International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **11(4)**(2023), 181–188
Available Online: http://ijmaa.in

Study on Pseudo Semiconformally Symmetric Manifolds

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Abstract

In this paper, a type of Riemannian manifold, namely Generalized pseudo semiconformally symmetric manifold is studied. Several geometric properties of such spaces are studied. By imposing different restrictions on the semiconformal curvature tensor, we have obtained several properties. If the semiconformal curvature tensor is harmonic, then the forms of the scalar curvature of such spaces are obtained. Also, the relations among the 1-forms under various conditions are obtained.

Keywords: Riemannian manifolds; second Bianchhi identity; semiconformal curvature tensor.

1. Introduction

The geometry of a space mainly depends on the curvature of the space. One of the most important geometric properties of a space is symmetry. Cartan began the study of local symmetry of Riemannian spaces and studied elaborately ([2], [3]). According to him, a Riemannian manifold is said to be locally symmetric if $\nabla R = 0$. During the last sixty years, the notion of locally symmetric manifolds has been generalized by many authors in a weaker sense. They have weakened in different directions with several defining conditions by giving some curvature restrictions. Various weaker symmetries are studied as generalizations or extensions of Cartan's notion, such as recurrent manifolds by Walker [18], semi-symmetric manifolds by Szabó ([17]), pseudosymmetric manifolds in the sense of Deszcz [7], pseudosymmetric manifolds in the sense of Chaki [4], generalized pseudosymmetric manifolds by Chaki [6], weakly symmetric manifolds by Selberg [11] and weakly symmetric manifolds by Támassy and Binh [16]. According to Chaki a Riemannian manifold is said to be pseudo symmetric if

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= 2\alpha(X)R(Y,Z,U,V) + \alpha(Y)R(X,Z,U,V) + \alpha(Z)R(Y,X,U,V) \\ &+ \alpha(U)R(Y,Z,X,V) + \alpha(V)R(Y,Z,U,X) \qquad (1)\end{aligned}$$

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where and α is a 1-form, $X, Y, Z, U, V \in \chi(M)$. Pseudo symmetric manifolds are studied by many authors ([5] , [6], [12]). Ishii [8], introduced the notion of conharmonic transformation under which a harmonic function transforms into a harmonic function. $\overline{C}$, the conharmonic curvature tensor of type (0,4) on an (M^n, g) is defined as follows

$$\overline{C} = R - \frac{1}{n-2} g \wedge S, \tag{2}$$

in terms of local coordinates

$$\overline{C}_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{jk}r_{il} - g_{ik}r_{jl} + g_{il}r_{jk} - g_{jl}r_{ik}) \tag{3}$$

which remains invariant under conharmonic transformation where R and S are the Riemannian curvature and Ricci curvature tensor respectively. $g \wedge S$ is the Kulkarni-Nomizu product [14]. In [13], Shaikh and Hui showed that the conharmonic curvature tensor satisfies the symmetric and skew-symmetric properties of the Riemannian curvature tensor as well as cyclic ones. They also studied it ellaborately [15]. The conharmonic curvature tensor has many applications in the theory of general relativity. The conformal curvature tensor of type(0,4) is defined by

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{jk}r_{il} - g_{ik}r_{jl} + g_{il}r_{jk} - g_{jl}r_{ik}) + \frac{s}{(n-1)(n-2)}(g_{il}g_{jk} - g_{ik}g_{jl}) \tag{4}$$

It should be noted that the conformal curvature tensor C_{ijkl} remains invariant under conformal transformation. Kim [9] introduced semiconformal curvature tensor P of type (1,3) which is defined as

$$P^i_{jkl} = -(n-2)C^i_{jkl} + [a + (n-2)b]\overline{C}^i_{jkl}, \tag{5}$$

where a, b are constants. The beauty of this tensor is that it reduces to different curvature tensor for diiferent values of a and b. The semiconformal curvature tensor remains invariant under conformal transformation. Also, The semiconformal curvature tensor P_{ijkl} of type (0,4) possesses the several symmetric and skew symmetric properties. In 2021, Ali, Khan and Vasiulla [1] introduced generalized pseudo symmetric manifold and studied various properties. Also in 2017, Kim introduced pseudo semiconformally symmetric manifolds [10] and studied various properties. According to him, a Riemannian manifold (M^n, g) is said to be pseudo semiconformally symmetric if

$$P_{ijkl;m} = 2A_m P_{ijkl} + A_i P_{mjkl} + A_j P_{imkl} + A_k P_{ijml} + A_l P_{ijkm}, \tag{6}$$

where A is a non zero 1-form. Motivating by the above studies in this paper, I would like to study generalized pseudo semiconformally symmetric manifold, which is defined by

$$(\nabla_X P)(Y, Z, U, V) \quad = \quad 2\alpha(X)P(Y, Z, U, V) + \beta(Y)P(X, Z, U, V) + \gamma(Z)P(Y, X, U, V) \tag{7}$$

$$+ \quad \delta(U)P(Y,Z,X,V) + \eta(V)P(Y,Z,U,X)$$

where and α, β, γ, δ, η are 1-forms.
In terms of local coordinates

$$P_{ijkl;m} = 2\alpha_m P_{ijkl} + \beta_i P_{mjkl} + \gamma_j P_{imkl} + \delta_k P_{ijml} + \eta_l P_{ijkm} \tag{8}$$

