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Applications and Common Coupled Fixed Point Results in Ordered Partial Metric Spaces

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Abstract

In this paper, we obtain a unique common coupled fixed point theorem by using $(\psi,\alpha,\beta)-$contraction in ordered partial metric spaces. We give an application to integral equations as well as homotopy theory. Also we furnish an example which supports our theorem.

Keywords: coupled fixed point; ordered partial metric space; $(\psi,\alpha,\beta)-$contraction.

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1. Introduction and Preliminaries

The aim of this paper is to study unique common coupled fixed point theorems of Jungck type maps by using a $(\psi,\alpha,\beta)-$contraction condition over partially ordered PMSs. First we recall some basic definitions and lemmas which play a crucial role in the theory of PMSs.

Definition 1.1. *A partial metric on a non-empty set X is a function $p : X \times X \to R^+$ such that, for all $x, y, z \in X$,*

$(p_1):\ x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y),$

$(p_2):\ p(x,x) \leq p(x,y), p(y,y) \leq p(x,y),$

$(p_3):\ p(x,y) = p(y,x),$

$(p_4):\ p(x,y) \leq p(x,z) + p(z,y) - p(z,z).$

The pair (X,p) is called a PMS. If p is a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}^+$, given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y), \tag{1}$$

is a metric on X.

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Example 1.2. *Consider* $X = [0, \infty) with p(x, y) = \max\{x, y\}$. *Then* (X, p) *is a PMS. It is clear that* p *is not a (usual) metric. Note that in this case* $d_p(x, y) = |x - y|$.

Example 1.3. *Let* $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ *and define* $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. *Then* (X, p) *is a PMS. Each partial metric* p *on* X *generates a* T_0 *topology* τ_p *on* X *which has as a base the family of open p-balls* $\{B_p(x, \varepsilon), x \in X, \varepsilon > 0\}$, *where* $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ *for all* $x \in X$ *and* $\varepsilon > 0$.

We now state some basic topological notions (such as convergence, completeness, continuity) on PMSs.

Definition 1.4.

(1) *A sequence* $\{x_n\}$ *in the PMS* (X, p) *converges to the limit* x *if and only if* $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

(2) *A sequence* $\{x_n\}$ *in the PMS* (X, p) *is called a Cauchy sequence if* $\lim_{n,m\to\infty} p(x_n, x_m)$ *exists and is finite.*

(3) *A PMS* (X, p) *is called complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges with respect to* τ_p, *to a point* $x \in X$ *such that* $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

(4) *A mapping* $F : X \to X$ *is said to be continuous at* $x_0 \in X$ *if, for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $F(B_p(x_0, \delta)) \subseteq B_p(Fx_0, \epsilon)$.

The following lemma is one of the basic results as regards PMS.

Lemma 1.5.

(1) *A sequence* $\{x_n\}$ *is a Cauchy sequence in the PMS* (X, p) *if and only if it is a Cauchy sequence in the metric space* (X, d_p).

(2) *A PMS* (X, p) *is complete if and only if the metric space* (X, d_p) *is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x, x_n) = 0 \quad \Leftrightarrow \quad p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2}$$

Next, we give two simple lemmas which will be used in the proofs of our main results. For the proofs we refer [13].

Lemma 1.6. *Assume* $x_n \to z as n \to \infty$ *in a PMS* (X, p) *such that* $p(z, z) = 0$. *Then* $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ *for every* $y \in X$.

Lemma 1.7. *Let* (X, p) *be a PMS. Then*

(A) *if* $p(x, y) = 0$, *then* $x = y$,

(B) *if* $x \neq y$, *then* $p(x, y) > 0$.

Remark 1.8. *If* $x = y$, $p(x, y)$ *may not be* 0.

Definition 1.9. *Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. Then the map F is said to have mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y; that is, for any $x, y \in X$,*

$$x_1 \preceq x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y) \text{ for all } y \in X \text{ and}$$
$$y_1 \preceq y_2 \text{ implies } F(x, y_2) \preceq F(x, y_1) \text{ for all } x \in X.$$

Definition 1.10. *An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Definition 1.11. *An element $(x, y) \in X \times X$ is called*

(g_1) : *a coupled coincident point of mappings $F : X \times X \to X$ and $f : X \to X$ if $fx = F(x, y)$ and $fy = F(y, x)$,*

(g_2) : *a common coupled fixed point of mappings $F : X \times X \to X$ and $f : X \to X$ if $x = fx = F(x, y)$ and $y = fy = F(y, x)$.*

Definition 1.12. *The mappings $F : X \times X \to X$ and $f : X \to X$ are called $w-$ compatible if $f(F(x, y)) = F(fx, fy)$ and $f(F(y, x)) = F(fy, fx)$ whenever $fx = F(x, y)$ and $fy = F(y, x)$.*

Inspired by Definition 2.9, Lakshmikantham and Ćirić in [31] introduced the concept of a g-mixed monotone mapping.

Definition 1.13. *Let $(X, \preceq)$ be a partially ordered set, $F : X \times X \to X$ and $g : X \to X$ be mappings. Then the map F is said to have a mixed $g-$monotone property if $F(x, y)$ is monotone g-non-decreasing in x as well as monotone $g-$non-increasing in y; that is, for any $x, y \in X$,*

$$gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y) \text{ for all } y \in X \text{ and}$$
$$gy_1 \preceq gy_2 \text{ implies } F(x, y_2) \preceq F(x, y_1) \text{ for all } x \in X.$$

Now we prove our main results.

2. Results and Discussions

Definition 2.1. *Let (X, p) be a PMS, let $F : X \times X \to X$ and $g : X \to X$ be mappings. We say that F satisfies a $(\psi, \alpha, \beta)-$contraction with respect to g if there exist $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ satisfying the following:*

(2.1.1): *ψ is continuous and monotonically non-decreasing, α is continuous and β is lower semi continuous,*

(2.1.2): *$\psi(t) = 0$ if and only if $t = 0, \alpha(0) = \beta(0) = 0$,*

(2.1.3): *$\psi(t) - \alpha(t) + \beta(t) > 0$ for $t > 0$,*

(2.1.4): $\psi(p(F(x,y),F(u,v))) \le \alpha(M(x,y,u,v)) - \beta(M(x,y,u,v)), \forall x,y,u,v \in X, gx \preceq gu, gy \succeq gv$ *and*

$$M(x,y,u,v) = \max\left\{\begin{array}{c} p(gx,gu),p(gy,gv),p(gx,F(x,y)),p(gy,F(y,x)),p(gu,F(u,v)),p(gv,F(v,u)), \\ \frac{p(gx,F(x,y))p(gy,F(y,x))}{1+p(gx,gu)+p(gy,gv)+p(F(x,y),F(u,v))}, \frac{p(gu,F(u,v))p(gv,F(v,u))}{1+p(gx,gu)+p(gy,gv)+p(F(x,y),F(u,v))} \end{array}\right\}.$$

Theorem 2.2. *Let $(X,\preceq)$ be a partially ordered set and p be a partial metric such that (X,p) is a PMS. Let $F: X\times X \to X$ and $g: X\to X$ be such that*

(2.2.1): *F satisfies a $(\psi,\alpha,\beta)-$contraction with respect to g,*

(2.2.2): *$F(X\times X)\subseteq g(X)$ and $g(X)$ is a complete subspace of X,*

(2.2.3): *F has a mixed g-monotone property,*

(2.2.4): *(a) if a non-decreasing sequence $\{x_n\}\to x$, then $x_n \preceq x$ for all n,*

(b) if a non-increasing sequence $\{y_n\}\to y$, then $y\preceq y_n$ for all n.

