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The Definitive Applications of Fibonacci Numbers: A Study

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Abstract : In Mathematics Fibonacci Numbers are the numbers in the following integer sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... By definition, the first two Fibonacci numbers are 1 and 1, and each subsequent number is the sum of the previous two. In this article, we discuss the various applications of Fibonacci numbers.

Keywords : Fibonacci Numbers, Fibonacci series.

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1 Introduction

Who was Fibonacci? The Fibonacci sequence, a remarkable numerical pattern, was introduced to the world by the distinguished Italian mathematician Leonardo Pisano Bigollo, who lived from 1170 to 1250. He is widely recognised in mathematical history by multiple names, including Leonard of Pisano—where "Pisano" signifies "from Pisa" and Fibonacci, which translates to "son of Bonacci." Fibonacci's early life was shaped by his experiences as the son of a prosperous Italian businessman in the bustling city of Pisa. His formative years unfolded in a vibrant trading colony located in North Africa during the height of the Middle Ages. During this dynamic period, Italians emerged as some of the most skilled traders and merchants in the Western world, demonstrating a keen need for practical mathematical tools to navigate their extensive commercial transactions. At that time, arithmetic calculations relied heavily on the cumbersome Roman numeral system, characterised by symbols like I, II, III, IV, V, etc. This outdated system posed significant challenges for merchants attempting to perform essential operations such as addition, subtraction, multiplication, and division. However, while living in North Africa, Fibonacci was fortunate to learn a more streamlined approach: the Hindu-Arabic numeral system. This innovative system, consisting of numerals like 1, 2, 3, etc., greatly simplified calculations and was taught him by an Arab instructor. In 1202, Fibonacci made a profound contribution to Mathematics by publishing his insights in a groundbreaking book titled "Liber Abaci." This work highlighted the undeniable superiority of the Hindu-Arabic arithmetic system over the Roman numeral system and illustrated practical applications of this advanced numerical method, demonstrating how it could significantly enhance the efficiency and accuracy of transactions for Italian merchants. This paper deliberates on the multiple applications of Fibonacci numbers.

Origin of Fibonacci sequence: In mathematics, Fibonacci Numbers are the numbers in the following integer sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... By definition, the first two Fibonacci numbers are 1 and 1 and each subsequent number is the sum of the previous two. In mathematical terms, the sequence a_n of Fibonacci numbers is defined by the recurrence relation $a_n = a_{n-1} + a_{n-1}$ with seed values $a_0 = 1$ and $a_1 = 1$. Fibonacci sequence was the outcome of a mathematical rabbit breeding problem posted in Liber abaci. The rabbit problem is as follows

"*A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many* *pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"*

The following table shows how the rabbit's population grows.

Month	Baby rabbit pairs	Adult rabbit pairs	Total
0 (at the beginning)	1	0	1
1	0	1	1
2	1	1	2
3	1	2	3
4	2	3	5
5	3	5	8
6	5	8	13
7	8	13	21
8	13	21	34
9	21	34	55
10	34	55	89
11	55	89	144
12	89	144	233

The Fibonacci sequence closely connects with nature, science, and real life. Therefore, it is widely used in many fields and well worth exploring. This paper includes two mathematical problems that the Fibonacci numbers could express. The recurrence relation concept was used to show the connection between the problems and Fibonacci numbers. This paper also includes two properties of Fibonacci numbers hidden in a game and some applications of Fibonacci numbers in nature.

2 Application of Fibonacci Numbers to Mathematical Problems

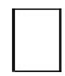
Consider two mathematical problems that can be explained more efficiently using Fibonacci numbers. The issues are a). Determine the number of patterns for bricks rearrangement and b). Determine the number of subsets of $\{1, 2, \ldots, n\}$ which do not contain two consecutive numbers. The recurrence relation concept was used to show the connection between the problems and

Fibonacci numbers. Solutions to the issues considered are shown as follows.

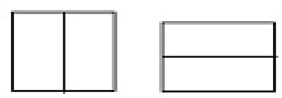
(a) Determine the number of patterns for bricks rearrangement

Suppose that there is a $2 \times n$ hole on the wall, and we wish to fill it using 1×2 boards. A board may be placed in either orientation. What is the number of different ways of filling in this hole?

