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Null Cesàro Summable Difference Sequence Space

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Abstract

In the present paper, extending the notion of difference sequence spaces of Kizmaz [8], we avail an opportunity to have null Cesàro summable difference sequence spaces $C^0(\Delta)$ where C^0 is the spaces of all null $(C,1)$ summable sequences. It is observed that $C^0(\Delta)$ is a separable space.

Keywords: Difference sequence space; Köthe-Toeplitz duals; Matrix maps.
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1. Introduction

Kizmaz [8] in 1981, initiated the theory of difference sequence spaces $E(\Delta)$ as follows

$$E(\Delta) = \{(\xi_k) \in s : (\Delta\xi_k) = (\xi_k - \xi_{k+1}) \in E\},\ E \in \{\ell_\infty,\, c,\, c_0\}$$

where s, c_0, c and ℓ_∞ denotes the spaces of all, null, convergent and bounded scalar sequences. Since then, a huge amount of work has been carried out by many mathematicians regarding various generalizations of difference sequence spaces, one may refer to [1,4,10,12,14,15]. In the present paper we introduce a new difference sequence space $C^0(\Delta)$, where C^0 is space of null $(C,1)$ summable scalar sequence defined as follows:
A sequence $\xi = (\xi_k)$ of complex numbers is said to be null $(C,1)$ summable (or Cesàro Summable to 0) if $lim_k \sigma_k = 0$, where $\sigma_k = \frac{1}{k}\sum\limits_{i=1}^{k}\xi_i$. By C^0 we shall denote the linear space of all null $(C,1)$ summable sequences of complex numbers over $\mathbb{C}$, i.e.,

$$C^0 = \left\{ \xi = (\xi_k) \in s : \left(\frac{1}{k}\sum_{i=1}^{k}\xi_i \right) \in c_0 \right\}.$$

In the present work, we take the opportunity to introduce a difference sequence space with underlying space as C^0. We observe that

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(i) $C^0 \nsubseteq c_0(\Delta)$ as $((-1)^k) \in C^0$ but $((-1)^k) \notin c_0(\Delta)$,

(ii) $c_0(\Delta) \nsubseteq C^0$ as $(\xi_k) = (1, 1, 1, , \ldots) \in c_0(\Delta)$ but $(\xi_k) \notin C^0$ and

(iii) $c_0 \subset c_0(\Delta) \cap C^0$.

Thus the sequence spaces C^0 and $c_0(\Delta)$ overlap but do not contain each other. Similarly, C^0 and ℓ_∞ also overlap without containing each other as is clear from the fact that $C^0 \nsubseteq \ell_\infty$, $\ell_\infty \nsubseteq C^0$ and $c_0 \subset C^0 \cap \ell_\infty$. Note that the sequence $((-1)^{k-1}\sqrt{k})$ is null $(C, 1)$ summable but not bounded whereas the sequence $\xi = (\xi_k)$ given by $\xi_1 = 1, \xi_2 = 0$ and

$$\xi_k = \begin{cases} 1, & \text{if} \quad 2^{i-1} < k \leq 3(2^{i-2}), \quad (i = 2, 3, \ldots); \\ 0, & \text{otherwise.} \end{cases}$$

is bounded but not null $(C, 1)$ summable. This has motivated the authors to look for a new sequence space which properly includes the spaces C^0, $c_0(\Delta)$ and ℓ_∞.

We now introduce a sequence space $C^0(\Delta)$, Cesàro summable difference sequence space, as follows:

$$\begin{aligned} C^0(\Delta) &= \left\{\xi = (\xi_k) \in s : (\Delta\xi_k) \in C^0\right\} \\ &= \left\{\xi = (\xi_k) \in s : \left(\frac{1}{k}\sum_{i=1}^{k} \Delta\xi_i\right) \in c_0\right\} \\ &= \left\{\xi = (\xi_k) \in s : \left(\frac{\xi_1 - \xi_{k+1}}{k}\right) \in c_0\right\} \\ &= \left\{\xi = (\xi_k) \in s : \left(\frac{\xi_k}{k}\right) \in c_0\right\} \end{aligned}$$

The definitions given below may be conveniently found in [4–7,10]. A complete metric linear space is called a Frèchet space. Let E be a linear subspace of s such that E is a Frèchet space with continuous coordinate projections. Then we say that E is an *FK* space. If the metric of an *FK* space is given by a complete norm then we say that E is a *BK* space. We say that an *FK* space E has *AK*, or has the *AK* property, if (e_k), the sequence of unit vectors, is a Schauder basis for E. A sequence space E is

(i) perfect if $E^{\alpha\alpha} = E$.

