International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **12(4)**(2024), 65–74
Available Online: http://ijmaa.in

Some Arithmetical Functions Like Functions in the Polynomial Ring Over $\mathbb{Z}_p$

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Abstract

An analogous study of arithmetical functions on the ring of polynomials $\mathbb{Z}_p[t]$ called Arithmetical functions like functions (AFL) like μ_p, I_p and ϕ_p are developed in this paper and some of their properties are described.

Keywords: Arithmetical functions; Polynomials over $\mathbb{Z}_p$; Euler function $\phi(n)$.

2020 Mathematics Subject Classification: 94A60, 11T71.

1. Introduction

The Euler function $\phi(n)$ defined on the set of positive integers gives the number of units in the residue ring $\mathbb{Z}/(n)$ of ring of integers $\mathbb{Z}$. Generalizing this to the study of count of units in the residue ring of the ring of polynomials $\mathbb{Z}_p[t]$ in the variable t and for p a prime number it is noted that it revolves around the study of functions that are analogues to arithmetical functions like Mobius function, Euler function etc. defined on $\mathbb{Z}_p[t]$. In this context the functions analogous to arithmetical functions like Mobius function, Euler function are developed and some related properties are discussed in this paper. This study is useful from the perspective of developing a generalization to Riemann zeta function with polynomials. In this context of generalization we consider the polynomial ring $\mathbb{Z}_p[t]$ for p a prime number and define the relation $\sim$ on $\mathbb{Z}_p[t]$ given as, for any $\alpha(t)$, $g(t) \in \mathbb{Z}_p[t]$, $\alpha(t) \sim g(t)$ if and only if $g(t) = c.\alpha(t)$ for some $c \neq 0$ in $\mathbb{Z}_p$. Note this relation $\sim$ on $\mathbb{Z}_p[t]$ is an equivalence relation on $\mathbb{Z}_p[t]$. Denote each equivalence class $[\alpha(t)]$ as $\overline{\alpha}(t)$ and the set of all equivalence classes be denoted as $\wp_c\mathbb{Z}_p[t]$. Then $\wp_c\mathbb{Z}_p[t]$ is given as

$$\wp_c\mathbb{Z}_p[t] = \{\overline{\alpha}(t) : \alpha(t) \neq 0 \in \mathbb{Z}_p[t]\}$$

Definition 1.1. *An Arithmetical function like (AFL) function on $\wp_c\mathbb{Z}_p[t]$ is a complex or real valued function defined on $\wp_c\mathbb{Z}_p[t]$.*

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2. AFL μ_p and I_p on $\wp_c\mathbb{Z}_p[t]$

Definition 2.1. *A real valued function μ_p on $\wp_c\mathbb{Z}_p[t]$ is defined as follows : For any $\overline{\alpha}(t) \in \wp_c\mathbb{Z}_p[t]$, we define*

$$\mu_p(\overline{\alpha}(t)) = 1 \text{ if } \deg \ (\mathrm{ff(t)}) = 0.$$

and if $\deg(\mathrm{ff(t)}) > 0$ *then for* $\alpha(t) = \alpha_1^{a_1}(t)\alpha_2^{a_2}(t)\alpha_3^{a_3}(t)\ldots\alpha_k^{a_k}(t)$, *where* $\mathrm{ff_i(t)}$ *are irreducible polynomials in* $\mathbb{Z}_p[t]$, *we define*

$$\mu_p(\overline{\alpha}(t)) = \begin{cases} (-1)^k, & \text{if } a_1 = a_2 = a_3 = \ldots a_k = 1 \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.2. *For any $\overline{\alpha}(t) \in \wp_c\mathbb{Z}_p[t]$, the function $I_p(\overline{\alpha}(t))$ given as*

$$I_p(\overline{\alpha}(t) = \left[\frac{1}{deg\alpha(t)+1}\right] = \begin{cases} 1, \text{ if } \deg(\mathrm{ff(t)}) = 0, \\ 0, \text{ if } \deg(\mathrm{ff(t)}) > 0. \end{cases}$$

is an Arithmetic function like function.