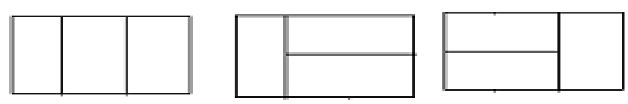
Example 2.1. *There is only one way of filling 2×1 hole.*



There are two ways of filling 2×2 hole.



There are three ways of filling 2×3 hole.



To solve this problem, we start on letting the number of ways be a_n. We have $a_0 = 1$ vacuously, and from the diagram above, $a_1 = 1, a_2 = 2$ and $a_3 = 3$. For $n \geq 2$, we classify a filling as type I if two horizontal boards are touching the right edge of the hole, and as type II if one vertical board is touching the right edge. Each filling of type I can be obtained from a filling of the $2 \times (n-2)$ hole by adding two horizontal boards. Hence the number of type I filling is a_{n-2}. Similarly, the number of type II filling is a_{n-1}, and we have $a_n = a_{n-1} + a_{n-2}$. Thus the solution of this problem is a sequence of Fibonacci numbers likewise, $a_0 = a_1 = 1$, $a_2 = a_1 + a_0 = 2$, $a_3 = a_2 + a_1 = 3$, $a_4 = a_3 + a_2 = 5$ and $a_n = a_{n-1} + a_{n-2}$ for $n = 2, 3, 4, \ldots$.

(b) Determine the number of subsets of $\{1, 2, \ldots, n\}$ which do not contain two consecutive numbers.

Example 2.2. *Determine the number of subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ of size 3 which do not contain two consecutive numbers. The size of this problem is small enough for us to work out the desirable subsets, of which there are 10. They are listed in the chart below on the left.*

135	123
136	124
137	125
146	134
147	135
157	145
246	234
247	235
257	245
357	345

We cannot do this when the size of the problem is significantly larger. We use an innovative transformation. In the chart above, the first digits of the numbers on the right are the same as the corresponding ones on the left. The second digits are 1 less, and the third digits are 2 less. The digits in the numbers on the left differ by at least 2 from column to column. Thus, the three digits are different in the numbers on the right. Since the last digit on the left is 7, the largest digit on the right is 5. It follows that the numbers on the right represent subsets of $\{1, 2, 3, 4, 5\}$*. Since this transformation is reversible, the answer to this is just the number of all subsets of size 3 of* $\{1, 2, 3, 4, 5\}$*, which is* $nC_r(5, 3) = 5!/(2!3!) = 10$*. By the given idea, we found that numbers of subsets of* $\{1, 2, 3, 4, 5, 6, 7\}$ *of different sizes are in the following chart.*

size	number
0	$nC_r(8, 0) = 1$
1	$nC_r(7, 1) = 7$
2	$nC_r(6, 2) = 15$
3	$nC_r(5, 3) = 10$
4	$nC_r(4, 4) = 1$
Total	34

In general, let S_n *be a set of all subsets of* $\{1, 2, \ldots, n\}$ *which do not contain two consecutive numbers. We have* $S_0 = \{\}$ *where* $|S_0| = 1$*,* $S_1 = \{, \{1\}\}$ *where* $|S_1| = 2$*,* $S_2 = \{, \{1\}, \{2\}\}$ *where*

$|S_2| = 3$ *and* $S_3 = \{, \{1\}, \{2\}, \{3\}, \{1, 3\}\}$ *where* $|S_3| = 5$*. For* $n \geq 2$*, we classify an element* S_n *of as type I if it is a subset of* $\{1, 2, \ldots, n\}$ *containing n, and as type II if the subset contains no n.*

Because we do not want the consecutive number, the type I subset can be obtained from a union of an element of S_{n-2} *with* $\{n\}$*, where the type II subset can be obtained directly from an element of* S_{n-1}*. For example, type I elements of* S_3 *are* $\{3\}$ *and* $\{1, 3\}$ *where they are produced from the union of elements of* S_1*(and* $\{1\}$*) with* $\{3\}$*. Type II elements of* S_3 *are* $, \{1\}$ *and* $\{2\}$*, which are elements of* S_2*. Thus number of elements of* S_n *is equal to* $|S_{n-2}| + |S_{n-1}|$*, i.e.* $|S_n| = |S_{n-2}| + |S_{n-1}|$ *for* $n = 2, 3, 4, \ldots$*. It shows that the solution of this problem has the same structure with a sequence of Fibonacci numbers.*