(ii) Solid(normal) if $(\eta_k) \in E$ whenever $|\eta_k| \leq |\xi_k|$, $k \geq 1$, for $(\xi_k) \in E$.

The study of sequence spaces is considered to be incomplete without computation of dual. The introduction of dual spaces is due to Köthe and Toeplitz [9] and for a sequence space E, the following

$$E^\alpha = \left\{(a_k) \in s : \sum_{k=1}^{\infty} |a_k\xi_k| < \infty \ \ \forall\ \xi = (\xi_k) \in E\right\}$$

and

$$E^{\beta} = \left\{ (a_k) \in s : \sum_{k=1}^{\infty} a_k \xi_k < \infty \ \ \forall \ \xi = (\xi_k) \in E \right\}$$

are called $\alpha-$ and $\beta-$duals spaces of E respectively. Also for $E \subseteq F$, we have $F^{\Theta} \subseteq E^{\Theta}$ for $\Theta \in \{\alpha, \beta\}$. For more detailed account of duals spaces one may refer to [2,3,6,11,13] where many more references can be found.

2. Algebraic and Topological Properties of $C^0(\Delta)$.

In this section, we establish that $C^0(\Delta)$ along with BK space, is also a separable space. Apart from this, schauder basis and various inclusion relations are established for this space.

Theorem 2.1. $\ell_\infty \subset C^0(\Delta)$, *the inclusion being strict.*

Proof. Let $\xi = (\xi_k) \in \ell_\infty$. Then there exists $M > 0$ such that $|\xi_1 - \xi_{k+1}| \leq M$ for all $k \geq 1$ and so $\frac{1}{k} \sum_{i=1}^{k} \Delta\xi_i \to 0$ as $k \to \infty$. For strict inclusion, take $(\xi_k) = (0, -\sqrt{1}, 0, -\sqrt{2}, 0, -\sqrt{3}, \ldots)$ then $(\Delta\xi_k) = (\sqrt{1}, -\sqrt{1}, \sqrt{2}, -\sqrt{2}, \ldots) \in C^0$, i.e., $(\xi_k) \in C^0(\Delta)$ but $(\xi_k) \notin \ell_\infty$. □

Theorem 2.2. $C^0 \subset C^0(\Delta)$, *the inclusion being strict.*

Proof. For $\xi = (\xi_k) \in C^0$, we have $\lim_k \frac{1}{k}\xi_k = 0$ and so $\frac{1}{k} \sum_{i=1}^{k} \Delta\xi_i \to 0$ as $k \to \infty$. For strict inclusion, take $(\xi_k) = (1, 1, 1, 1, \ldots) \in C^0(\Delta)$ but $(\xi_k) \notin C^0$ □

Theorem 2.3. $c_0(\Delta) \subset C^0(\Delta)$, *the inclusion being strict.*

Proof. Inclusion is obvious since $c_0 \subset C^0$. For inclusion is strict, consider the sequence $(\xi_k) = (0, -\sqrt{1}, 0, -\sqrt{2}, 0, -\sqrt{3}, \ldots)$ then $(\Delta\xi_k) = (\sqrt{1}, -\sqrt{1}, \sqrt{2}, -\sqrt{2}, \ldots) \in C^0$, i.e., $(\xi_k) \in C^0(\Delta)$. But $(\Delta\xi_k) \notin c_0$, i.e., $(\xi_k) \notin c_0(\Delta)$. □

Theorem 2.4. *Let X and Y be sequence spaces. If $X \nsubseteq Y$, then $X(\Delta) \nsubseteq Y(\Delta)$.*

Proof. Since $X \nsubseteq Y$, there is a sequence $\xi = (\xi_k) \in X$ such that $\xi \notin Y$. Consider the sequence $\eta = (\eta_k) = (0, -\xi_1, -\xi_1 - \xi_2, -\xi_1 - \xi_2 - \xi_3, \ldots)$. Then $\eta \in X(\Delta)$ but $\eta \notin Y(\Delta)$. □

Remark 2.5. *We have already observed that $C^0 \nsubseteq \ell_\infty$ and $\ell_\infty \nsubseteq C^0$, so, neither $C^0(\Delta) \subseteq \ell_\infty(\Delta)$ nor $\ell_\infty(\Delta) \subseteq C^0(\Delta)$. Also we have $c_0(\Delta) \subset C^0(\Delta) \cap \ell_\infty(\Delta)$. In view of this and* Theorem 2.3, *we can say that $C^0(\Delta)$ strictly includes $c_0(\Delta)$ but overlaps with $\ell_\infty(\Delta)$.*