Theorem 2.3. *For any $\overline{\alpha}(t) \in \wp_c\mathbb{Z}_p[t]$ with $deg(\alpha(t)) \geq 0$ we have*

$$\sum_{\overline{d}(t)\big|\overline{\alpha}(t)} \mu_p(\overline{d}(t)) = \begin{cases} 1, \text{ if } \deg(\mathrm{ff(t)}) = 0, \\ 0, \text{ if } \deg(\mathrm{ff(t)}) > 0. \end{cases} = I_p(\overline{\alpha}(t))$$

Proof. Let $\overline{\alpha}(t) \in \wp_c\mathbb{Z}_p[t]$, if $deg(\alpha(t) = 0$ then $\alpha(t) = c$ for some $c \neq 0 \in \mathbb{Z}_p$ and $\overline{\alpha}(t) = \overline{c} = \overline{1}$ with

$$\sum_{\overline{d}(t)\big|\overline{\alpha}(t)} \mu_p(\overline{d}(t)) = \sum_{\overline{d}(t)\big|\overline{1}} \mu_p(\overline{d}(t)) = \mu_p(\overline{1}) = 1$$

If $deg(\alpha(t)) > 0$ and $\alpha(t) = \alpha_1^{a_1}\alpha_2^{a_2}\alpha_3^{a_3}\ldots\alpha_k^{a_k}$ for each α_i irreducible in $\mathbb{Z}_p[t]$, then take $D = \Big\{\overline{d}(t) \in \wp_c\mathbb{Z}_p[t] : \overline{d}(t)\Big|\overline{\alpha}(t)\Big\}$, the set of all divisors of $\overline{\alpha}(t)$. Now if

$$D_1 = \{\overline{d}(t) \in \wp_c\mathbb{Z}_p[t] : \overline{d}(t)\Big|\overline{\alpha}(t) \text{ and } \overline{\mathrm{d}}(\mathrm{t}) \text{ has no square irreducible factor in } \wp_c\mathbb{Z}_p[t]\}$$

$$D_2 = \{\overline{d}(t) \in \wp_c\mathbb{Z}_p[t] : \overline{d}(t)\Big|\overline{\alpha}(t) \text{ and } \mathrm{d}(\mathrm{t}) \text{ has square irreduciblefactor in } \wp_c\mathbb{Z}_p[t]\}$$

Then note $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \phi$

$$\sum_{\overline{d}(t)\big|\overline{\alpha}(t)} \mu_p(\overline{d}(t)) = \sum_{\overline{d}(t)\in D} \mu_p(\overline{d}(t))$$

$$= \sum_{\overline{d}(t) \in D_1 \cup D_2} \mu_p(\overline{d}(t))$$
$$= \sum_{\overline{d}(t) \in D_1} \mu_p(\overline{d}(t)) + \sum_{\overline{d}(t) \in D_2} \mu_p(\overline{d}(t))$$
$$= \sum_{\overline{d}(t) \in D_1} \mu_p(\overline{d}(t)) + 0$$

Now note D_1 consists of $\overline{d}(t) = \overline{1}$, $\overline{\alpha_1}(t)$, $\overline{\alpha_2}(t), \ldots, \overline{\alpha_k}(t)$, $(\overline{\alpha_1}(t)\overline{\alpha_2}(t))$, $(\overline{\alpha_1}(t)\overline{\alpha_3}(t))$, ... and $(\overline{\alpha_1}(t)\overline{\alpha_2}(t)$ $\overline{\alpha_3}(t)\ldots\overline{\alpha_k}(t))$, therefore we have

$$\sum_{\overline{d}(t) \mid \overline{\alpha}(t)} \mu_p(\overline{d}(t)) = \mu_p(\overline{1}) + \mu_p(\overline{\alpha_1}(t)) + \mu_p(\overline{\alpha_2}(t)) + \ldots, \mu_p(\overline{\alpha_k}(t)) + \mu_p((\overline{\alpha_1}(t)\overline{\alpha_2}(t)))$$
$$+ \mu_p((\overline{\alpha_1}(t))\overline{\alpha_3}(t)) + \ldots \text{ or } \bar{\ }_{\mathrm{p}}(\overline{\mathrm{ff}_1}(\mathrm{t})\overline{\mathrm{ff}_2}(\mathrm{t})\overline{\mathrm{ff}_3}(\mathrm{t})\ldots\overline{\mathrm{ff}_k}(\mathrm{t}))$$
$$= 1 + \binom{k}{1}(-1)^1 + \binom{k}{2}(-1)^2 + \cdots + \binom{k}{k}(-1)^k$$
$$= (1-1)^k = 0.$$

