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A Note on Standard Closed Ideals in Weighted Discrete Abelian Semigroup Algebras

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Abstract

Let S be an abelian semigroup and ω be a weight on S. If T is a semigroup ideal in S, then the closed linear subspace $\ell^1_T(S,\omega) := \{f \in \ell^1(S,\omega) : \operatorname{supp} f \subseteq T\}$ is a closed ideal in $\ell^1(S,\omega)$. Such ideals including $\{0\}$ and $\ell^1(S,\omega)$ are *standard closed ideals*; while the others are *non-standard closed ideals* in $\ell^1(S,\omega)$. The weight ω on S is an *unicellular weight* if every closed ideal in $\ell^1(S,\omega)$ is a standard closed ideal. In the case where $S = \mathbb{Z}_+$, it has been extensively studied by several mathematicians. In this article, we intend to study the standard closed ideals in the case where $S = \mathbb{Z}^2_+$.

Keywords: Commutative Banach algebra; Standard closed ideals; Standard elements; Semigroup; Semigroup ideal; and Weight.

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1. Introduction

Throughout, S is a unital, abelian semigroup with the binary operation $+$. A *weight* ω on S is a function $\omega : S \longrightarrow (0,\infty)$ satisfying the submultiplicativity $\omega(s+t) \leq \omega(s)\omega(t)$ for all $s,t \in S$. The ω is called a *radical weight* on S if $\inf\{\omega(ns)^{\frac{1}{n}} : n \in \mathbb{N}\} = \lim_{n\to\infty} \omega(ns)^{\frac{1}{n}} = 0$ for every non-unital element $s \in S$.
Let $l^1(S,\omega) = \{f : S \longrightarrow \mathbb{C} : \|f\|_\omega := \sum\{|f(s)|\omega(s) : s \in S\} < \infty\}$. For $f,g \in l^1(S,\omega)$, the *convolution product* $f * g$ is defined as

$$(f*g)(s) = \sum\{f(u)g(v) : u,v \in S \text{ and } u+v=s\} \quad (s \in S).$$

Then $l^1(S,\omega)$ is a unital, commutative Banach algebra with the convolution product $*$ and the weighted norm $\|\cdot\|_\omega$; it is so-called a *weighted discrete semigroup algebra* [4, P.159]. For $s,t \in S$, define $\delta_s(t) = 1$ for $s=t$ and $\delta_s(t) = 0$ for $s \neq t$. Then we can write $f \in l^1(S,\omega)$ as $f = \sum\{f(s)\delta_s : s \in S\}$.
Nikolskii has mentioned in [9, P.189] that Šilov asked the following question around 1940: Is every radical weight ω on $\mathbb{Z}_+$ unicellular? In the year 1973, Grabiner [7] gave a sufficient condition on a weight ω on $\mathbb{Z}_+$ such that ω is unicellular. Later on, several authors gave different sufficient conditions

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on ω to ensure the unicellularity of the weight ω on $\mathbb{Z}_+$ [2,5,8–10]. However, no one could find a non-unicellular weight ω on $\mathbb{Z}_+$ till 1984. Nikolskii *claimed* in his paper [9, Theorem-5, P.205] to have such a radical weight on $\mathbb{Z}_+$. However, there was a gap in his proof. In 1984, Mark Thomas succeeded to construct a non-unicellular weight on $\mathbb{Z}_+$ [11]; its proof is quite difficult.
So it is quite natural to ask the following question: Does there exist a (necessarily, radical) unicellular weight on the semigroup $\mathbb{Z}_+^2$? In this article, we find some partial results. The methods used in the proofs are similar to the proofs for the case $S = \mathbb{Z}_+$.

2. Standard Closed Ideals in $\ell^1(S, \omega)$

In this section, we define standard closed ideals in $\ell^1(S, \omega)$ for any semigroup S. First we note the following elementary result without proof.