Theorem 2.6. *$C^0(\Delta)$ is a BK space normed by*

$$\|\xi\|_\Delta = |\xi_1| + sup_k \frac{1}{k} \left| \sum_{i=1}^{k} \Delta\xi_i \right|, \quad \xi = (\xi_k) \in C^0(\Delta).$$

Proof. The proof is a routine verification by using 'standard' techniques and hence is omitted. □

Theorem 2.7. *Sequence space $(C^0(\Delta), \|\cdot\|_\Delta)$ is separable.*

Proof. Let $D = \{\eta = (\eta_k) : \eta_k \in \mathbb{Q} \; for \; 1 \leq k \leq n \; and \; \eta_k = 0 \; for \; k > n, \; n \in \mathbb{N}\}$. Then D is countable. Now

$$D \subseteq c_0 \subset c_0(\Delta) \subseteq C^0(\Delta)$$

i.e., D is a subset of $C^0(\Delta)$. We prove that D is dense in $(C^0(\Delta), \|\cdot\|_\Delta)$. Let $\xi = (\xi_k) \in C^0(\Delta)$ and $\varepsilon > 0$. Then $\dfrac{\xi_1 - \xi_{k+1}}{k} \to 0$ as $k \to \infty$. So there exists $p \in \mathbb{N}$ such that

$$\left| \frac{\xi_1 - \xi_{k+1}}{k} \right| < \frac{\varepsilon}{6} \quad \forall \, k \geq p.$$

Let m be a natural number such that $\dfrac{|\xi_1|}{m} < \dfrac{\varepsilon}{6}$. Take $t = \max\{p, m\}$. Then

$$\left| \frac{\xi_1 - \xi_{k+1}}{k} \right| < \frac{\varepsilon}{6} \quad \forall \, k \geq t \quad \text{and} \quad \frac{|\xi_1|}{t} < \frac{\varepsilon}{6}$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, so there are rational numbers η_k, $1 \leq k \leq t$ such that $|\xi_k - \eta_k| < \dfrac{\varepsilon}{6}$. We set $\eta_k = 0$ for all $k > t$. Let $\eta = (\eta_k)$. Then $\eta \in D$. We claim that $\|\xi - \eta\|_\Delta < \varepsilon$. Now

$$\|\xi - \eta\|_\Delta = |\xi_1 - \eta_1| + \sup_k \frac{1}{k} \left[|(\xi_1 - \eta_1) - (\xi_{k+1} - \eta_{k+1})| \right]$$

Now for $1 \leq k \leq t - 1$,

$$\begin{aligned} \frac{1}{k}|(\xi_1 - \eta_1) - (\xi_{k+1} - \eta_{k+1})| &\leq \frac{|\xi_1 - \eta_1|}{k} + \frac{|\xi_{k+1} - \eta_{k+1}|}{k} \\ &\leq \frac{\varepsilon}{6k} + \frac{\varepsilon}{6k} = \frac{\varepsilon}{3k} < \frac{\varepsilon}{3}. \end{aligned}$$

For $k \geq t$

$$\begin{aligned} \frac{1}{k}|(\xi_1 - \eta_1) - (\xi_{k+1} - \eta_{k+1})| &= \frac{1}{k}|\xi_1 - \eta_1 - \xi_{k+1}| \\ &\leq \frac{|\xi_1 - \eta_1|}{k} + \frac{|\xi_{k+1}|}{k} \\ &\leq \frac{\varepsilon}{6k} + \frac{1}{k}|\xi_{k+1} - \xi_1| + \frac{1}{k}|\xi_1| \\ &\leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}. \end{aligned}$$

It follows that

$$\|\xi - \eta\|_\Delta = \frac{\varepsilon}{6} + \frac{\varepsilon}{2} < \varepsilon.$$

This implies D is dense in $C^0(\Delta)$. □

Theorem 2.8. *$C^0(\Delta)$ has Schauder basis namely $\{e, e_1, e_2, \ldots\}$, where $e = (1, 1, 1, \ldots)$ and $e_k = (0, 0, 0, \ldots, 1, 0, 0, \ldots)$, 1 is in the k^{th} place and 0 elsewhere for $k = 1, 2, \ldots$.*

Theorem 2.9. *The sequence space* $(\ell_\infty, \|\cdot\|_\Delta)$ *is separable.*

Proof. As $\ell_\infty \subset C^0(\Delta)$ and subspace of a separable metric space is separable, so $(\ell_\infty, \|\cdot\|_\Delta)$ is separable. □

Remark 2.10. *Because* $(\ell_\infty, \|\cdot\|_\infty)$ *is not separable so* $\|\cdot\|_\Delta$ *and* $\|.\|_\infty$ *where* $\|\xi\|_\infty = \sup\{|\xi_k| : k \geq 1\}$ *on the space* ℓ_∞ *are not equivalent.*