Therefore $\sum_{\overline{d}(t) \mid \overline{\alpha}(t)} \mu_p(\overline{d}(t)) = 0$ if $\deg(\mathrm{ff}(\mathrm{t})) > 0$. □

3. An Euler Function Like Function ϕ_p on $\wp_c\mathbb{Z}_p[t]$:

Definition 3.1. We define an AFL ϕ_p on $\wp_c\mathbb{Z}_p[t]$ as, for any $\overline{\alpha}(t) \in \wp_c\mathbb{Z}_p[t]$,

$$\phi_p(\overline{\alpha}(t)) = \begin{cases} 1 & \text{if } deg(\alpha(t)) = 0 \\ \#\left\{\overline{g}(t) \in \wp_c\mathbb{Z}_p[t] : deg(g(t)) < deg(\alpha(t)) \text{and } (\mathrm{g(t), ff(t)}) = \overline{1}\right\} & \text{if } deg(\alpha(t)) > 0 \end{cases}$$

Theorem 3.2. For any $\overline{\alpha}(t) \in \wp_c\mathbb{Z}_p[t]$, with $deg(\alpha(t)) \geq 1$, we have

$$\phi_p(\overline{\alpha}(t)) = \frac{1}{p-1} \sum_{\overline{d}(t) \mid \overline{\alpha}(t)} \mu_p(\overline{d}(t)).\frac{p^{deg(\alpha(t))}}{p^{deg(d(t))}}$$

Proof. Let $\overline{\alpha}(t) \in \wp_c\mathbb{Z}_p[t]$ and $deg(\alpha(t)) = s$ for $s \geq 1$. If $R_\alpha = \{\overline{g}(t) \in \wp_c\mathbb{Z}_p[t] : deg(g(t)) < deg(\alpha(t))\}$, then for any $\overline{g}(t) \in R_\alpha$ note $\overline{g}(t) = a_0 + a_1x + \ldots.. + a_{r-1}x^{r-1} + x^r$, for $a_i \in \mathbb{Z}_p$ for all $0 \leq i \leq r-1$ and $0 \leq r \leq s-1$, therefore note $\#R_\alpha = \frac{p^{deg(\alpha(t))}-p}{p-1}$

$$\phi_p(\overline{\alpha}(t)) = \sum_{\substack{\overline{g}(t) \in R_f \\ (\overline{g}(t),(\overline{\alpha}(t))=\overline{1})}} 1$$

$$= \sum_{i=1}^{\frac{p^{deg(\alpha(t))}-p}{p-1}} \left[\frac{1}{deg(\overline{g}_i(t),\overline{\alpha}(t))+1}\right]$$

$$= \sum_{i=1}^{\frac{p^{deg(\alpha(t))}-p}{p-1}} \sum_{\overline{d}(t)\Big|(\overline{g}_i(t),\overline{\alpha}(t)} \mu_p(d(t))$$

$$= \sum_{i=1}^{\frac{p^{deg(\alpha(t))}-p}{p-1}} \sum_{\substack{\overline{d}(t)|\overline{g}_i(t)\\(\overline{d}(t)|\overline{\alpha}(t)}} \mu_p(\overline{d}(t))$$