3 Game About Fibonacci Number

3.1 Game

Cut a 8×8 *square into four parts(as figure 1 shows), and rearrange the four parts into a new* 5×13 *rectangle as figure 2 shows.*

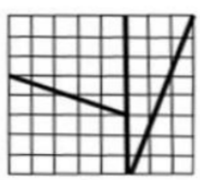


Figure 1 Original square

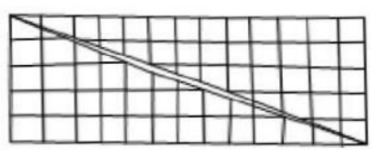


Figure 2 Rearrange into a new rectangle

When we calculate the area of figure 1, we can easily get that the area of the square equals to 64. However, the area of the new rectangle equals to 65. During this process, we have not abandoned or added any piece of paper into the new rectangle, so the area of the original square and the rectangle should equal to each other. What is the reason for the area change and what is the hidden mathematical principle behind it.

3.2 Mathematic principle behind

Observe the square carefully it can be found that the length of the sides of the square and the rectangle are all Fibonacci numbers. We have $n = 6$

$$a_n = a_6 = 8, a_{n-1} = a_5 = 5, a_{n+1} = a_7 = 13$$

In fact, Fibonacci sequence has the following property:

$$a_{n+1}a_{n-1} = a_n^2 + (-1)^n \tag{3.1}$$

In our game, the product $a_{n+1}a_{n-1}$ *actually represents the area of the rectangle* $S_{rectangle}$*, and* a_n^2 *represents the area of the square* S_{square}*.*

$$S_{rectangle} = a_7 a_5 = 5 \times 13 = 65,$$

$$S_{square} + (-1)^6 = a_6^2 + (-1)^6 = 64 + (-1)^6 = 65$$

If we observe the new rectangle carefully, we will find there is a gap on the rectangle. So the area of the original square has never changed. The area change only caused by adding extra part into the area of the rectangle. We use the mathematical induction method to prove this property:

Proof.

Step 1: *Let* $n = 1$, $a_0 = 0$, $a_1 = 1$, $a_2 = 1$. *Then equality (1) holds.*

Step 2: *Suppose it is true for* $n = k$. *Then we get the equality*

$$a_{k-1}a_{k+1} = a_k^2 + (-1) \tag{3.2}$$

Step 3: *Now we show it is true for* $n = \mathrm{k} + 1$. *According to the definition of the Fibonacci sequence, we have the following recursion formula:*

$$a_{k+1} = a_k + a_{k-1}$$

$$ak + 2 = a_{k+1} + a_k = 2a_k + a_{k-1}$$

Then we get

$$a_{k-1}\left(a_k + a_{k-1}\right) = a_k^2 + (-1)^k$$

$$a_k^2 - a_{k-1}a_k - a_{\mathrm{k}-1}^2 = (-1)^{k^{+1}}$$

$$a_{\mathrm{k}}\left(2a_k + a_{k-1}\right) = \left(a_k + a_{k-1}\right)^2 + (-1)^{k^{+1}}$$

Thus

$$a_k a_{k+2} = a_{k+1}^2 + (-1)^{+1}$$

It is true for $n = k + 1$. *Thus property (1) is proved.*

3.3 Another situation

When we reshape the four parts of the original square in another way, as Figure 3 shows, we get a new polygon. The intersection in the new polygon changes its area to 63. We can also use another property of the Fibonacci sequence to explain why the change happens.