Lemma 2.1. *Let ω be a weight on a semigroup S and let $T \subset S$. Define $\ell^1_T(S,\omega) = \{f \in \ell^1(S,\omega) : suppf \subset T\}$. Then*

1. *$\ell^1_T(S,\omega)$ is a closed linear subspace of $\ell^1(S,\omega)$.*
2. *T is a subsemigroup of S iff $\ell^1_T(S,\omega)$ is a closed subalgebra of $\ell^1(S,\omega)$.*
3. *T is a semigroup ideal in S iff $\ell^1_T(S,\omega)$ is a closed ideal in $\ell^1(S,\omega)$.*

Definition 2.2. *A closed ideal $\mathcal{I}$ in $\ell^1(S,\omega)$ is a* standard closed ideal *if either $\mathcal{I} = \{0\}$ or there exists a semigroup ideal T in S such that $\mathcal{I} = \ell^1_T(S,\omega)$; otherwise, it is a* non-standard closed ideal. *An element $f \in \ell^1(S,\omega)$ is a* standard element *if $\mathcal{I}_f := \overline{f * \ell^1(S,\omega)}$ is a standard closed ideal in $\ell^1(S,\omega)$.*

Definition 2.3. *A weight ω on S is* unicellular *if $\ell^1(S,\omega)$ has only standard closed ideals.*

Lemma 2.4. *Let S be cancellative, ω be a weight on S, and $\mathcal{I}$ be a non-zero closed ideal in $\ell^1(S,\omega)$. Then there exists a smallest semigroup ideal T in S such that $\mathcal{I} \subseteq \ell^1_T(S,\omega)$.*

Proof. Let $T = \cup\{\text{supp}f : f \in \mathcal{I}\}$. First, we prove that T is a semigroup ideal in S. Let $s \in S$ and $t \in T$. Then $t \in \text{supp}f$ for some $f \in \mathcal{I}$. Since $\mathcal{I}$ is an ideal, $\delta_s * f \in \mathcal{I}$. Since S is cancellative, $(\delta_s * f)(s + t) = f(t) \neq 0$. Thus $s + t \in \text{supp}(\delta_s * f) \subseteq T$. So T is a semigroup ideal. Also it is clear that $\mathcal{I} \subseteq \ell^1_T(S,\omega)$. From the definition of T itself, it follows that T is the smallest semigroup ideal such that $\mathcal{I} \subseteq \ell^1_T(S,\omega)$. □

Proposition 2.5. *Let T be a semigroup ideal in S. Let $f \in \ell^1(S,\omega)$ be a non-zero element. Then $suppf + S \subseteq T$ iff $\mathcal{I}_f \subseteq \ell^1_T(S,\omega)$.*

Proof. Assume that $\text{supp}f + S \subseteq T$. Let $g \in \ell^1(S,\omega)$. Then $\text{supp}(f * g) \subseteq \text{supp}f + \text{supp}g \subseteq \text{supp}f + S \subseteq T$. Hence $f * g \in \ell^1_T(S,\omega)$. Since $\ell^1_T(S,\omega)$ is closed in $\ell^1(S,\omega)$, $\mathcal{I}_f \subseteq \ell^1_T(S,\omega)$. The converse is clear, because $f \in \mathcal{I}_f \subseteq \ell^1_T(S,\omega)$ implies $\text{supp}f + S \subseteq T + S \subseteq T$ as T is a semigroup ideal. □

The next result should be compared with [4, Proposition 4.6.19]. But it could not be considered as its generalization even though its proof is borrowed here.

Theorem 2.6. *Let ω be a radical weight on S. Then following are equivalent.*

1. *ω is unicellular;*

2. *Each $f \in \ell^1(S,\omega)$ is a standard element;*

3. *If $f \in \ell^1(S,\omega)$ non-zero and $s \in suppf$, then $\delta_s \in \mathcal{I}_f$;*

4. *If $\mathcal{I}$ is a non-zero closed ideal in $\ell^1(S,\omega)$ and $f \in \mathcal{I}$ is non-zero, then $\delta_s \in \mathcal{I}$ for all $s \in suppf$.*