Now to compute the sum above we first sum over fixed divisor $\overline{d}(t)$ of $\overline{\alpha}(t)$ then vary over $\overline{d}(t)$. Note for fixed divisor $\overline{d}(t)$ of $\overline{\alpha}(t)$ the sum is over all $\overline{g}_i(t)$ with $deg(g_i(t)) \geq 0$ such that $\overline{g}_i(t) = \overline{d}(t)\ .\overline{q}_i(t)$

$$= \sum_{\overline{d}(t)|\overline{\alpha}(t)} \sum_{\overline{q}_i(t)} \mu_p(\overline{d}(t)$$

$$= \sum_{\overline{d}(t)|\overline{\alpha}(t)} \sum_{i=1}^{\frac{p^{deg(\alpha(t)-deg(d(t)}-p}{p-1}} \mu_p(\overline{d}(t)$$

$$= \sum_{\overline{d}(t)\Big|\overline{\alpha}(t)} \mu_p(\overline{d}(t)). \sum_{i=1}^{\frac{p^{deg(\alpha(t)-deg(d(t)}-p}{p-1}} 1$$

$$= \frac{1}{p-1} \sum_{\overline{d}(t)\Big|\overline{\alpha}(t)} \mu_p(\overline{d}(t)).\frac{p^{deg(\alpha(t))}}{p^{deg(d(t))}} - \frac{p}{p-1} \sum_{\overline{d}(t)\Big|\overline{\alpha}(t)} \mu_p(\overline{d}(t))$$

$$= \frac{1}{p-1} \sum_{\overline{d}(t)\Big|\overline{\alpha}(t)} \mu_p(\overline{d}(t)).\frac{p^{deg(\alpha(t))}}{p^{deg(d(t))}} - \frac{p}{p-1} I_p(\overline{\alpha}(t))$$

$$= \frac{1}{p-1} \sum_{\overline{d}(t)\Big|\overline{\alpha}(t)} \mu_p(\overline{d}(t)).\frac{p^{deg(\alpha(t))}}{p^{deg(d(t))}} - \frac{p}{p-1}.0$$

$$\therefore \phi_p(\overline{\alpha}(t)) = \frac{1}{p-1} \sum_{\overline{d}(t)\Big|\overline{\alpha}(t)} \mu_p(\overline{d}(t)).\frac{p^{deg(\alpha(t))}}{p^{deg(d(t))}}$$

□

Theorem 3.3. *For any $\overline{\alpha}(t) \in \wp_c\mathbb{Z}_p[t]$, with $deg(\alpha(t)) \geq 1$, we have*

$$\phi_p(\overline{\alpha}(t)) = \frac{p^{deg(\alpha(t))}}{(p-1)} \prod_{\overline{g}(t)\Big|\overline{\alpha}(t)} \left(1 - \frac{1}{p^{deg(g(t))}}\right)$$

where the product runs over the irreducible factors of $\overline{\alpha}(t)$.

Proof. Let $\overline{\alpha}(t) \in \wp_c\mathbb{Z}_p[t]$, with $deg(\alpha(t)) \geq 1$ then we have

$\overline{\alpha}(t) = \overline{g}_1^{e_1} \ldots \overline{g}_r^{e_r}$ for some irreducible $g_i \in \wp_c \mathbb{Z}_p[t]$ and

$$\begin{aligned}
\prod_{\overline{g}(t)\Big|\overline{\alpha}(t)} 1 - \frac{1}{p^{deg(g(t))}} &= \prod_{\overline{g}_i(t)\Big|\overline{\alpha}(t)} \left(1 - \frac{1}{p^{deg(g_i(t))}}\right) \\
&= \prod_{i=1}^{r} \left(1 - \frac{1}{p^{deg(g_i(t)}}\right) \\
&= 1 - \sum_i \frac{1}{p^{deg(g_i(t))}} + \sum_{i,j} \frac{1}{p^{deg(g_i(t))}.p^{deg(g_j(t))}} + ... + \frac{(-1)^r}{p^{deg(g_1(t))}....p^{deg(g_r(t))}} \\
&= 1 - \sum_i \frac{1}{p^{deg(g_i(t))}} + \sum_{i,j} \frac{1}{p^{deg(g_i(t))+deg(g_j(t))}} + ... + \frac{(-1)^r}{p^{deg(g_1(t))+..+deg(g_r(t))}} \\
&= 1 - \sum_i \frac{1}{p^{deg(g_i(t))}} + \sum_{i,j} \frac{(-1)^2}{p^{deg(g_i(t).g_j(t))}} + ... + \frac{(-1)^r}{p^{deg(g_1(t)...g_r(t))}} \\
&= \sum_{\overline{d}(t)\Big|\overline{\alpha}(t)} \frac{\mu_p(\overline{d}(t))}{p^{deg(d(t))}} \\
&= \frac{1}{p^{deg(\alpha(t))}} \sum_{\overline{d}(t)\Big|\overline{\alpha}(t)} \frac{\mu_p(\overline{d}(t)).p^{deg(\alpha(t))}}{p^{deg(d(t))}} \\
&= \frac{(p-1)}{p^{deg(\alpha(t))}}.\phi_p(\overline{\alpha}(t))
\end{aligned}$$

