Bhavale and Waphare [4] introduced the concept of an adjunct of ears in graphs as follows. Let G be a directed graph and P be any directed path (ear) from c to d with $V(G) \cap V(P) = \phi$. Let a be a vertex of G. Then the *u-adjunct* of P to G at a to be a directed graph, denoted by $G]_a P$, having vertex set $V(G) \cup V(P)$ and arc set $A(G) \cup A(P) \cup \{(a, c)\}$. And the *d-adjunct* of P to G at a to be a directed graph, denoted by $G]^a P$, having vertex set $V(G) \cup V(P)$ and arc set $A(G) \cup A(P) \cup \{(d, a)\}$. Also the *ud-adjunct* of P to G at (a, b), where (a, b) is a pair of vertices in G such that there is a directed path from a to b in G of length at least 2, to be a directed graph, denoted by $G]_a^b P$ having vertex set $V(G) \cup V(P)$ and the arc set $A(G) \cup A(P) \cup \{(a, c), (d, b)\}$.
A directed graph G is *adjunct of directed ears* if it can be obtained by u-adjunction or d-adjunction or ud-adjunction of directed ears starting with a directed path. An underlying graph of a directed graph which is adjunct of directed ears is called adjunct of ears.

Definition 2.4 ([10])**.** *For an ordered set P, a pair of elements* $\{a, b\}$ *of P is an edge of the comparability graph of P if* $a < b$*.*

Definition 2.5 ([10])**.** *For an ordered set P, a pair* $\{a, b\}$ *is an edge of a covering graph of P if b covers a. Covering graph of P is denoted by* $C(P)$*.*

Definition 2.6 ([5])**.** *A graph G is said to be an orientable as an ordered set P if G and* $C(P)$ *are isomorphic as graphs.*

The hypercube graph is the graph formed from the vertices and edges of an n-dimensional hypercube. A retract of a hypercube is an induced subgraph of a hypercube that has an edge-preserving map from the hypercube onto the subgraph. A connected graph in which each vertex is of degree 2 is called a *cycle graph*. A connected graph that has no cycle is called a *tree*.
Tree Decomposition: A tree decomposition of a graph G is a pair $(T, \{X_t\}_{t \in T})$, where T is a tree and $\{X_t\}_{t \in T}$ is a collection of subsets of vertices of G, such that

1. Every vertex of G is in at least one subset.

2. For each edge (u, v) there is at least one subset containing both u and v.

3. For each vertex v of G, the subset containing v form a connected subtree of T.

The *tree width* of a graph G is the minimum width over all possible tree decompositions of G.

Definition 2.7 ([11]). *A Halin graph H is a plane graph obtained by drawing a tree T in the plane, where T has no vertex of degree 2, and then drawing a cycle C through all leaves in the plane so $H = T \cup C$.*

For more details and the other definitions see [9–11,23,28,31].

3. Orientability

The conjuncture of Bollobas [29] was, there are graphs of arbitrarily large girth that are not covering graphs, which was proved by Nešetřil and Rödl (see [26] and [27]) using probabilistic methods. Also Nešetřil and Rödl proved that, the recognition of cover graphs of finite posets is an NP-hard problem. In 1964 Gilmore and Hoffman [15] proved a characterization of those graphs, finite or infinite, which are comparability graphs that is a graph is comparability graph if and only if each odd cycle has at least one triangular chord. Another proof of the same characterization has been given in [14] and a related question examined in [30]. Their proof of the sufficiency of the characterization yields a very simple algorithm for directing all the edges of a comparability graph in such a way that the resulting graph partially orders its vertices.

In 1983 Duffus and Rival [10] stated as there are two types of graphs commonly associated with finite partially ordered sets: the comparability graph and the covering graph. While the first type has been characterized, only partial descriptions of the second are known. Duffus and Rival [10] proved that the covering graphs of distributive lattices are precisely those graphs which are retracts of hypercubes. Many researchers have investigated connections between dimension for posets and planarity for graphs. In 1993 Brightwell [6] extend this line of research to the structural graph theory parameter tree-width by proving that the dimension of a finite poset is bounded in terms of its height and the tree-width of its cover graph. In contrast little is known about this question [28] when is a graph the covering graph of an ordered set? Also it is an NP-complete to test whether a graph is a cover graph (see [14] and [15]). This question is already solved for a finite distributive lattice. Similar type of characterizations are obtained for modular lattices (see [21]) and geometric lattices (see [9]). In 1958 Grotzsch (see [16]) proved that triangle-free planar graphs are 3-chromatic; consequently they are orientable. In 2014 Bhavale and Waphare [5] have given a partial solution to the open problem of orientability by characterizing covering graphs of posets dismantlable by doubly irreducibles. Also Bhavale and Waphare [5] gave the following results related to orientability.