Proof. (1) $\Rightarrow$ (2): This is clear.
(2) $\Rightarrow$ (3): Let $f \in \ell^1(S,\omega)$ be non-zero. Since f is a standard element, $\mathcal{I}_f = \ell^1_T(S,\omega)$ for some semigroup ideal T in S. Because $f \in \mathcal{I}_f = \ell^1_T(S,\omega)$, we have supp$f \subseteq T$. Thus $\delta_s \in \ell^1_T(S,\omega) = \mathcal{I}_f$ for all $s \in$ supp$f$.
(3) $\Rightarrow$ (4): Let $\mathcal{I}$ be a non-zero, closed ideal in $\ell^1(S,\omega)$. Let $f \in \mathcal{I}$ be non-zero and let $s \in$ supp$f$. It is clear that $\mathcal{I}_f \subseteq \mathcal{I}$ because $\mathcal{I}$ is a closed ideal. By the assumption, $\delta_s \in \mathcal{I}_f \subseteq \mathcal{I}$.
(4) $\Rightarrow$ (1): Let $\mathcal{I}$ be a non-zero closed ideal and $T = \cup\{\text{supp}f : f \in \mathcal{I}\}$. As per the proof of Lemma 2.4, T is a semigroup ideal in S. Clearly, $\mathcal{I} \subseteq \ell^1_T(S,\omega)$. Now let $f \in \ell^1_T(S,\omega)$ and $s \in$ supp$f$. Then $s \in T$. By the definition of T, there exists a non-zero element $g \in \mathcal{I}$ such that $s \in$ supp$g$. By the assumption, $\delta_s \in \mathcal{I}$. Since $s \in$ supp$f$ is arbitrary and $\mathcal{I}$ is a closed ideal, $f \in \mathcal{I}$. Thus $\ell^1_T(S,\omega) \subseteq \mathcal{I}$ and so $\mathcal{I}$ is a standard closed ideal. Hence ω is unicellular. □

3. Standard Closed Ideals in $\ell^1(\mathbb{Z}^2_+,\omega)$

There are two famous theorems on standard closed ideals in $\ell^1(\mathbb{Z}_+,\omega)$; one is proved by Domar [4, Theorem 4.6.28] and another one is proved by Thomas [4, Theorem 4.6.29]. In this section, we have made an attempt to prove analogous results in $\ell^1(\mathbb{Z}^2_+,\omega)$. We start with the following notations.

Definition 3.1. *Let $c, d \in \mathbb{Z}_+$ and ω be a weight on $\mathbb{Z}^2_+$. Define*

$$\begin{aligned} U(c,d) &= \{(m,n) \in \mathbb{Z}^2_+ : c \le m \text{ and } d \le n\} \\ V(c,d) &= \{(m,n) \in \mathbb{Z}^2_+ : c \le m \text{ or } d \le n\} \\ U_{(c,d)}(\mathbb{Z}^2_+,\omega) &= \{f \in \ell^1(\mathbb{Z}^2_+,\omega) : suppf \subset U(c.d)\} \\ V_{(c,d)}(\mathbb{Z}^2_+,\omega) &= \{f \in \ell^1(\mathbb{Z}^2_+,\omega) : suppf \subset V(c.d)\} \end{aligned}$$

Clearly, $U(c,d)$ and $V(c,d)$ are semigroup ideals in $\mathbb{Z}^2_+$. Some basic properties of $U_{(c,d)}(\mathbb{Z}^2_+,\omega)$ and $V_{(c,d)}(\mathbb{Z}^2_+,\omega)$ are listed out in the following lemma.

Lemma 3.2. *Let $c, d \in \mathbb{Z}_+$, and ω be a weight on $\mathbb{Z}^2_+$. Then*

1. $U_{(c,d)}(\mathbb{Z}_+^2,\omega) \subseteq V_{(c,d)}(\mathbb{Z}_+^2,\omega)$;
2. $U_{(c,d)}(\mathbb{Z}_+^2,\omega) = \ell_T^1(\mathbb{Z}_+^2,\omega)$, *where* $T = U(c,d)$;
3. $V_{(c,d)}(\mathbb{Z}_+^2,\omega) = \ell_T^1(\mathbb{Z}_+^2,\omega)$, *where* $T = V(c,d)$;
4. $U_{(c,d)}(\mathbb{Z}_+^2,\omega)$ *and* $V_{(c,d)}(\mathbb{Z}_+^2,\omega)$ *are standard closed ideals in* $\ell^1(\mathbb{Z}_+^2,\omega)$.

Proof. The proofs follow immediately from Definition 3.1. □

Proposition 3.3. *Consider the totally ordered semigroup* $(\mathbb{Z}_+^2,+,\leq)$ *with respect to the dictionary order. Let* T *be a semigroup ideal in* $\mathbb{Z}_+^2$ *and* ω *be a weight on* $\mathbb{Z}_+^2$. *Then*

1. *There exist a finite sequence* $(c_1,d_1) < \cdots < (c_n,d_n)$ *in* $\mathbb{Z}_+^2$ *such that*

$$\ell_T^1(\mathbb{Z}_+^2,\omega) = U_{(c_1,d_1)}(\mathbb{Z}_+^2,\omega) + \cdots + U_{(c_n,d_n)}(\mathbb{Z}_+^2,\omega);$$