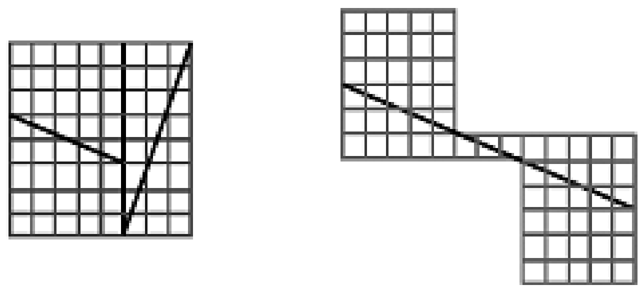


Figure 3 Rearrange four pars into a new polygon

The Fibonacci sequence also has the following property

Proof. From (1) we have

$$a_n a_{n+2} = a_{n+1}^2 + (-1)^{+1} 4a_{n-1}a_{n-2} + a_{n-2} \qquad a_{n-4} = 4a_{n-1}a_{n-2} + a_{n-2}\left(2a_{n-2} - a_{n-1}\right)$$

$$= a_{n-2}\left(3a_{n-1} + 2a_{n-2}\right)$$

$$= a_{n-2}\left(a_n + a_{n-2} + 2a_{n-1}\right)$$

$$= a_{n-1}^2 + (-1)^{-1} + a_{n-2}^2 + 2a_{n-2}a_{n-1}$$

$$= a_n^2 + (-1)^{-1}$$

Thus the property is proved.

When we calculate the area of the new polygon, we do not involve the intersection part. So the area of the new polygon we get is smaller than the original square.

3.4 Perfect situation

After we have researched the two situations above, it is reasonable to ask how we can cut the original square to get a perfect rectangle. Suppose the length of the sides are x and y as figure 4 shows, and we can get a new rectangle without any gap or intersection.

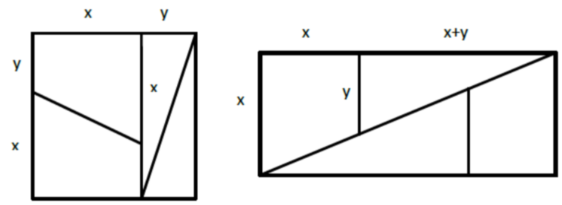


Figure 4 The situation we can get a perfect rectangle

Suppose the area of the original square is S_{square} *and the area of rectangle is* $S_{rectangle}$*. Then we have*

$$\begin{aligned} S_{Square} &= (x+y)^2 \\ S_{rectangle} &= (2x+y) \end{aligned}$$

Let $S_{Square} = S_{rectangle}$*. Then we have*

$$(x+y)^2 = (2x+y)x$$

$$i.e.\ (x/y)^2 - (x/y) - 1 = 0$$

Thus we get the solution $\frac{x}{y} = \frac{(1\pm\sqrt{5})}{2}$*. Because* x *and* y *are the lengths of the sides, so we only take the positive ones. It is easy to realise that* $\frac{y}{x} = \frac{2}{(1+\sqrt{5})} \approx 0.618$*.*

In fact, if y *equals to* a_{n-2} *and* x *equals to* a_{n-1}*, the proportion of* x *for* y *represents the proportion of* a_{n-1} *for* a_{n-2}*. When* n *tends to infinity, the ratio tends to be 0.618 (golden ratio). So the perfect situation above happened when* x *equals to* a_{n-1}*,* y *equals to* a_{n-2} *and* n *tends to be infinity.*

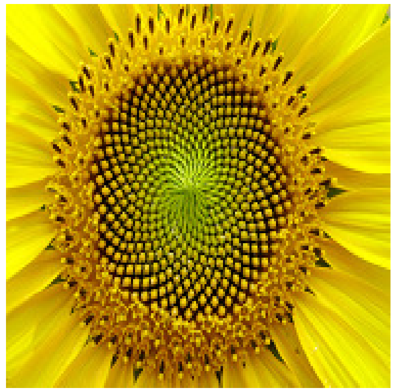
4 Fibonacci Sequence in Nature

One of the most spectacular examples of the Fibonacci sequence in nature is in the head of the sunflower. Scientists have measured the number of spirals in the sunflower head. They found one set of short spirals going clockwise from the centre and another set of longer spirals going anticlockwise; these two beautiful sinuous spirals of the sunflower head reveal the astonishing double connection with the Fibonacci series.