If there exist $x_0,y_0\in X$ such that $gx_0 \preceq F(x_0,y_0)$ and $gy_0\succeq F(y_0,x_0)$, then F and g have a coupled coincidence point in $X\times X$.

Proof. Let $x_0,y_0\in X$ be such that $gx_0\preceq F(x_0,y_0)$ and $gy_0\succeq F(y_0,x_0)$. Since $F(X\times X)\subseteq g(X)$, we choose $x_1,y_1\in X$ such that $gx_0\preceq F(x_0,y_0)=gx_1$ and $gy_0\succeq F(y_0,x_0)=gy_1$ and choose $x_2,y_2\in X$ such that $gx_2=F(x_1,y_1)$ and $gy_2=F(y_1,x_1)$. Since F has the mixed $g-$ monotone property, we obtain $gx_0\preceq gx_1\preceq gx_2$ and $gy_0\succeq gy_1\succeq gy_2$. Continuing this process, we construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_{n+1}=F(x_n,y_n)$ and $gy_{n+1}=F(y_n,x_n)$, $n=0,1,2,\ldots$ with

$$\left.\begin{array}{l} gx_0\preceq gx_1\preceq gx_2\preceq\cdots \quad \text{and} \\ gy_0\succeq gy_1\succeq gy_2\succeq\cdots. \end{array}\right\} \tag{3}$$

Case (a): $If gx_m = gx_{m+1}$ and $gy_m = gy_{m+1}$ for some m, then (x_m,y_m) is a coupled coincidence point in $X\times X$.
Case (b): Assume $gx_n\neq gx_{n+1}$ or $gy_n\neq gy_{n+1}$ for all n. Since $gx_n\preceq gx_{n+1} and gy_n\succeq gy_{n+1}$, from (2.2.1), we obtain

$$\begin{aligned}\psi(p(gx_n,gx_{n+1})) &= \psi(p(F(x_{n-1},y_{n-1}),F(x_n,y_n))) \\ &\le \alpha(M(x_{n-1},y_{n-1},x_n,y_n)) - \beta(M(x_{n-1},y_{n-1},x_n,y_n)),\end{aligned}$$

$$M(x_{n-1},y_{n-1},x_n,y_n) = \max\left\{\begin{array}{c} p(gx_{n-1},gx_n),p(gy_{n-1},gy_n),p(gx_{n-1},gx_n), \\ p(gy_{n-1},gy_n),p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1}), \\ \frac{p(gx_{n-1},gx_n)p(gy_{n-1},gy_n)}{1+p(gx_{n-1},gx_n)+p(gy_{n-1},gy_n)+p(gx_n,gx_{n+1})}, \\ \frac{p(gx_n,gx_{n+1})p(gy_n,gy_{n+1})}{1+p(gx_{n-1},gx_n)+p(gy_{n-1},gy_n)+p(gx_n,gx_{n+1})} \end{array}\right\}.$$

But

$$\frac{p(gx_{n-1},gx_n)p(gy_{n-1},gy_n)}{1+p(gx_{n-1},gx_n)+p(gy_{n-1},gy_n)+p(gx_n,gx_{n+1})} \leq \max\{p(gx_{n-1},gx_n),p(gx_n,gx_{n+1})\}$$

and

$$\frac{p(gx_n,gx_{n+1})p(gy_n,gy_{n+1})}{1+p(gx_{n-1},gx_n)+p(gy_{n-1},gy_n)+p(gx_n,gx_{n+1})} \leq p(gy_n,gy_{n+1}).$$

Therefore

$$M(x_{n-1},y_{n-1},x_n,y_n) = \max\left\{\begin{array}{c} p(gx_{n-1},gx_n),p(gy_{n-1},gy_n), \\ p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1}) \end{array}\right\}.$$

Hence

$$\begin{aligned}\psi(p(gx_n,gx_{n+1})) \leq \alpha &\left(\max\left\{\begin{array}{c} p(gx_{n-1},gx_n),p(gy_{n-1},gy_n), \\ p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1}) \end{array}\right\}\right) \\ -\beta &\left(\max\left\{\begin{array}{c} p(gx_{n-1},gx_n),p(gy_{n-1},gy_n), \\ p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1}) \end{array}\right\}\right).\end{aligned}$$

Similarly

$$\begin{aligned}\psi(p(gy_n,gy_{n+1})) \leq \alpha &\left(\max\left\{\begin{array}{c} p(gx_{n-1},gx_n),p(gy_{n-1},gy_n), \\ p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1}) \end{array}\right\}\right) \\ -\beta &\left(\max\left\{\begin{array}{c} p(gx_{n-1},gx_n),p(gy_{n-1},gy_n), \\ p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1}) \end{array}\right\}\right).\end{aligned}$$

Put $R_n = \max\{p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1})\}$. Let us suppose that

$$R_n \neq 0 \quad \text{for all } n \geq 1. \tag{4}$$

Let, if possible, for some n, $R_{n-1} < R_n$. Now

$$\begin{aligned}\psi(R_n) &= \psi(\max\{p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1})\}) \\ &= \max\{\psi(p(gx_n,gx_{n+1})),\psi(p(gy_n,gy_{n+1}))\} \\ &\leq \alpha\left(\max\left\{\begin{array}{c} p(gx_{n-1},gx_n),p(gy_{n-1},gy_n), \\ p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1}) \end{array}\right\}\right) - \beta\left(\max\left\{\begin{array}{c} p(gx_{n-1},gx_n),p(gy_{n-1},gy_n), \\ p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1}) \end{array}\right\}\right) \\ &= \alpha(\max\{R_{n-1},R_n\}) - \beta(\max\{R_{n-1},R_n\}) \\ &= \alpha(R_n) - \beta(R_n).\end{aligned}$$

From (2.1.2) and (2.1.3), it follows that $R_n = 0$, a contradiction. Hence

$$R_n \leq R_{n-1}. \tag{5}$$

Thus $\{R_n\}$ is a non-increasing sequence of non-negative real numbers and must converge to a real number $r \geq 0$. Also $\psi(R_n) \leq \alpha(R_{n-1}) - \beta(R_{n-1})$. Letting $n \to \infty$, we get $\psi(r) \leq \alpha(r) - \beta(r)$. From (3.1.2) and (3.1.3), we get $r = 0$. Thus

$$\begin{aligned} &\lim_{n\to\infty} \max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\} = 0, \\ &\lim_{n\to\infty} p(gx_n, gx_{n+1}) = 0 = \lim_{n\to\infty} p(gy_n, gy_{n+1}). \end{aligned} \tag{6}$$

