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Tri Topological spaces - An Overview

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Abstract

A N-topological space is a set equipped with N arbitrary topologies. If $T_1, T_2, ..., T_N$ are N topologies defined on a nonempty set X, then the $N-$topological space is denoted by $(X, T_1, T_2, ..., T_N)$. To denote a topology we use the symbol T for convenience. This article is a study on tri topological space. Some definitions related to topological space are discussed first. Subsequently, the tri topological space with example, characterisation, tri α continuous functions, tri α homeomorphisms, tri α biconnected spaces and discussed.

Keywords: hypercube; embedding; treewidth; topological invariant.
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1. Introduction

Topology, as a mathematical discipline, studies the properties of spaces that are preserved under continuous transformations. Over the years, several generalizations and extensions of classical topology have been developed to address complex structures and relationships in various fields. One such intriguing concept is tri-topological spaces. This article delves into the definition, properties, and significance of tri-topological spaces, emphasizing their role in advancing mathematical understanding. J.C Kelly [1] introduced bitopological spaces in 1963. The study of tri topological spaces was first initiated by Martin M Kovar in 2000, where a nonempty set X with three topologies is called tri topological space. In 1965, Njastad introduced a generalization of open set in topological space called α open set. A subset A of a topological space is called α open set if $A \subset$ int cl int A. In the definition of α open set, the same topology is used thrice. In this paper we use three different topologies and extend this concept to a tri topological space. The concept of tritopological spaces represents an advanced and relatively modern generalization in topology, wherein a single set is equipped with three distinct topologies. Unlike classical topological spaces, which involve a single topology, or bitopological spaces [2], which involve two, tritopological spaces introduce a more

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intricate structure that allows for the simultaneous study of three interrelated or independent topological properties. While the history of tritopological spaces is not as extensive as that of classical topology, it emerges naturally as a progression from earlier work in generalizations of topological spaces [3,4].

The idea of tritopological spaces [5] has its roots in the generalization of classical topological spaces. The study of a single topology on a set began in earnest with the foundational work of mathematicians such as Felix Hausdorff in the early 20th century.In 1963, J.C. Kelly introduced the concept of bitopological spaces, which involve two distinct topologies on the same set. This work was motivated by the need to study dual structures, such as lower and upper semicontinuity in analysis or distinct convergence criteria in probability theory. Bitopology provided a natural stepping stone to tritopological spaces, where an additional topology allows for the exploration of systems with even more complexity. The formal study of tritopological spaces began as an extension of bitopological spaces, addressing problems that required more than two interacting topologies. The motivations for defining and studying such spaces include Mathematical Necessity and Generalization of Existing Theories.Systems with three distinct, possibly interacting, criteria of "closeness" or continuity demanded a framework to describe their structure. Fields like multi-objective optimization, triadic decision-making, and certain physical systems naturally involve three overlapping but distinct perspectives. Tritopological spaces arose as natural extensions of results in bitopology and single-topology spaces, allowing researchers to investigate whether classical theorems (such as separation axioms, compactness, or continuity) could be extended to these more complex spaces.

2. Definition and Preliminaries

Definition 2.1. *A topological space is a pair (X, T),where X is a set and T is a family of subsets of X satisfying:*

(1). $\phi \in T$ and $X \in T$.

(2). T is closed under arbitrary union.

(3). $T$is closed under finite intersection.

The family T is said to be a topology on the set X.

Definition 2.2. *Let (X, T) be a topological space. Each element of T is open in X. A subset of X is closed in X if its complement is open in X.*

Definition 2.3. *The closure of a subset of a topological space is defined as the intersection of all closed subsets containing it.*

Definition 2.4. *Let (X, T) be a topological space, $x \in X$ and $N \subset X$. Then x is said to be an interior point of N if there exist an open set V such that $x \in V \subset N$.*

Definition 2.5. *Let (X, T) be a topological space and $A \subset X$. Then the interior of A is defined to be the set of all interior points of A.*

Definition 2.6. *Let A be a subset of a topological space X and $y \in X$. Then y is said to be an accumulation point of A if every open set containing y contains at least one point of A other than y.*

Definition 2.7. *Let $A \subset X$. Then the derived set of A is the set of all accumulation points of A in X.*

Definition 2.8. *Let $f : X \to Y$ be a function and $x \in X$ and T_1, T_2 be topologies on X, Y respectively. Then f is said to be continuous at x if for every $V \in T_2$ such that $f(x) \in V$, there exist $U \in T_1$ such that $x \in U$ and $f(U) \subset V$.*

Definition 2.9. *Let X be a non empty set and $T_1, T_2 and T_3$ be three topologies on X. X together with three topologies is called a tri topological space, it is denoted by (X, T_1, T_2, T_3).*