Theorem 3.1 ([5]). *A graph is orientable as a poset dismantlable by doubly irreducible if and only if it is (non-trivial) adjunct of ears.*

Theorem 3.2 ([5]). *A graph is orientable as a dismantlable lattice if and only if it is ud-adjunct of ears.*

Theorem 3.3 ([5]). *A graph G is orientable as a lattice in which all the reducible elements are comparable then G is connected and contains a chordless path passing through all the higher degree* (≥ 3) *vertices.*

However the converse is not true.

4. Types of Orientability

In this section, we are going to discuss three types of orientability.

4.1 Cyclic Orientability

Graph G is called cyclically orientable (CO) if it admits an orientation in which every simple chordless cycle is cyclically oriented. This family of graphs was introduced by Barot et al. [3] in 2006 and use it to devise a test for whether a cluster algebra is of finite type. Barot et al. [3] work leaves open question of giving an efficient characterization of cyclically orientable graphs. Barot et al. [3] gave a simple recursive description of cyclically orientable graphs, and use this to give an $O(n)$ algorithm to test whether a graph on n vertices is cyclically orientable.

In 2008 Gurvich [17] obtained several nice characterizations of CO-graphs, being motivated primarily by their applications in cluster algebras. Gurvich [17] obtained several new characterizations that provide algorithms for recognizing CO-graphs and obtaining their cyclic orientations in linear time. Gurvich [17] showed that the edge maximal CO-graphs are 2-trees; that is, $G = (V, E)$ is a 2-tree if and only if G is CO and $G' = (V, E')$ is not CO whenever E is a proper subset of E'.

4.2 Fully Orientability

An orientation D of a graph G assigns a direction to each edge of G [18] . If there does not exist any directed cycle in D, then D is said to be acyclic orientation. Let D be an acyclic orientation of a graph G. An arc of D is dependent if its reversal creates a directed cycle. Let $d_{min}(G)(d_{max}(G))$ denote the minimum (maximum) of the number of dependent arcs over all acyclic orientations of G. A graph G is called fully orientable if G has an acyclic orientation with exactly d dependent arcs for every d satisfying $d_{min}(G) \leq d \leq d_{max}(G)$. In 2009 Lai et al. [19] studied the conditions under which fully orientability of graph can be preserved when the graph is extended by attaching new paths or cycles. Preservation theorems are applied to prove fully orientability of subdivisions of Halin graphs and graphs of maximum degree at most three and gave interpretation to their $d_{min}(G)$. Lai et al. [19] showed that a connected graph G is fully orientable if $d_{min}(G) \leq 1$.

In 1997 Fisher et al. [12] gave a formula to a find maximum of the number of dependent arcs over all acyclic orientations of G, that is, $d_{max}(G) = |E(G)| - |V(G)| + k$, where $E(G)$ is the edge set, $V(G)$ is the vertex set, and k is the number of component in G, which is nothing but the nullity of a graph. Also Fisher et al. [12] proved that if G is a connected graph with $\chi(G) < g(G)$ then G is fully orientable and $d_{min}(G) = 0$. In 1995 West [32] proved that complete bipartite graphs are fully orientable. In 2006 Lai et al. [20] determine $d_{min}(G)$ for an outerplanar graph G and proved that G is fully orientable. In 2008 Lai et al. [24] proved that every 2-degenerate graph is fully orientable also gave an interpretations to their d_{min}.

Let $K_{r(n)}$ be the complete $r-$partite graph each of whose partite sets has n vertices. In 2009 Chang et al. [7] determined the fully orientability of $K_{r(n)}$ and proved that $K_{r(n)}$ is non-fully-orientable when $r \geq 3$ and $n \geq 2$. Also Chang et al. [7] determined that these are the only non-fully-orintable graphs known so far and they all have girth 3. An immediate consequence of this result is proved by Tong [25], that is, when m is composite number, there exists m-degenerate graphs which are non-fully-orientable. Moreover $K_{3(2)}$ is the smallest non-fully orientable and any acyclic orientation of $K_{3(2)}$ has 4, 6 or 7 vertices.