2. Generalized Pseudo Semiconfrmally Symmetric Manifolds

Semiconformal curvature tensor P of a Riemannian manifold (M^n, g) is said to be harmonic if the divergence of P is zero, i.e.,

$$P^h_{jkl;h} = 0. \tag{9}$$

By virtue of second Bianchhi Identity we have

$$R^h_{jkl;h} = r_{jk;l} - r_{jl;k}. \tag{10}$$

And then

$$r^k_{l;k} = \frac{1}{2}s;l. \tag{11}$$

We have

$$P^h_{jkl;h} = -(n-2)bC^h_{jkl;h} + [a+(n-2)b]\overline{C}^h_{jkl;h} \tag{12}$$

Or,

$$\begin{aligned} P^h_{jkl;h} &= -(n-2)b\left(\frac{n-3}{n-2}\right)\left[(r_{jk;l} - r_{jl;k}) - \frac{1}{2(n-2)}(g_{jk}s_{;l} - g_{jl}s_{;k})\right] \\ &+ [a+(n-2)b]\left[(r_{jk;l} - r_{jl;k}) - \frac{1}{2(n-2)}(g_{jk}s_{;l} - g_{jl}s_{;k})\right]. \end{aligned} \tag{13}$$

Multiplying (13) by g^{jk} and using the condition (9) we get

$$0 = [a+(n-2)b]\frac{s_{;l}}{n-2}. \tag{14}$$

Which on siplification gives $s_{;l} = 0$ provided $[a+(n-2)b] \neq 0$, that is the scalar curvature is constant. Hence we have the following:

Theorem 2.1. *If the semiconfrmal curvature tensor of a generalized pseudo semiconfrmally symmetric Riemannian manifold is harmonic and $[a+(n-2)b] \neq 0$, then the scalar curvature of the space is constant.*

Let the semiconfrmal curvature tensor of a generalized pseudo semiconfrmally symmetric Riemannian manifold is harmonic. Then we have

$$0 = 2\alpha_m P^m_{jkl} + \beta^m P_{mjkl} + \gamma_j P^m_{mkl} + \delta_k P^m_{mjl} + \eta_l P^m_{jkm}. \tag{15}$$

Multiplying the above equation by g^{jk} we have

$$[a + (n-2)b]\frac{s}{n-2}[2\alpha_l + \beta_l - \delta_l + n\eta_l] = 0. \tag{16}$$

If the scalar curvature does not vanishes and $[a + (n-2)b] \neq 0$ then we have

$$2\alpha_l + \beta_l - \delta_l + n\eta_l = 0. \tag{17}$$

Thus we can state the following:

Theorem 2.2. *Let the semiconfrmal curvature tensor of a generalized pseudo semiconfrmally symmetric Riemannian manifold is harmonic. If $[a + (n-2)b] \neq 0$ and $2\alpha + \beta - \delta + n\eta \neq 0$, then the scalar curvature of the space vanishes.*

If the space is pseudo semiconfrmally symmetric Riemannian manifold then $\alpha = \beta = \gamma = \delta = \eta = A$ then we have:

Corollary 2.3. *If the semiconfrmal curvature tensor of a pseudo semiconfrmally symmetric Riemannian manifold is harmonic and $[a + (n-2)b] \neq 0$, then the scalar curvature of the space vanishes.*

Definition 2.4. *A Riemannian manifold (M^n, g) is said to be recurrent if its curvature tensor R_{ijkl} of type (0,4) satisfies the condition*

$$R_{ijkl;m} = B_m R_{ijkl} \tag{18}$$

where 1-form B_m is non zero.

Multiplying the equation (18) by g^{il} and then multiplying by g^{jk} we get

$$r_{jk;m} = B_m r_{jk} \tag{19}$$

and then

$$s_{;m} = B_m s \tag{20}$$

Using (19), (20) and (3) we get

$$g^{il} g^{jk} P_{ijkl;m} = -\frac{n}{n-2}[a + (n-2)b] B_m s. \tag{21}$$

From (8)

$$g^{il}g^{jk}P_{ijkl;m} = -\frac{s}{n-2}[a+(n-2)b][2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m]. \tag{22}$$

If $[a+(n-2)b] \neq 0$, then from the above two equations we get

$$B_m = \frac{2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m}{n} \tag{23}$$

Thus we can state that:

Theorem 2.5. *If a generalized pseudo semiconformally symmetric Riemannian manifold is recurrent, then the 1-forms B, α, β, γ, δ, η satisfy the relation* $B = \frac{2n\alpha+\beta+\gamma+\delta+\eta}{n}$.