Example 2.10. *Let $X = \{a, b, c, d\}$, $T_1 = \{\phi, X\}$, $T_2 = P(X)$, $T3 = \{\phi, \{a\}, X\}$. Then (X, T_1, T_2, T_3) is a tri topological space.*

Theorem 2.11. *Any topological space is a tri topological space. Let (X, T) be a topological space, then (X, T, T, T) is a tri topological space.*

Theorem 2.12. *Any tri topological space is not a topological space, but any tri topological space induces a topological space . If we take the intersection of all topologies, then we get a topological space.*

Example 2.13. *Let $X = \{a, b, c, d\}$, $T_1 = \{\phi, \{a\}, \{b\}, \{a, b\}X\}$, $T_2 = \{\phi, \{a\}, \{c\}, \{a, c\}X\}$, $T_3 = \{\phi, \{a\}, \{d\}, \{a, d\}X\}$. Let $T = T_1 \cap T_2 \cap T_3$, $T = \{\phi, \{a\}, X\}$. Then (X, T) is a topological space.*

Theorem 2.14.

- *A is tri open iff A is open with respect to each topology.*
- *A is tri closed iff A is closed with respect to each topology.*
- *A is tri closed iff A^c is tri open.*
- *ϕ is always tri open.*
- *X is always tri open.*

Definition 2.15. *Let (X, T_1, T_2, T_3) be a tri topological space. A subset A of X is called tri α open in X, if $A \subset T_1$ int T_2 cl T_3 int A. The complement of tri α open set is called tri α closed set.*

Theorem 2.16. *A is tri open $\dashrightarrow$ A is tri α open.*

Theorem 2.17. *The converse is not true.*

3. Applications

By the mid-20th century, topology had become a central field in mathematics, influencing and being influenced by other areas such as:

Differential Topology: Studying smooth structures on manifolds, with figures like John Milnor and Stephen Smale making groundbreaking contributions.

Algebraic Topology: Bridging algebra and topology through invariants like the fundamental group and cohomology.

Set-Theoretic Topology: Examining spaces with intricate structures, often involving infinite sets.

Problems in areas like computer science, network theory, and theoretical physics often involve interacting systems with three distinct components (e.g., time, space, and state). Tritopological spaces provide a mathematical framework for modeling these phenomena.

The formalization and study of tritopological spaces were developed in the late 20th and early 21st centuries. Researchers generalized the axiomatic approach of topological spaces [6] to handle three topologies, exploring their interactions and how they influence concepts like continuity, convergence, compactness, and separation.

Functions defined between tritopological spaces are studied under the condition of tri-continuity, where a function is continuous with respect to all three topologies. The relationships among the three topologies (independence, refinement, or mutual exclusivity) have been a subject of interest, influencing how tritopological spaces are classified and analyzed.

Tritopological spaces [7] have found applications in modeling systems with triple criteria or dimensions, including: Multi-modal optimization in mathematics and economics. Complex systems in biology and ecology, where three types of interactions are significant. Network systems with three interacting layers, such as social, informational, and physical networks.

4. Challenges and Scope for Further Research

Today, topological spaces serve as the backbone for many mathematical disciplines, from pure mathematics to theoretical physics and computer science. Concepts like topological data analysis (TDA) and topological quantum computing highlight the relevance of topology in cutting-edge research.

As a relatively recent concept, tritopological spaces present challenges and opportunities for further research:

Extension of Classical Results: Theorems and principles in classical topology often require significant reformulation to apply to tritopological spaces. Questions about compactness, connectedness, and separation properties in tritopological spaces remain active areas of investigation.

Computational Complexity: The study of tritopological spaces involves higher computational

complexity due to the interaction of three topologies, particularly in applications.
Interdisciplinary Applications: Identifying and formalizing real-world systems that naturally align with tritopological structures is an ongoing challenge.

5. Conclusion

The history of topological spaces is a testament to the power of abstraction in mathematics. What began as a study of geometric shapes and physical continuity has evolved into a vast field with applications across science and engineering. The journey from Euler's bridges to abstract spaces demonstrates the dynamic and ever-evolving nature of mathematics, where new challenges continually inspire deeper insights and generalizations. The history of tritopological spaces is one of natural progression and generalization in the field of topology. Emerging from the foundational work on classical and bitopological spaces, tritopological spaces represent a sophisticated framework for studying systems with triple-layered complexity. Although still a developing field, their theoretical richness and potential applications suggest that tritopological spaces will play an increasingly significant role in both pure and applied mathematics. As research advances, the exploration of these spaces promises to yield deeper insights into the interplay of multiple structures and their influence on mathematical and real-world systems.

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