Therefore

$$\phi_p(\overline{\alpha}(t)) = \frac{p^{deg(\alpha(t))}}{(p-1)}. \prod_{\overline{g}(t)\Big|\overline{\alpha}(t)} \left(1 - \frac{1}{p^{deg(g(t))}}\right)$$

where the product is over irreducible factors $\overline{g}(t)$ of $\overline{\alpha}(t)$. □

Theorem 3.4. *The AFL ϕ_p function has following properties: For any, $\overline{a}(t), \overline{b}(t) \in \wp_c \mathbb{Z}_p[t]$*

(1). $\phi_p(\overline{g}(t)^\alpha) = \frac{1}{(p-1)}(p^{deg(g(t))})^\alpha - (p^{deg(g(t))})^{\alpha-1}$

(2). $\phi_p(\overline{a}(t).\overline{b}(t)) = \phi_p(\overline{a}(t)).\phi_p(\overline{b}(t).\frac{p^{deg(d(t))}}{\phi_p(\overline{d}(t))}$ *if* $gcd(\overline{a}(t), \overline{b}(t)) = \overline{d}(t)$

(3). $\overline{a}(t)\Big|\overline{b}(t)$ *implies* $\phi_p(\overline{a}(t))\Big|\phi_p(\overline{b}(t))$

(4). If p is odd then $\phi_p(\overline{a}(t))$ is even whenever $a(t)$ has atleast two irreducible factors and for $r \geq 2a(t)$ has r distinct irreducible factors if and only if $2^{r-}\Big|\phi_p(\overline{a}(t))$.

Proof.

(1). We have

$$\phi_p(\overline{\alpha}(t)) = \frac{p^{deg(\alpha(t))}}{(p-1)}. \prod_{\overline{g}(t)\Big|\overline{\alpha}(t)} \left(1 - \frac{1}{p^{deg(g(t))}}\right)$$

then for $\overline{\alpha}(t) = (\overline{g}(t)^{\alpha}$ above we have

$$\begin{aligned}\phi_p(\overline{g}(t)^{\alpha}) =& \frac{1}{(p-1)}(p^{deg(g(t)})^{\alpha}.\left(1 - \frac{1}{p^{deg(g(t)}}\right)\\ =& \frac{1}{(p-1)}(p^{\alpha.deg(g(t)}).\left(\frac{p^{deg(g(t)}-1}{p^{deg(g(t)}}\right)\\ =& \frac{1}{(p-1)}(p^{\alpha.deg(g(t))} - p^{(\alpha-1)deg(g(t))})\end{aligned}$$

(2). We have

$$\frac{\phi_p(\overline{a}(t)}{p^{deg(\overline{a}(t)}} = \frac{1}{(p-1)}\prod_{\overline{g}(t)\Big|\overline{a}(t)}.\left(1 - \frac{1}{p^{deg(\overline{g}(t))}}\right)$$

$$\frac{\phi_p(\overline{b}(t)}{p^{deg(\overline{b}(t)}} = \frac{1}{(p-1)}\prod_{\overline{h}(t)\Big|\overline{b}(t)}.\left(1 - \frac{1}{p^{deg(\overline{h}(t)}}\right)$$