Hence from (p_2), we have

$$\lim_{n\to\infty} p(gx_n, gx_n) = 0 = \lim_{n\to\infty} p(gy_n, gy_n). \tag{7}$$

From 6 and 7 and by the definition of d_p, we get

$$\lim_{n\to\infty} d_p(gx_n, gx_{n+1}) = 0 = \lim_{n\to\infty} d_p(gy_n, gy_{n+1}). \tag{8}$$

Now we prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. To the contrary, suppose that $\{gx_n\}$ or $\{gy_n\}$ is not Cauchy. This implies that $d_p(gx_m, gx_n) \not\to 0$ or $d_p(gy_m, gy_n) \not\to 0$ as $n, m \to \infty$. Consequently $\max\{d_p(gx_m, gx_n), d_p(gy_m, gy_n)\} \not\to 0$ as $n, m \to \infty$. Then there exist an $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k > k$. We have

$$\max\{d_p(gx_{m_k}, gx_{n_k}), d_p(gy_{m_k}, gy_{n_k})\} \geq \epsilon \text{ and} \tag{9}$$

$$\max\{d_p(gx_{m_k}, gx_{n_k-1}), d_p(gy_m, gy_{n_k-1})\} < \epsilon. \tag{10}$$

From (9) and (10), we have

$$\begin{aligned} \epsilon &\leq \max\{d_p(gx_{m_k}, gx_{n_k}), d_p(gy_{m_k}, gy_{n_k})\} \\ &\leq \max\{d_p(gx_{m_k}, gx_{n_k-1}), d_p(gy_{m_k}, gy_{n_k-1})\} + \max\{d_p(gx_{n_k-1}, gx_{n_k}), d_p(gy_{n_k-1}, gy_{n_k})\} \\ &< \epsilon + \max\{d_p(gx_{n_k-1}, gx_{n_k}), d_p(gy_{n_k-1}, gy_{n_k})\}. \end{aligned}$$

Letting $k \to \infty$ and using 8, we get

$$\lim_{k\to\infty} \max\{d_p(gx_{m_k}, gx_{n_k}), d_p(gy_{m_k}, gy_{n_k})\} = \epsilon. \tag{11}$$

By the definition of d_p and using 7 we get

$$\lim_{k\to\infty} \max\{p(gx_{m_k}, gx_{n_k}), p(gy_{m_k}, gy_{n_k})\} = \frac{\epsilon}{2}. \tag{12}$$

From (9), we have

$$\begin{aligned}\epsilon &\leq \max\{d_p(gx_{m_k}, gx_{n_k}), d_p(gy_{m_k}, gy_{n_k})\} \\ &\leq \max\{d_p(gx_{m_k}, gx_{m_k-1}), d_p(gy_{m_k}, gy_{m_k-1})\} + \max\{d_p(gx_{m_k-1}, gx_{n_k}), d_p(gy_{m_k-1}, gy_{n_k})\} \\ &\leq 2\max\{d_p(gx_{m_k}, gx_{m_k-1}), d_p(gy_{m_k}, gy_{m_k-1})\} + \max\{d_p(gx_{m_k}, gx_{n_k}), d_p(gy_{m_k}, gy_{n_k})\}.\end{aligned} \tag{13}$$

Letting $k \to \infty$, using (8), (11) and (13), we get

$$\lim_{k\to\infty} \max\{d_p(gx_{m_k-1}, gx_{n_k}), d_p(gy_{m_k-1}, gy_{n_k})\} = \epsilon. \tag{14}$$

Hence, we get

$$\lim_{k\to\infty} \max\{p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k})\} = \frac{\epsilon}{2}. \tag{15}$$

From (10), we have

$$\begin{aligned}\epsilon &\leq \max\{d_p(gx_{m_k}, gx_{n_k}), d_p(gy_{m_k}, gy_{n_k})\} \\ &\leq \max\{d_p(gx_{m_k}, gx_{m_k-1}), d_p(gy_{m_k}, gy_{m_k-1})\} + \max\{d_p(gx_{m_k-1}, gx_{n_k+1}), d_p(gy_{m_k-1}, gy_{n_k+1})\} \\ &+ \max\{d_p(gx_{n_k+1}, gx_{n_k}), d_p(gy_{n_k+1}, gy_{n_k})\} \\ &\leq 2\max\{d_p(gx_{m_k}, gx_{m_k-1}), d_p(gy_{m_k}, gy_{m_k-1})\} + \max\{d_p(gx_{m_k}, gx_{n_k}), d_p(gy_{m_k}, gy_{n_k})\} \\ &+ 2\max\{d_p(gx_{n_k}, gx_{n_k+1}), d_p(gy_{n_k}, gy_{n_k+1})\}.\end{aligned} \tag{16}$$

Letting $k \to \infty$, using (8), (11) and (16), we get

$$\lim_{k\to\infty} \max\{d_p(gx_{m_k-1}, gx_{n_k+1}), d_p(gy_{m_k-1}, gy_{n_k+1})\} = \epsilon. \tag{17}$$

Hence, we have

$$\lim_{k\to\infty} \max\{p(gx_{m_k-1}, gx_{n_k+1}), p(gy_{m_k-1}, gy_{n_k+1})\} = \frac{\epsilon}{2}. \tag{18}$$

Now from (9), we have

$$\begin{aligned}\epsilon &\leq \max\{d_p(gx_{m_k}, gx_{n_k}), d_p(gy_{m_k}, gy_{n_k})\} \\ &\leq \max\{d_p(gx_{m_k}, gx_{n_k+1}), d_p(gy_{m_k}, gy_{n_k+1})\} + \max\{d_p(gx_{n_k+1}, gx_{n_k}), d_p(gy_{n_k+1}, gy_{2n_k})\}.\end{aligned}$$

Letting $k \to \infty$ and using (8), we obtain

$$\begin{aligned}\epsilon &\le \lim_{k\to\infty} \max\{d_p(gx_{m_k}, gx_{n_k+1}), d_p(gy_{m_k}, gy_{n_k+1})\} + 0 \\ &\le \lim_{k\to\infty} \max\left\{\begin{array}{c} 2p(gx_{m_k}, gx_{n_k+1}) - p(gx_{m_k}, gx_{m_k}) - p(gx_{n_k+1}, gx_{n_k+1}), \\ 2p(gy_{m_k}, gy_{n_k+1}) - p(gy_{m_k}, gy_{m_k}) - p(gy_{n_k+1}, gy_{n_k+1}) \end{array}\right\} \\ &= 2\lim_{k\to\infty} \max\{p(gx_{m_k}, gx_{n_k+1}), p(gy_{m_k}, gy_{n_k+1})\}, \quad \text{from (6)}.\end{aligned}$$