If a generalized pseudo semiconformally symmetric Riemannian manifold is pseudo semiconformally symmetric then $\alpha = \beta = \gamma = \delta = \eta$. Then we can state that :

Corollary 2.6. *If a pseudo semiconformally symmetric Riemannian manifold is recurrent, then the 1-forms B and α are related by* $B = \frac{2(n+2)}{n}\alpha$.

Definition 2.7. *The semiconformal curvature tensor is said to be harmonic if the divergence of the curvature tensor* P^i_{jkl} *of type (1,3) vanishes, i.e.,*

$$P^h_{jkl;h} = 0. \tag{24}$$

From the Ricci identity and a parallel vector field V, it follows that

$$0 = V^t_{;jk} - V^t_{;kj} = V^m R^t_{mjk}. \tag{25}$$

Taking covariant derivative of the above equation we get

$$V^m R^t_{mjk;l} = 0. \tag{26}$$

Multiplying by g_{ti} we get

$$V^m R_{imjk;l} = 0 \tag{27}$$

Using second Bianchi identity we obtain

$$V^m R_{jkli;m} = 0. \tag{28}$$

Multiplying by g^{ji} and then multiplying by g^{kl} we have

$$V^m r_{kli;m} = 0 \tag{29}$$

$$V^m s_{;m} = 0. \tag{30}$$

Using the above equations it follows that

$$V^m P_{ijkl;m} = 0. \tag{31}$$

Or,

$$[2\alpha_m P_{ijkl} + \beta_i P_{mjkl} + \gamma_j P_{imkl} + \delta_k P_{ijml} + \eta_l P_{ijkm}]V^m = 0. \tag{32}$$

Or,

$$[a + (n-2)b]\left[\frac{2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m}{n}\right] V^m = 0. \tag{33}$$

Which leads the following:

Theorem 2.8. *If a generalized pseudo semiconformally symmetric manifold (M^n, g) admits a parallel vector field V and $[a + (n-2)b] \neq 0$, then either s=0 or $\left[\frac{2n\alpha_m+\beta_m+\gamma_m+\delta_m+\eta_m}{n}\right] V^m = 0$.*

Let a generalized pseudo semiconformally symmetric manifold is pseudo semiconformally symmetric then $\alpha = \beta = \gamma = \delta = \eta$. Then we can state that:

Corollary 2.9. *If a pseudo semiconformally symmetric manifold (M^n, g) admits a parallel vector field V and $[a + (n-2)b] \neq 0$, then either s=0 or $\alpha_m V^m = 0$.*

and then Multiplying (8) by g^{il} and then multiplying the relation thus obtained by g^{jk}, we obtain

$$-[a + (n-2)]\left(\frac{s_{;m}}{n-2}\right) n = -[a + (n-2)]\left(\frac{s}{n-2}\right)[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m]. \tag{34}$$

Since $[a + (n-2)] \neq 0$, we have

$$s_{;m} = \frac{[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m]}{n} s. \tag{35}$$

Taking covariant derivative of (35), we get

$$s_{;mt} = \frac{[2n\alpha_{m;t} + \beta_{m;t} + \gamma_{m;t} + \delta_{m;t} + \eta_{m;t}]}{n} s + \frac{[2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m)s_{;t}]}{n} \tag{36}$$

Or,

$$\begin{aligned} s_{;mt} &= \frac{[2n\alpha_{m;t} + \beta_{m;t} + \gamma_{m;t} + \delta_{m;t} + \eta_{m;t}]}{n} s \\ &+ \frac{[(2n\alpha_m + \beta_m + \gamma_m + \delta_m + \eta_m)(2n\alpha_t + \beta_t + \gamma_t + \delta_t + \eta_t)]s}{n} \end{aligned} \tag{37}$$

Therefore from the above relation we can write

$$0 = s_{;mt} - s_{;tm} = \frac{s}{n}[2n(\alpha_{m;t} - \alpha_{t;m}) + (\beta_{m;t} - \beta_{t;m}) + (\gamma_{m;t} - \gamma_{t;m}) + (\delta_{m;t} - \delta_{t;m}) + (\eta_{m;t} - \eta_{t;m})] \quad (38)$$

Thus we can state that:

Theorem 2.10. *Let the scalar curvature of a generalized pseudo semiconformally symmetric manifold does not vanish and* $[a + (n-2)] \neq 0$*. Then if four 1-forms are closed then all the 1-forms are closed.*

If the manifold pseudo semiconformally symmetric manifold then, $\alpha = \beta = \gamma = \delta = \eta = A$. Then we have from (38)

$$0 \quad = \quad s_{;mt} - s_{;tm} = \frac{2n+4}{n} s[A_{m;t} - A_{t;m}]. \quad (39)$$

If $s \neq 0$ then

$$A_{m;t} - A_{t;m} = 0. \quad (40)$$

Thus we have the following

Corollary 2.11. *If the scalar curvature of a generalized pseudo semiconformally symmetric manifold does not vanish, then the 1-form A is closed.*

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