Therefore

$$\frac{\phi_p(\overline{a}(t)).\phi_p(\overline{b}(t))}{p^{deg(\overline{a}(t)}.p^{deg(\overline{b}(t)}} = \frac{1}{(p-1)}\prod_{\overline{g}(t)\Big|\overline{b}(t)}.\left(1 - \frac{1}{p^{deg(\overline{g}(t)}}\right).\frac{1}{(p-1)}\prod_{\overline{h}(t)\Big|\overline{b}(t)}\left(1 - \frac{1}{p^{deg(\overline{\alpha}(t)}}\right)$$

and as
$$\begin{aligned}\frac{Œ_p(\overline{a}(t).\overline{b}(t))}{p^{deg(\overline{a}(t).\overline{b}(t)}} &= \frac{1}{(p-1)}\prod_{\overline{g}(t)\Big|(\overline{a}(t).\overline{b}(t)}.\left(1 - \frac{1}{p^{deg(a(t).b(t))}}\right)\\ &= \frac{\frac{1}{(p-1)}\prod_{\overline{g}(t)\Big|\overline{a}(t)}(1 - \frac{1}{p^{deg(\overline{g}(t))}}).\frac{1}{(p-1)}\prod_{\overline{h}(t)\Big|\overline{a}(t)}(1 - \frac{1}{p^{deg(\overline{h}(t)}})}{\frac{1}{(p-1)}\prod_{\overline{t}(t)\Big|\overline{d}(t)}.(1 - \frac{1}{p^{deg(\overline{h}(t))}})}\\ &= \frac{\frac{\phi_p(\overline{a}(t)).\phi_p(\overline{b}(t))}{p^{deg(\overline{a}(t)}.p^{deg(\overline{b}(t))}}}{\frac{\phi_p(\overline{d}(t)}{p^{deg(\overline{d}(t))}}}\end{aligned}$$

we have

$$\phi_p(\overline{a}(t).\overline{b}(t)) = \phi_p(\overline{a}(t)).\phi_p(\overline{b}(t)).\frac{p^{deg(d(t))}}{\phi_p(\overline{d}(t))}$$

(3). Let $\overline{a}(t)\Big|\overline{b}(t)$ then $\overline{b}(t) = \overline{a}(t).\overline{c}(t)$ for $0 \leq deg(c(t) \leq deg(b(t))$. If $deg(c(t)) = deg(b(t))$ then $deg(a(t) = 0$ and $\overline{a}(t) = \overline{1}$ and by definition as $\phi_p(\overline{1}) = \overline{1}$ we have $\phi_p(\overline{a}(t))\Big|\phi_p(\overline{b}(t))$. Therefore we now assume $deg(c(t)) < deg(b(t))$, then we have

$$\phi_p(\overline{b}(t) = \phi_p(\overline{a}(t).\overline{c}(t)) = \phi_p(\overline{a}(t).\phi_p(\overline{c}(t).\frac{p^{deg(d(t)}}{\phi_p(d(t))}$$

for $\overline{d}(t) = gcd(\overline{a}(t), \overline{c}(t))$, now note the result follows by induction on $deg(b(t))$.

For $deg(b(t)) = 1$, the result holds trivially and suppose that the result holds for all polynomials $\bar{t}(t)$ with $deg(t(t)) < deg(b(t))$, in particular it holds for $\bar{c}(t)$ So $\phi_p(\bar{d}(t))\Big|\phi_p(\bar{c}(t))$ as $\bar{d}(t)\Big|\bar{c}(t)$, then we have $\phi_p(\bar{b}(t) = \phi_p(\bar{a}(t).\bar{c}(t)) = \phi_p(\bar{a}(t).p^{deg(d(t)}\dfrac{\phi_p(\bar{c}(t)}{\phi_p(\bar{d}(t))}$. Hence $\phi_p(\bar{b}(t)$ is a multiple of $\phi_p(\bar{a}(t)$. Therefore $\phi_p(\bar{a}(t)\Big|\bar{b}(t))$, whenever $\bar{a}(t)\Big|\bar{b}(t)$.