Thus,

$$\frac{\epsilon}{2} \le \lim_{k\to\infty} \max\{p(gx_{m_k}, gx_{n_k+1}), p(gy_{m_k}, gy_{n_k+1})\}.$$

By the properties of ψ,

$$\begin{aligned}\psi\left(\frac{\epsilon}{2}\right) &\le \lim_{k\to\infty} \psi\big(\max\{p(gx_{m_k}, gx_{n_k+1}), p(gy_{m_k}, gy_{n_k+1})\}\big) \\ &= \lim_{k\to\infty} \max\{\psi(p(gx_{m_k}, gx_{n_k+1})), \psi(p(gy_{m_k}, gy_{n_k+1}))\}.\end{aligned} \tag{19}$$

Now

$$\begin{aligned}\psi(p(gx_{m_k}, gx_{n_k+1})) &= \psi(p(F(x_{m_k-1}, y_{m_k-1}), F(x_{n_k}, y_{n_k}))) \\ &\le \alpha(M(x_{m_k-1}, y_{m_k-1}, x_{n_k}, y_{n_k})) - \beta(M(x_{m_k-1}, y_{m_k-1}, x_{n_k}, y_{n_k})) \\ &= \alpha\left(\max\left\{\begin{array}{c} p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k}), p(gx_{m_k-1}, gx_{m_k}), \\ p(gy_{m_k-1}, gy_{m_k}), p(gx_{n_k}, gx_{n_k+1}), p(gy_{n_k}, gy_{n_k+1}), \\ \frac{p(gx_{m_k-1}, gx_{m_k})p(gy_{m_k-1}, gy_{m_k})}{1+p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k})+p(gx_{m_k}, gx_{n_k+1})}, \\ \frac{p(gx_{n_k}, gx_{n_k+1})p(gy_{n_k}, gy_{n_k+1})}{1+p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k})+p(gx_{m_k}, gx_{n_k+1})} \end{array}\right\}\right) \\ &\quad - \beta\left(\max\left\{\begin{array}{c} p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k}), p(gx_{m_k-1}, gx_{m_k}), \\ p(gy_{m_k-1}, gy_{m_k}), p(gx_{n_k}, gx_{n_k+1}), p(gy_{n_k}, gy_{n_k+1}), \\ \frac{p(gx_{m_k-1}, gx_{m_k})p(gy_{m_k-1}, gy_{m_k})}{1+p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k})+p(gx_{m_k}, gx_{n_k+1})}, \\ \frac{p(gx_{n_k}, gx_{n_k+1})p(gy_{n_k}, gy_{n_k+1})}{1+p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k})+p(gx_{m_k}, gx_{n_k+1})} \end{array}\right\}\right).\end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} \psi(p(gx_{m_k}, gx_{n_k+1})) \le \alpha\left(\frac{\epsilon}{2}\right) - \beta\left(\frac{\epsilon}{2}\right).$$

Similarly, we obtain

$$\lim_{k\to\infty} \psi(p(gy_{m_k}, gy_{n_k+1})) \le \alpha\left(\frac{\epsilon}{2}\right) - \beta\left(\frac{\epsilon}{2}\right).$$

Hence from 19, we have

$$\psi\left(\frac{\epsilon}{2}\right) \le \alpha\left(\frac{\epsilon}{2}\right) - \beta\left(\frac{\epsilon}{2}\right).$$

From (3.1.2) and (3.1.3), we get $\frac{\epsilon}{2} = 0$, a contradiction. Hence $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in the metric space (X, d_p). Hence we have $\lim_{n,m\to\infty} d_p(gx_n, gx_m) = 0 = \lim_{n,m\to\infty} d_p(gy_n, gy_m)$. Now from the definition of d_p and from 7, we have

$$\lim_{n\to\infty} p(gx_n, gx_m) = 0 = \lim_{n\to\infty} p(gy_n, gy_m). \tag{20}$$

Suppose $g(X)$ is a complete subspace of X. Since $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in a complete metric space $(g(X), d_p)$. Then $\{gx_n\}$ and $\{gy_n\}$ converges to some u and v in $g(X)$ respectively. Thus

$$\lim_{n\to\infty} d_p(gx_n, u) = 0$$
$$\lim_{n\to\infty} d_p(gy_n, v) = 0$$

for some u and v in $g(X)$. Since $u, v \in g(X)$, there exist $x, y \in X$ such that $u = gx$ and $v = gy$. Since $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences, $gx_n \to u$, $gy_n \to v$, $gx_{n+1} \to u$ and $gy_{n+1} \to v$. From Lemma 1.5, (2) and (20), we obtain

$$p(u, u) = \lim_{n\to\infty} p(gx_n, u) = p(v, v) = \lim_{n\to\infty} p(gy_n, v) = 0. \tag{21}$$

Now we prove that $\lim_{n\to\infty} p(F(x,y), gx_n) = p(F(x,y), u)$. By definition of d_p,

$$d_p\big(F(x,y), gx_n\big) = 2p\big(F(x,y), gx_n\big) - p\big(F(x,y), F(x,y)\big) - p(gx_n, gx_n).$$

Letting $n \to \infty$, we have

$$d_p\big(F(x,y), u\big) = 2\lim_{n\to\infty} p\big(F(x,y), gx_n\big) - p\big(F(x,y), F(x,y)\big) - 0, \text{ from 7 }.$$

By Definition of d_p and 20, we have

$$\lim_{n\to\infty} p\big(F(x,y), gx_n\big) = p\big(F(x,y), u\big).$$

Similarly,

$$\lim_{n\to\infty} p(F(y,x), gy_n) = p(F(y,x), v).$$

From (p_4), we have

$$\begin{aligned} p\big(u, F(x,y)\big) &\le p(u, gx_{n+1}) + p\big(gx_{n+1}, F(x,y)\big) - p(gx_{n+1}, gx_{n+1}) \\ &= p(u, gx_{n+1}) + p\big(gx_{n+1}, F(x,y)\big). \end{aligned}$$

Letting $n \to \infty$, we have

$$p\big(u, F(x,y)\big) \le 0 + \lim_{n\to\infty} p\big(F(x_n, y_n), F(x,y)\big).$$