4.3 Semitransitive Orientability

In 2019 Choi et al. [8] investigated graph operations and graph products that preserve semitransitive orientability of graphs. The main theme of the paper [8] is to determine which graph operations satisfies the following statement: if a graph operation is possible on a semitransitively orientable graph, then the same graph operation can be executed on the graph while preserving the semi-transitive orientability. Choi et al. [8] were able to prove that this statement is true for edge-deletions, edge-additions, and edge-liftings in graphs. Moreover, for all three graph operations, Choi et al. [8] proved that the initial semi-transitive orientation can be extended to the new graph obtained by the graph operation. Also, Kitaev and Lozin [23] explicitly asked if certain graph products preserve the semitransitive orientability. Choi et al. [8] answer their question in the negative for the tensor product, lexicographic product, and strong product. Choi et al. [8] also push the investigation further and initiate the study of sufficient conditions that guarantee a certain graph operation to preserve the semi-transitive orientability.

In 2012 Adrian Tanasa [2], Group Field Theories (GFT) are quantum field theories over group manifolds; they can be seen as a generalization of matrix models. GFT Feynman graphs are tensor graphs generalizing ribbon graphs (or combinatorial maps); these graphs are dual not only to manifolds. In order to simplify the topological structure of these various singularities, colored GFT was introduced and intensively studied in 2011. Adrian Tanasa [2] proposed a different simplification of GFT, which he called multi-orientable GFT. Adrian Tanasa [2] studied the relation between multi-orientable GFT Feynman graphs and colorable graphs. Adrian Tanasa [2] proved that tadfaces and some generalized tadpoles are absent. Some Feynman amplitude computations are performed. A few remarks on the renormalizability of both multi-orientable and colorable GFT are made. A generalization from three-dimensional to four-dimensional theories is also proposed.

A graph $G = (V, E)$ is said to be word-representable (See [13]) if a word w can be formed using the letters of the alphabet V such that for every pair of vertices x and $y, xy \in E$ if and only if x and y alternate in w. A semi-transitive orientation is an acyclic directed graph where for any directed path $v_0 \to v_1 \to \cdots \to v_m, m \geq 2$ either there is no arc between v_0 and v_m or for all $1 \leq i < j \leq m$ there is an arc between v_i and v_j. An undirected graph is semi-transitive if it admits a semi-transitive orientation. For a given positive integers $n, a_1, a_2, \cdots, a_k$, Srinivasan and Hariharasubramanian [13] consider the undirected circulant graph with set of vertices $\{0, 1, 2, \ldots, n-1\}$ and the set of edges $\{ij \mid (i-j) \pmod{n} \text{ or } (j-i) \pmod{n} \in \{a_1, a_2, \ldots, a_k\}\}$, where $0 < a_1 < a_2 < \cdots < a_k < (n+1)/2$. Recently, Kitaev and Pyatkin [22] have proved that every 4-regular circulant graph is semi-transitive. Further, [22] have posed an open problem regarding the semi-transitive orientability of circulant graphs for which the elements of the set $\{a_1, a_2, \ldots, a_k\}$ are consecutive positive integers. Kitaev and Pyatkin [22] solved the problem mentioned above. In addition, Kitaev and Pyatkin [22] proved that under certain assumptions, some $k(\geq 5)$-regular circulant graphs are semi-transitive, and some are not. Moreover, since a semi-transitive orientation is a characterisation of word-representability, Kitaev and Pyatkin [22] gave some upper bound for the representation number of certain k-regular circulant graphs.

Open Problems

The following open problems are due to Li-Da Tong [25].

1. For a given odd prime p, does there exist a non-fully-orientable $p-$degenerate graph that is not $(p-1)$-degenerate?
2. For any given integer $g \geq 4$, does there exist a non-fully-orientable graph G whose girth is g?
3. Does there exist a non-fully-orientable graph G whose $d_{min}(G)$ is 2 or 3?
4. $K_{3(2)}$ shows that a maximal planar graph can be non-fully-orientable. How to characterize all fully orientable planar graphs?
5. How to characterize those complete multipartite graphs that are fully orientable?

We raise the open problem, namely, whether the cover graph of a poset dismantlable by irreducibles is orientable?

Acknowledgement

The authors are thankful to the management of PES, Modern College of Arts, Science and Commerce (Autonomous), Pune 5, for granting the project entitled "Ear Decomposition and Orientability of

Graphs" under “Seed Money Project 2023-24" of the college.

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