(4). For $\bar{\alpha}(t) = \prod_{i=1}^{r}(\bar{g}_i(t))^{\alpha_i}$, where $\bar{g}_i$ are irreducible factors of $\bar{\alpha}(t)$ for all $i = 1, 2, \ldots, r$, we have $\phi_p(\bar{\alpha}(t))$ given as

$$\begin{aligned}
\phi_p(\bar{\alpha}(t)) &= \frac{1}{(p-1)} \prod_{\bar{g_i}(t)\big|\bar{a}(t)} (p^{\alpha_i.deg(g_i(t))} - p^{(\alpha_i-1)deg(g_i(t))}) \\
&= p^{(\alpha_1-1).deg(g_1(t))} \prod_{i=2}^{r} p^{(\alpha_i-1).deg(g_i(t))}(p-1)
\end{aligned}$$

therefore if $\bar{a}(t)$ has atleast two irreducible factors then for p odd as $2\Big|p-1$ we have $\phi_p(\bar{a}(t))$ is even and for $r \geq 2$ $a(t)$ has r distinct irreducible factors if and only if and $2^{r-1}\Big|\phi_p(\bar{a}(t))$.

□

Theorem 3.5. *If $deg(\alpha(t) \geq 0$, we have, $\sum_{\bar{d}(t)|\bar{\alpha}(t)} \phi_p(\bar{d}(t)) = p^{deg(\alpha(t))}$.*

Proof. Let $\bar{\alpha}(t)$ be any polynomial in $\wp_c\mathbb{Z}_p[t]$ with $deg(\alpha(t) \geq 0$. Let $S = \{\bar{k}(t) : \bar{k}(t) \in \wp_c\mathbb{Z}_p[t]$ with $0 \leq$ deg(k(t) $<$ deg(ff(t)$\}$. For any fixed divisor $\bar{d}(t)$ of $\bar{\alpha}(t)$ define the set $A(\bar{d}(t))$ as

$$A(\bar{d}(t)) = \{\bar{k}(t) : \bar{k}(t) \in S \text{ such that } \mathrm{gcd}(\bar{\mathrm{k}}(\mathrm{t}), \bar{\mathrm{ff}}(\mathrm{t})) = \bar{\mathrm{d}}(\mathrm{t})\}$$

Then note $A(\bar{d}_1(t)) \cap A(\bar{d}_2(t)) = \phi$ and $S = \cup A(d(t))$, therefore

$$|S| = \sum_{\bar{d}(t)\big|\bar{\alpha}(t)} (A(\bar{d}(t))$$

Now as $gcd(\bar{k}(t), \bar{\alpha}(t)) = \bar{d}(t)$ if and only if $gcd(\dfrac{\bar{k}(t)}{\bar{d}(t)}, \dfrac{\bar{\alpha}(t)}{\bar{d}(t)}) = \bar{1}$, we have $|A(d(t)| = \phi_p(\frac{\bar{\alpha}(t)}{\bar{d}(t)})$, therefore

$$\sum_{\bar{d}(t)|\bar{\alpha}(t)} \phi_p(\bar{d}(t)) = |S| = p^{deg(\alpha(t))}$$

□

Example 3.6. *Let $\alpha(t) = x^2 + x + 2 \in \mathbb{Z}_3[x]$, then $\alpha(t)$ is an irreducible polynomial over $\mathbb{Z}_3$ therefore by the definition of $\phi_p(\alpha(t))$ note*

$$\phi_p\overline{(x^2+x+2)} = \#\{\bar{g}(t) \in \wp_c\mathbb{Z}_p[t] : deg(g(t)) < deg(x^2+x+2) \text{ and } (\mathrm{g(t)}, \mathrm{x}^2+\mathrm{x}+2) = \bar{1}\}$$

$$
\begin{aligned}
&= \#\{\overline{g}(t) \in \wp_c\mathbb{Z}_p[t] : g(t) \neq 0 \text{ and } \mathrm{g(t) = ax + b; a, b} \in \mathbb{Z}_3 \}\\
&= \#\{\overline{g}(t) : g(x) = 1, 2, 3, x, x+1, x+2, 2x, 2x+1, 2x+2\}\\
&= \#\{\overline{1}, \overline{x}, \overline{x+1}, \overline{x+2}\}\\
&= 4.
\end{aligned}
$$