Also from (2.2.4), we get $gx_n \preceq gx$ and $gy_n \succeq gy$. Since ψ is a continuous and non-decreasing function, we get

$$\begin{aligned}
\psi(p(u,F(x,y)) &\le \lim_{n\to\infty} \psi(p(F(x_n,y_n),F(x,y))) \\
&\le \lim_{n\to\infty} [\alpha(M(x_n,y_n,x,y)) - \beta(M(x_n,y_n,x,y))], \\
M(x_n,y_n,x,y) &= \max \left\{ \begin{array}{c} p(gx_n,u), p(gy_n,v), p(gx_n,gx_{n+1}), \\ p(gy_n,gy_{n+1}), p(u,F(x,y)), p(v,F(y,x)), \\ \frac{p(gx_n,gx_{n+1})p(gy_n,gy_{n+1})}{1+p(gx_n,u)+p(gy_n,v)+p(gx_{n+1},F(x,y))}, \\ \frac{p(u,F(x,y))p(v,F(y,x))}{1+p(gx_n,u)+p(gy_n,v)+p(gx_{n+1},F(x,y))} \end{array} \right\} \\
&\to \max\{p(u,F(x,y)), p(v,F(y,x))\} \quad \text{as } n\to\infty.
\end{aligned}$$

Therefore

$$\psi(p(u,F(x,y))) \le \alpha\left(\max\left\{ \begin{array}{c} p(u,F(x,y)), \\ p(v,F(y,x)) \end{array} \right\}\right) - \beta\left(\max\left\{ \begin{array}{c} p(u,F(x,y)), \\ p(v,F(y,x)) \end{array} \right\}\right).$$

Similarly,

$$\psi(p(v,F(y,x))) \le \alpha\left(\max\left\{ \begin{array}{c} p(u,F(x,y)), \\ p(v,F(y,x)) \end{array} \right\}\right) - \beta\left(\max\left\{ \begin{array}{c} p(u,F(x,y)), \\ p(v,F(y,x)) \end{array} \right\}\right).$$

Hence

$$\begin{aligned}
&\psi(\max\{p(u,F(x,y)), p(v,F(y,x))\}) \\
&= \max\{\psi(p(u,F(x,y))), \psi(p(v,F(y,x)))\} \\
&\le \alpha\left(\max\left\{ \begin{array}{c} p(u,F(x,y)), \\ p(v,F(y,x)) \end{array} \right\}\right) - \beta\left(\max\left\{ \begin{array}{c} p(u,F(x,y)), \\ p(v,F(y,x)) \end{array} \right\}\right).
\end{aligned}$$

It follows that $\max\{p(u,F(x,y)), p(v,F(y,x))\} = 0$. So $F(x,y) = u$ and $F(y,x) = v$. Hence $F(x,y) = gx = u$ and $F(y,x) = gy = v$. Hence F and g have a coincidence point in $X \times X$. □

Theorem 2.3. *In addition to the hypothesis of Theorem 2.2, we suppose that for every $(x,y),(x^1,y^1) \in X \times X$ there exists $(u,v) \in X \times X$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x^1,y^1),F(y^1,x^1))$. If (x,y) and (x^1,y^1) are coupled coincidence points of F and g, then*

$$F(x,y) = gx = gx^1 = F(x^1,y^1) \quad \text{and}$$
$$T(y,x) = gy = gy^1 = F(y^1,x^1).$$

Moreover, if (F,g) is w -compatible, then F and g have a unique common coupled fixed point in $X \times X$.

Proof. The proof follows from Theorem 2.2 and the definition of comparability. □

Theorem 2.4. *Let $(X,\preceq)$ be a partially ordered set and p be a partial metric such that (X,p) is a complete PMS. Let $F : X \times X \to X$ be such that*

(3.4.1): $\psi\big(p\big(F(x,y),F(u,v)\big)\big) \le \alpha\big(\max\{p(x,u),p(y,v)\}\big) - \beta\big(\max\{p(x,u),p(y,v)\}\big)$, $\forall\ x,y,u,v \in X$, $x \preceq u$ *and* $y \succeq v$, *where* ψ,α *and* β *are defined in Definition 2.1 and*

(3.4.2): *(a) If a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \preceq x$ *for all* n, *and*

(b) If a non-increasing sequence $\{y_n\} \to y$, *then* $y \preceq y_n$ *for all* n.

If there exist $x_0, y_0 \in X$ *such that* $x_0 \preceq F(x_0,y_0)$ *and* $y_0 \succeq F(y_0,x_0)$, *then* F *has a unique coupled fixed point in* $X \times X$.

Example 2.5. *Let* $X = [0,1]$, *let* $\preceq$ *be partially ordered on* X *by* $x \preceq y \quad \Leftrightarrow \quad x \ge y$. *The mapping* $F : X \times X \to X$ *defined by* $F(x,y) = \frac{x^2+y^2}{8(x+y+1)}$ *and* $p : X \times X \to [0,\infty)$ *by* $p(x,y) = \max\{x,y\}$ *is a complete partial metric on* X. *Define* $\psi,\alpha,\beta : [0,\infty) \to [0,\infty)$ *by* $\psi(t) = t$, $\alpha(t) = \frac{t}{2}$ *and* $\beta(t) = \frac{t}{4}$. *We have*

$$\begin{aligned}
p\big(F(x,y),F(u,v)\big) =& \max\left\{\frac{x^2+y^2}{8(x+y+1)}, \frac{u^2+v^2}{8(u+v+1)}\right\} \\
=& \frac{1}{4}\left[\max\left\{\frac{x^2}{x+y+1}, \frac{u^2}{u+v+1}\right\} + \max\left\{\frac{y^2}{x+y+1}, \frac{v^2}{u+v+1}\right\}\right] \\
\le& \frac{1}{8}\left[\max\left\{\frac{x^2}{x+1}, \frac{u^2}{u+1}\right\} + \max\left\{\frac{y^2}{y+1}, \frac{v^2}{v+1}\right\}\right] \\
\le& \frac{1}{8}\left[\max\left\{\frac{x}{x+1}, \frac{u}{u+1}\right\} + \max\left\{\frac{y}{y+1}, \frac{v}{v+1}\right\}\right] \\
\le& \frac{1}{8}\left[\max\{x,u\} + \max\{y,v\}\right] \\
=& \frac{1}{8}\left[p(x,u) + p(y,v)\right] \\
\le& \frac{1}{4}\max\{p(x,u),p(y,v)\} \\
=& \alpha\big(\max\{p(x,u),p(y,v)\}\big) - \beta\big(\max\{p(x,u),p(y,v)\}\big).
\end{aligned}$$

Hence all conditions of Theorem 2.2 hold. From Theorem 2.4, $(0,0)$ *is a unique coupled fixed point of* F *in* $X \times X$.

3. Application to Integral Equations

In this section, we study the existence of a unique solution to an initial value problem, as an application to Theorem 2.2. Consider the initial value problem

$$\begin{aligned}
&x^1(t) = f\big(t,x(t),x(t)\big), \quad t \in I = [0,1], \\
&x(0) = x_0,
\end{aligned} \tag{22}$$

where $f : I \times [\frac{x_0}{4},\infty) \times [\frac{x_0}{4},\infty) \to [\frac{x_0}{4},\infty)$ and $x_0 \in \mathbb{R}$.