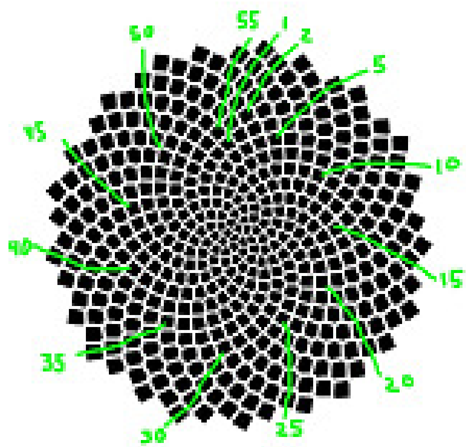
- *The pairs are always adjacent numbers in the Fibonacci series. One pair could be 21 and 34, and the next pair could be 34 and 55.*

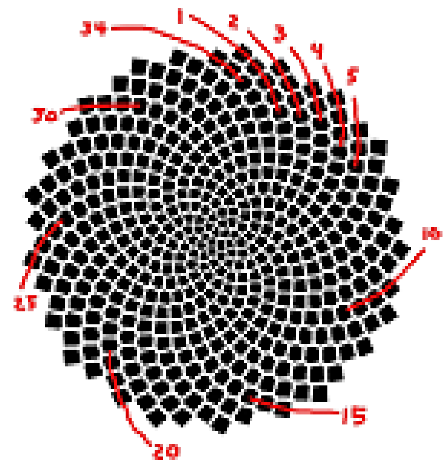
- *The adjacent numbers divided yield the golden proportion* $55/34 = 1.618$*.*

The sunflower seed pattern used by the Museum of Mathematics contains many spirals.

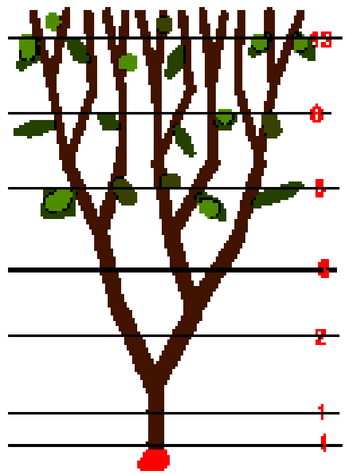


Below are the two natural ways to find the spirals in this pattern.





Note that the black pattern is identical in both images. The clockwise direction shows 55 spirals of seeds, and the anticlockwise direction shows 34 spirals. Sunflowers have a golden spiral seed arrangement. This provides a biological advantage because it maximises the number of seeds packed into a seed head. Pinecones have a Fibonacci number of spirals and petals at every set of "winding". A pineapple has more oversized shapes on the bottom of it and gets smaller as you go up. If you take one of the shapes on the bottom and go diagonally upward (to your right or left), you see these shapes are the same, only scaled to a smaller size. If we say the bottom is a size 13, then we will notice that the shape directly to the upper left is size 8. The size you need to start with on the pineapple depends on the size of the pineapple and how many shapes are in a sequence. This sequence is another example of the Fibonacci sequence. Many flowers have a Fibonacci number of petals. One possible reason for why this phenomenon happens is that they try to decrease the overlapped area to get more sunlight. The number of leaves on a plant in *each turn as you goes from bottom to top follow the Fibonacci sequence. Inside the fruit of many plants we can observe the presence of Fibonacci order. The banana has 3 sections, the apple has 5 sections.*



The number of branches follows the Fibonacci numbers: Botanists find that the number of branches of trees is always a Fibonacci number. They find that after a certain period, each old tree branch will get a new one and need one more period to turn to an old one. If the tree has only one branch at the very beginning, it will have two branches after a year, and in the next circle, it will have three branches. Every year, the total branches of the tree compose a Fibonacci sequence. One possible reason is that every new branch needs one year to mature.

5 Conclusion

Plants are believed to utilise Fibonacci numbers and the golden ratio in their growth patterns to optimise their use of space. This mathematical approach enhances their efficiency in packing leaves and flowers, allows them to capture sunlight more effectively, and manage resources better as they develop. Following these principles, plants can achieve an ideal arrangement that maximises their exposure to light and nutrients, ultimately supporting their overall growth and survival.

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