2. *There exists a minimum element* (c_0,d_0) *in* T *such that*

$$U_{(c_0,d_0)}(\mathbb{Z}_+^2,\omega) \subseteq \ell_T^1(\mathbb{Z}_+^2,\omega) \subseteq V_{(c_0,d_0)}(\mathbb{Z}_+^2,\omega).$$

Proof. (1) By [3, Theorem 2.3], there exists a finite sequence $(c_1,d_1) < \ldots < (c_n,d_n)$ in $\mathbb{Z}_+^2$ such that $T = \bigcup_{i=1}^{n} U(c_i,d_i)$. Clearly, $U_{(c_i,d_i)}(\mathbb{Z}_+^2,\omega) \subseteq \ell_T^1(\mathbb{Z}_+^2,\omega)$ for each $1 \leq i \leq n$ and so $U_{(c_1,d_1)}(\mathbb{Z}_+^2,\omega) + \ldots + U_{(c_n,d_n)}(\mathbb{Z}_+^2,\omega) \subseteq \ell_T^1(\mathbb{Z}_+^2,\omega)$. Conversely, let $f \in \ell_T^1(\mathbb{Z}_+^2,\omega)$. Then, there exist $f_1,\ldots,f_n \in \ell^1(\mathbb{Z}_+^2,\omega)$ such that $f = f_1 + \cdots + f_n$, $\text{supp} f_1 \subseteq U_{(c_1,d_1)}$ and $\text{supp} f_k \subseteq U_{(c_k,d_k)} \setminus \bigcup_{i=1}^{k-1} U(c_i,d_i)$ for each $2 \leq k \leq n$. So that $f = f_1 + \cdots + f_n \in U_{(c_1,d_1)}(\mathbb{Z}_+^2,\omega) + \cdots + U_{(c_n,d_n)}(\mathbb{Z}_+^2,\omega)$. This completes the proof of (1).
(2) Again, by [3, Theorem 2.3], there is a finite sequence $(c_1,d_1) < \ldots < (c_n,d_n)$ in $\mathbb{Z}_+^2$ such that $T = \bigcup_{i=1}^{n} U(c_i,d_i)$. Take $(c_0,d_0) = (c_1,d_1)$. Then (c_0,d_0) is the minimum element in T and $U(c_0,d_0) \subseteq T \subseteq V(c_0,d_0)$. Hence, by basic definitions, $U_{(c_0,d_0)}(\mathbb{Z}_+^2,\omega) \subseteq \ell_T^1(\mathbb{Z}_+^2,\omega) \subseteq V_{(c_0,d_0)}(\mathbb{Z}_+^2,\omega)$. This completes the proof of (2). □

Proposition 3.4. *Let* ω *be a radical weight on* $\mathbb{Z}_+^2$. *Let* $f \in \ell_T^1(\mathbb{Z}_+^2,\omega)$ *have a finite support and* (c,d) *be the minimum element of supp*f. *Then* $\mathcal{I}_f \subseteq \ell_T^1(S,\omega)$, *where* $T = U(c,d)$.

Proof. By the hypothesis, there exists $g \in \ell^1(\mathbb{Z}_+^2,\omega)$ such that g has a finite support and $f = \delta_{(c,d)} * g$. Then $g(0) = f(c,d) \neq 0$. Since $\ell^1(\mathbb{Z}_+^2,\omega)$ is a local algebra, g is invertible in $\ell^1(\mathbb{Z}_+^2,\omega)$. Hence $\delta_{(c,d)} = f * g^{-1} \in \mathcal{I}_f$. This implies $\delta_{(c+m,d+n)} = \delta_{(c,d)} * \delta_{(m,n)} \in \mathcal{I}_f$ for any $(m,n) \in \mathbb{Z}_+^2$. So that $\ell_T^1(S,\omega) \subseteq \mathcal{I}_f$. The reverse inclusion is clear. Hence f is a standard element. □

Conjecture 3.5. *We believe that there is no unicellular weight on* $\mathbb{Z}_+^2$. *More specifically,* $f = \delta_{(1,0)} + \delta_{(0,1)}$ *is not a standard element in* $\ell^1(\mathbb{Z}_+^2,\omega)$ *for any weight* ω *on* $\mathbb{Z}_+^2$.

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