Theorem 3.1. *Consider the initial value problem 22 with* $f \in C(I \times [\frac{x_0}{4},\infty) \times [\frac{x_0}{4},\infty))$ *and*

$$\int_0^t f\big(s,x(s),y(s)\big)\,ds \le \max\left\{\begin{array}{c} \frac{1}{4}\int_0^t f\big(s,x(s),x(s)\big)\,ds - \frac{9x_0}{16}, \\ \frac{1}{4}\int_0^t f\big(s,y(s),y(s)\big)\,ds - \frac{9x_0}{16} \end{array}\right\}.$$

Then there exists a unique solution in $C(I,[\frac{x_0}{4},\infty))$ for the initial value problem 22.

Proof. The integral equation corresponding to initial value problem 22 is

$$x(t) = x_0 + \int_0^t f\big(s, x(s), x(s)\big)\, ds. \tag{23}$$

Let $X = C(I,[\frac{x_0}{4},\infty))$ and $p(x,y) = \max\{x - \frac{x_0}{4}, y - \frac{x_0}{4}\}$ for $x, y \in X$. Define $\psi, \alpha, \beta : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$, $\alpha(t) = \frac{1}{2}t$ and $\beta(t) = \frac{1}{4}t$. Define $F : X \times X \to X$ by

$$F(x,y)(t) = x_0 + \int_0^t f\big(s, x(s), y(s)\big)\, ds.$$

Now

$$\begin{aligned}
p\big(F(x,y)(t), F(u,v)(t)\big) &= \max\Big\{F(x,y) - \frac{x_0}{4}, F(u,v) - \frac{x_0}{4}\Big\} \\
&= \max\Big\{\frac{3x_0}{4} + \int_0^t f\big(s,x(s),y(s)\big)\,ds, \frac{3x_0}{4} + \int_0^t f\big(s,u(s),v(s)\big)\,ds\Big\} \\
&\le \max\left\{\begin{array}{l} \frac{3x_0}{4} + \max\left\{\begin{array}{l} \frac{1}{4}\int_0^t f(s,x(s),x(s))\,ds - \frac{9x_0}{16}, \\ \frac{1}{4}\int_0^t f(s,y(s),y(s))\,ds - \frac{9x_0}{16} \end{array}\right\}, \\ \frac{3x_0}{4} + \max\left\{\begin{array}{l} \frac{1}{4}\int_0^t f(s,u(s),u(s))\,ds - \frac{9x_0}{16}, \\ \frac{1}{4}\int_0^t f(s,v(s),v(s))\,ds - \frac{9x_0}{16} \end{array}\right\} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} \max\left\{\frac{x(t)}{4} - \frac{x_0}{16}, \frac{y(t)}{4} - \frac{x_0}{16}\right\}, \\ \max\left\{\frac{u(t)}{4} - \frac{x_0}{16}, \frac{v(t)}{4} - \frac{x_0}{16}\right\} \end{array}\right\} \\
&= \frac{1}{4}\max\Big\{\max\Big\{x(t) - \frac{x_0}{4}, u(t) - \frac{x_0}{4}\Big\}, \max\Big\{y(t) - \frac{x_0}{4}, v(t) - \frac{x_0}{4}\Big\}\Big\} \\
&= \frac{1}{4}\max\{p(x,u), p(y,v)\} \\
&= \alpha\big(\max\{p(x,u), p(y,v)\}\big) - \beta\big(\max\{p(x,u), p(y,v)\}\big).
\end{aligned}$$

Thus F satisfies the condition (3.4.1) of Theorem 2.4. From Theorem 2.4, we conclude that F has a unique coupled fixed point (x, y) with $x = y$. In particular $x(t)$ is the unique solution of the integral equation 23. □

4. Application to Homotopy

In this section, we study the existence of a unique solution to homotopy theory.

Theorem 4.1. *Let (X, p) be a complete PMS, U be an open subset of X and $\overline{U}$ be a closed subset of X such that $U \subseteq \overline{U}$. Suppose $H : \overline{U} \times \overline{U} \times [0,1] \to X$ is an operator such that the following conditions are satisfied:*

(i) $x \neq H(x, y, \lambda)$ and $y \neq H(y, x, \lambda)$ for each $x, y \in \partial U$ and $\lambda \in [0,1]$ (here ∂U denotes the boundary of U in X),

(ii) $\psi(p(H(x,y,\lambda),H(u,v,\lambda))) \leq \alpha(\max\{p(x,y),p(u,v)\}) - \beta(\max\{p(x,y),p(u,v)\})$ $\forall x,y \in \overline{U}$ *and* $\lambda \in [0,1]$, *where* $\psi,\alpha : [0,\infty) \to [0,\infty)$ *is continuous and non-decreasing and* $\beta : [0,\infty) \to [0,\infty)$ *is lower semi continuous with* $\psi(t) - \alpha(t) + \beta(t) > 0$ *for* $t > 0$,

(iii) *there exists* $M \geq 0$ *such that* $p\big(H(x,y,\lambda),H(x,y,\mu)\big) \leq M|\lambda - \mu|$ *for every* $x \in \overline{U}$ *and* $\lambda,\mu \in [0,1]$.

Then $H(\cdot,0)$ *has a coupled fixed point if and only if* $H(\cdot,1)$ *has a coupled fixed point.*

Proof. Consider the set

$$A = \big\{\lambda \in [0,1] : (x,y) = H(x,y,\lambda) \text{ for some } x,y \in U\big\}.$$

Since $H(\cdot,0)$ has a coupled fixed point in U, we have $0 \in A$, so that A is a non-empty set. We will show that A is both open and closed in $[0,1]$ so by the connectedness we have $A = [0,1]$. As a result, $H(\cdot,1)$ has a fixed point in U. First we show that A is closed in $[0,1]$. To see this let $\{\lambda_n\}_{n=1}^{\infty} \subseteq A$ with $\lambda_n \to \lambda \in [0,1]$ as $n \to \infty$. We must show that $\lambda \in A$. Since $\lambda_n \in A$ for $n = 1,2,3,\ldots$, there exist $x_n, y_n \in U$ with $u_n = (x_n,y_n) = H(x_n,y_n\lambda_n)$. Consider

$$\begin{aligned}
p(x_n,x_{n+1}) &= p\big(H(x_n,y_n,\lambda_n),H(x_{n+1},y_{n+1},\lambda_{n+1})\big)\\
&\leq p\big(H(x_n,y_n,\lambda_n),H(x_{n+1},y_{n+1}\lambda_n)\big)\\
&\quad + p\big(H(x_{n+1},y_{n+1},\lambda_n),H(x_{n+1},y_{n+1},\lambda_{n+1})\big) - p\big(H(x_{n+1},y_{n+1},\lambda_n),H(x_{n+1},y_{n+1},\lambda_n)\big)\\
&\leq p\big(H(x_n,y_n,\lambda_n),H(x_{n+1},y_{n+1},\lambda_n)\big) + M|\lambda_n - \lambda_{n+1}|.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} p(x_n,x_{n+1}) \leq \lim_{n\to\infty} p\big(H(x_n,y_n,\lambda_n),H(x_{n+1},y_{n+1},\lambda_n)\big) + 0.$$