Now verify this by the above formula: By the above formula we have

$$\phi_p(\overline{\alpha}(t)) = \frac{p^{deg(\alpha(t))}}{(p-1)} \prod_{\overline{g}(t)\big|\overline{\alpha}(t)} \left(1 - \frac{1}{p^{deg(g(t))}}\right)$$

where the product runs over the irreducible factors of $\overline{\alpha}(t)$. Now for $\alpha(t) = x^2 + x + 2$ as $\alpha(t)$ is irreducible, we have

$$
\begin{aligned}
\phi_3(x^2 + x + 2)) &= \frac{3^2}{3-1}.\left(1 - \frac{1}{3^2}\right)\\
&= \frac{3^2 - 1}{3 - 1} = 4.
\end{aligned}
$$

Example 3.7. *Let $\alpha(t) = x^2 + 8$ and $\mathbb{Z}_{11}[x]$, then $\alpha(t) = x^2 + 8$* is reducible over $\mathbb{Z}_{11}$ and $\alpha(t) = x^2 + 8 = (x+6)(x+5)$ and by the definition of $\phi_p(\alpha(t))$ note

$$
\begin{aligned}
\phi_p\overline{(x^2+8)} &= \#\{\overline{g}(t) \ \in \ \wp_c\mathbb{Z}_p[t] : deg(g(t)) < deg(x^2+8) \text{ and } (\mathrm{g(t), x^2 + 8}) = \overline{1}\}\\
&= \#\{\overline{g}(t) \in \wp_c\mathbb{Z}_p[t] : g(t) \neq 0 \text{ and } \mathrm{g(t) = ax + b; a, b} \in \mathbb{Z}_11 \}\\
&= \#\{\overline{g}(t) : g(x) = 1, 2, 3, \ldots, 10, x, (x+1), (x+2), \ldots, (x+10), 2x, (2x+1),
\end{aligned}
$$

$(2x+2), \ldots, (2x+10), \ldots, (10x+10)$ such that $\overline{ax+b}$ is not $\overline{(x+6)}$ and

$$
\begin{aligned}
\overline{(x+5)} &= \#\{\overline{1}, \overline{x}, \overline{x+1}, \overline{x+2}, \overline{x+3}, \overline{x+4}, \overline{x+7}, \overline{x+8}, \overline{x+9}, \overline{x+10}\}\\
&= 10.
\end{aligned}
$$

Now verify this by the above formula:

By the above formula we have

$$\phi_p(\overline{\alpha}(t)) = \frac{p^{deg(\alpha(t))}}{(p-1)} \prod_{\overline{g}(t)\big|\overline{\alpha}(t)} \left(1 - \frac{1}{p^{deg(g(t))}}\right)$$

where the product runs over the irreducible factors of $\overline{\alpha}(t)$. Now for $\alpha(t) = x^2 + 8$ as $\overline{\alpha(t)}$ is reducible with irreducible factors as $\overline{(x+5)}$ and $\overline{(x+6)}$ we have

$$\begin{aligned}\phi_p(x^2+8) &= \frac{p^{deg(x^2+8))}}{(11-1)}.\left(1-\frac{1}{11^{deg(x+5)}}\right)\left(1-\frac{1}{11^{deg(x+5)}}\right)\\ &= \frac{11^2}{(11-1)}\left(1-\frac{1}{11}\right)\left(1-\frac{1}{11}\right)\\ &= \frac{1}{(11-1)}(11-1)(11-1)\\ &= 10.\end{aligned}$$

4. Conclusion

The formula for $\phi_p(\alpha(t))$ is useful in the count of the units in the residue $\mathbb{Z}_p[t]/(\alpha(t))$ which is the order of the multiplicative group $\mathbb{Z}_p[t]/(\alpha(t))$ for $\alpha(t)$ a polynomial and not necessarily irreducible in $\mathbb{Z}_p[t]$; In this context The product formula for $\phi_p(\alpha(t))$ is obtained applying the relation between mobius fuction $\mu_p(\alpha(t))$ and $\phi_p(\alpha(t))$. Extending this study to developing analogues to other arithmetical functions and the related properties, and then employ the Dirichlet product and study, the multiplicative and completely multiplicative properties is useful in understanding a generalised Riemann zeta function with Polynomials $\mathbb{Z}_p[t]$.

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