Since ψ is continuous and non-decreasing we obtain

$$\begin{aligned}
\lim_{n\to\infty} \psi\big(p(x_n,x_{n+1})\big) &\leq \lim_{n\to\infty} \psi\big(p\big(H(x_n,y_n,\lambda_n),H(x_{n+1},y_{n+1},\lambda_n)\big)\big)\\
&\leq \lim_{n\to\infty} \big[\alpha\big(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\}\big) - \beta\big(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\}\big)\big].
\end{aligned}$$

Similarly

$$\lim_{n\to\infty} \psi\big(p(y_n,y_{n+1})\big) \leq \lim_{n\to\infty} \big[\alpha\big(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\}\big) - \beta\big(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\}\big)\big].$$

It follows that

$$\lim_{n\to\infty} p(x_n,x_{n+1}) = 0 = \lim_{n\to\infty} p(y_n,y_{n+1}). \tag{24}$$

From (p_2),

$$\lim_{n\to\infty} p(x_n, x_n) = 0 = \lim_{n\to\infty} p(y_n, y_n). \tag{25}$$

By the definition of d_p, we obtain

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0 = \lim_{n\to\infty} d_p(y_n, y_{n+1}). \tag{26}$$

Now we prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d_p). Contrary to this hypothesis, suppose that $\{x_n\}$ or $\{s_n\}$ is not Cauchy. There exists an $\epsilon > 0$ and a monotone increasing sequence of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$,

$$\max\{d_p(x_{m_k}, x_{n_k}), d_p(y_{m_k}, y_{n_k})\} \geq \epsilon \tag{27}$$

and

$$\max\{d_p(x_{m_k}, x_{n_k-1}), d_p(y_{m_k}, y_{n_k-1})\} < \epsilon. \tag{28}$$

From (27) and (28), we obtain

$$\begin{aligned}\epsilon &\leq \max\{d_p(x_{m_k}, x_{n_k}), d_p(y_{m_k}, y_{n_k})\} \\ &\leq \max\{d_p(x_{m_k}, x_{n_k-1}), d_p(y_{m_k}, y_{n_k-1})\} + \max\{d_p(x_{n_k-1}, x_{n_k}), d_p(y_{n_k-1}, y_{n_k})\} \\ &< \epsilon + \max\{d_p(x_{n_k-1}, x_{n_k}), d_p(y_{n_k-1}, y_{n_k})\}.\end{aligned}$$

Letting $k \to \infty$ and then using 26, we get

$$\lim_{k\to\infty} \max\{d_p(x_{m_k}, x_{n_k}), d_p(y_{m_k}, y_{n_k})\} = \epsilon. \tag{29}$$

Hence from the definition of d_p and from 25, we get

$$\lim_{k\to\infty} \max\{p(x_{m_k}, x_{n_k}), p(y_{m_k}, y_{n_k})\} = \frac{\epsilon}{2}. \tag{30}$$

Letting $k \to \infty$ and then using 29 and 26 in

$$|d_p(x_{m_k}, x_{n_k+1}) - d_p(x_{m_k}, x_{n_k})| \leq d_p(x_{n_k+1}, x_{n_k}),$$

we get

$$\lim_{k\to\infty} d_p(x_{n_k+1}, x_{m_k}) = \epsilon. \tag{31}$$

Hence, we have

$$\lim_{k\to\infty} p(x_{n_k+1}, x_{m_k}) = \frac{\epsilon}{2}. \tag{32}$$

Similarly

$$\lim_{k\to\infty} p(y_{n_k+1}, y_{m_k}) = \frac{\epsilon}{2}. \tag{33}$$

Consider

$$\begin{aligned}
p(x_{m_k}, x_{n_k+1}) &= p\big(H(x_{m_k}, y_{m_k}, \lambda_{m_k}), H(x_{n_k+1}, y_{n_k+1}, \lambda_{n_k+1})\big) \\
&\le p\big(H(x_{m_k}, y_{m_k}, \lambda_{m_k}), H(x_{m_k}, y_{m_k}, \lambda_{n_k+1})\big) \\
&\quad + p\big(H(x_{m_k}, y_{m_k}, \lambda_{n_k+1}), H(x_{n_k+1}, y_{n_k+1}, \lambda_{n_k+1})\big) - p\big(H(x_{m_k}, y_{m_k}, \lambda_{n_k+1}), H(x_{m_k}, y_{m_k}, \lambda_{n_k+1})\big) \\
&\le M|\lambda_{m_k} - \lambda_{n_k+1}| + p\big(H(x_{m_k}, y_{m_k}, \lambda_{n_k+1}), H(x_{n_k+1}, y_{n_k+1}, \lambda_{n_k+1})\big).
\end{aligned}$$

Since $\{\lambda_n\}$ is Cauchy, letting $k \to \infty$ in the above, we get

$$\frac{\epsilon}{2} \le \lim_{k\to\infty} p\big(H(x_{m_k}, y_{m_k}, \lambda_{n_k+1}), H(x_{n_k+1}, y_{n_k+1}, \lambda_{n_k+1})\big).$$

Since ψ is continuous and non-decreasing we obtain

$$\begin{aligned}
\psi\left(\frac{\epsilon}{2}\right) &\le \lim_{k\to\infty} \psi\big(p\big(H(x_{m_k}, y_{m_k}, \lambda_{n_k+1}), H(x_{n_k+1}, y_{n_k+1}, \lambda_{n_k+1})\big)\big) \\
&\le \lim_{k\to\infty} \big[\alpha\big(\max\{p(x_{m_k}, x_{n_k+1}), p(y_{m_k}, y_{n_k+1})\}\big) - \beta\big(\max\{p(x_{m_k}, x_{n_k+1}), p(y_{m_k}, y_{n_k+1})\}\big)\big] \\
&= \alpha\left(\frac{\epsilon}{2}\right) - \beta\left(\frac{\epsilon}{2}\right).
\end{aligned}$$

It follows that $\epsilon \le 0$, which is a contradiction. Hence $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d_p) and

$$\lim_{n,m\to\infty} d_p(x_n, x_m) = 0 = \lim_{n,m\to\infty} d_p(y_n, y_m).$$

By the Definition of d_p and (25), we get

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0 = \lim_{n,m\to\infty} p(y_n, y_m).$$

From Lemma 1.5, we conclude (a) $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, p). Since (X, p) is complete, from Lemma 1.5(b), we conclude there exist $u, v \in U$ with

$$p(u, u) = \lim_{n\to\infty} p(x_n, u) = \lim_{n\to\infty} p(x_{n+1}, u) = \lim_{n,m\to\infty} p(x_n, x_m) = 0, \tag{34}$$

$$p(v,v) = \lim_{n\to\infty} p(x_n, v) = \lim_{n\to\infty} p(x_{n+1}, v) = \lim_{n,m\to\infty} p(y_n, y_m) = 0. \tag{35}$$

From Lemma 1.6, we get

$$\lim_{n\to\infty} p(x_n, H(u,v,\lambda)) = p(u, H(u,v,\lambda)).$$

Now,

$$\begin{aligned} p\big(x_n, H(u,v,\lambda)\big) &= p\big(H(x_n,y_n,\lambda_n), H(u,v,\lambda)\big) \\ &\leq p\big(H(x_n,y_n,\lambda_n), H(x_n,y_n,\lambda)\big) + p\big(H(x_n,y_n,\lambda), H(u,v,\lambda)\big) \\ &\quad - p\big(H(x_n,y_n,\lambda), H(x_n,y_n,\lambda)\big) \\ &\leq M|\lambda_n - \lambda| + p\big(H(x_n,y_n,\lambda), H(u,v,\lambda)\big). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$p\big(u, H(u,v,\lambda)\big) \leq \lim_{n\to\infty} p\big(H(x_n,y_n,\lambda), H(u,v,\lambda)\big).$$

Since ψ is continuous and non-decreasing, we obtain

$$\begin{aligned} \psi\big(p\big(u, H(u,v,\lambda)\big)\big) &\leq \lim_{n\to\infty} \psi\big(p\big(H(x_n,y_n,\lambda), H(u,v,\lambda)\big)\big) \\ &\leq \lim_{n\to\infty} \big[\alpha\big(\max\{p(x_n,u), p(y_n,v)\}\big) - \beta\big(\max\{p(x_n,u), p(y_n,v)\}\big)\big] \\ &= 0. \end{aligned}$$

It follows that $p(u, H(u,v,\lambda)) = 0$. Thus $u = H(u,v,\lambda)$. Similarly $v = H(v,u,\lambda)$. Thus $\lambda \in A$. Hence A is closed in $[0,1]$. Let $\lambda_0 \in A$. Then there exist $x_0, y_0 \in U$ with $x_0 = H(x_0, y_0, \lambda_0)$. Since U is open, there exists $r > 0$ such that $B_p(x_0, r) \subseteq U$. Choose $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ such that $|\lambda - \lambda_0| \leq \frac{1}{M^n} < \epsilon$. Then $x \in \overline{B_p(x_0,r)} = \{x \in X / p(x,x_0) \leq r + p(x_0,x_0)\}$. We have

$$\begin{aligned} p\big(H(x,y,\lambda), x_0\big) &= p\big(H(x,y,\lambda), H(x_0,x_0,\lambda_0)\big) \\ &\leq p\big(H(x,y,\lambda), H(x,y,\lambda_0)\big) + p\big(H(x,y,\lambda_0), H(x_0,y_0,\lambda_0)\big) - p\big(H(x,y,\lambda_0), H(x,y,\lambda_0)\big) \\ &\leq M|\lambda - \lambda_0| + p\big(H(x,y,\lambda_0), H(x_0,y_0,\lambda_0)\big) \\ &\leq \frac{1}{M^{n-1}} + p\big(H(x,y,\lambda_0), H(x_0,y_0,\lambda_0)\big). \end{aligned}$$

Letting $n \to \infty$, we obtain $p\big(H(x,y,\lambda), x_0\big) \leq p\big(H(x,y,\lambda_0), H(x_0,y_0,\lambda_0)\big)$. Since ψ is continuous and non-decreasing, we have

$$\begin{aligned} \psi\big(p\big(H(x,y,\lambda), x_0\big)\big) &\leq \psi\big(p\big(H(x,y,\lambda_0), H(x_0,y_0,\lambda_0)\big)\big) \\ &\leq \alpha\big(\max\{p(x,x_0), p(y,y_0)\}\big) - \phi\big(\max\{p(x,x_0), p(y,y_0)\}\big). \end{aligned}$$

Similarly

$$\psi(p(H(y,x,\lambda),y_0)) \leq \alpha(\max\{p(x,x_0),p(y,y_0)\}) - \phi(\max\{p(x,x_0),p(y,y_0)\}).$$

Thus

$$\begin{aligned}\psi(\max\{p(H(x,y,\lambda),x_0),p(H(y,x,\lambda),y_0)\}) &\leq \alpha(\max\{p(x,x_0),p(y,y_0)\}) - \phi(\max\{p(x,x_0),p(y,y_0)\}) \\ &\leq \psi(\max\{p(x,x_0),p(y,y_0)\}).\end{aligned}$$

Since ψ is non-decreasing, we have

$$\begin{aligned}\max\{p(H(x,y,\lambda),x_0),p(H(y,x,\lambda),y_0)\} &\leq \max\{p(x,x_0),p(y,y_0)\} \\ &\leq \max\{r+p(x_0,x_0),r+p(y_0,y_0)\}.\end{aligned}$$

Thus for each fixed

$$\lambda \in (\lambda_0-\epsilon,\lambda_0+\epsilon), H(\cdot,\lambda) : \overline{B_p(x_0,r)} \to \overline{B_p(x_0,r)}.$$

Since also (ii) holds and ψ and α are continuous and non-decreasing and β is continuous with $\psi(t) - \alpha(t) + \beta(t) > 0$ for $t > 0$, all conditions of Theorem 2.4 are satisfied. Thus we deduce that $H(\cdot,\lambda)$ has a coupled fixed point in $\overline{U}$. But this coupled fixed point must be in U since (i) holds. Thus $\lambda \in A$ for any $\lambda \in (\lambda_0-\epsilon,\lambda_0+\epsilon)$. Hence $(\lambda_0-\epsilon,\lambda_0+\epsilon) \subseteq A$ and therefore A is open in $[0,1]$. For the reverse implication, we use the same strategy. □

Corollary 4.2. *Let (X,p) be a complete PMS, U be an open subset of X and $H : \overline{U} \times \overline{U} \times [0,1] \to X$ with the following properties:*

(1) $x \neq H(x,y,t)$ and $y \neq H(y,x,t)$ for each $x,y \in \partial U$ and each $\lambda \in [0,1]$ (here ∂U denotes the boundary of U in X),

(2) there exist $x,y \in \overline{U}$ and $\lambda \in [0,1], L \in [0,1)$, such that $p(H(x,y,\lambda),H(u,v,\mu)) \leq L\max\{p(x,u),p(y,v)\}$,

(3) there exists $M \geq 0$, such that $p(H(x,\lambda),H(x,\mu)) \leq M\cdot|\lambda-\mu|$ for all $x \in \overline{U}$ and $\lambda,\mu \in [0,1]$.

If $H(\cdot,0)$ has a fixed point in U, then $H(\cdot,1)$ has a fixed point in U.

Proof. The proof follows by taking $\psi(x) = x$, $\phi(x) = x - Lx$ with $L \in [0,1)$ in Theorem 4.1. □

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