Theorem 2.11. $C^0(\Delta)$ *is normal or solid.*

Proof. Let (η_k) be such that $|\eta_k| \leq |\xi_k|$ for some $(\xi_k) \in C^0(\Delta)$. We prove that $(\eta_k) \in C^0(\Delta)$. Let $\varepsilon > 0$ be given. Then there is some natural number p such that $\left|\frac{\xi_1 - \xi_{k+1}}{k}\right| < \frac{\varepsilon}{6}$, for all $k \geq p$. Let m be a natural number such that $\frac{|\xi_1|}{m} < \frac{\varepsilon}{6}$. Let $t = \max\{p, m\}$. Then $\left|\frac{\xi_1 - \xi_{k+1}}{k}\right| < \frac{\varepsilon}{6}$, for all $k \geq t$ and $\frac{|\xi_1|}{t} < \frac{\varepsilon}{6}$. Now for all $k \geq t$,

$$\begin{aligned}
\left|\frac{\eta_1 - \eta_{k+1}}{k}\right| &< \frac{|\eta_1|}{k} + \frac{|\eta_{k+1}|}{k} \\
&< \frac{|\xi_1|}{k} + \frac{|\xi_{k+1}|}{k} \\
&< \frac{\varepsilon}{6} + \frac{|\xi_{k+1} - \xi_1|}{k} + \frac{|\xi_1|}{k} \\
&< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} < \varepsilon.
\end{aligned}$$

This implies $(\eta_k) \in C^0(\Delta)$. Thus $C^0(\Delta)$ is normal space. □

Theorem 2.12. $C^0(\Delta)$ *does not have AK property.*

Proof. As sequence of unit vectors $\{e_1, e_2, \ldots\}$ is not a Schauder basis for $C^0(\Delta)$, so $C^0(\Delta)$ does not have *AK* property. □

3. Köthe-Toeplitz Duals and Matrix Maps

In this section we compute the Köthe-Toeplitz duals of $C^0(\Delta)$ and show that $C^0(\Delta)$ is not perfect. Before proceeding in this section, we recall about matrix map.

Let X and Y be any two sequence spaces and let $A = (a_{nk})$ $(n, k = 1, 2, \ldots)$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_k a_{nk}x_k$ converges for each n. If $x = (x_k) \in X$ implies that $Ax = (A_n(x)) \in Y$, then we say that A define a matrix transformation from X into Y and we denote it by $A : X \longrightarrow Y$. The sequence Ax is called the A transformation of x. By (X, Y) we shall denote the set of all matrices A which map (or transform) X into Y. A good account of matrix transformations in sequence spaces can be found in [5,10].

Theorem 3.1.

$$[C^0(\Delta)]^\alpha = \left\{a = (a_k) : \sum_k k|a_k| < \infty\right\} = D_1.$$

Proof. Let $a = (a_k) \in D_1$. For any $\xi = (\xi_k) \in C^0(\Delta)$, we have $\dfrac{\xi_k}{k} \in c_0$, and so there exists some $M > 0$ such that $\left|\dfrac{\xi_k}{k}\right| \leq M$ for $k \geq 1$ and hence $\sup_k k^{-1}|\xi_k| < \infty$ which implies that

$$\sum_k |a_k \xi_k| = \sum_k (k\,|a_k|)\,(k^{-1}\,|\xi_k|) < \infty.$$

Thus $a = (a_k) \in [C^0(\Delta)]^\alpha$.

Conversely, let $a = (a_k) \in [C^0(\Delta)]^\alpha$. Then $\sum_k |a_k \xi_k| < \infty$ for all $\xi = (\xi_k) \in C^0(\Delta)$. To prove $\sum_k k|a_k| < \infty$. Let if possible, $\sum_k k|a_k| = \infty$, we can determine a sequence of integer $n_1 < n_2 < n_3 < \ldots$ such that

$$\sum_{k=1}^{n_1} k|a_k| > 2^1, \quad \sum_{k=n_1+1}^{n_2} k|a_k| > 2^2, \ldots, \sum_{k=n_p+1}^{n_{p+1}} k|a_k| > 2^{p+1}$$

Take

$$\xi_k = \begin{cases} \frac{k}{2^{p+1}}, & \text{if} \quad n_p + 1 \leq k \leq n_{p+1}, \quad \text{for all } p \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then $\xi = (\xi_k) \in c_0(\Delta) \subset C^0(\Delta)$ and

$$\sum_{k=n_p+1}^{n_{p+1}} |a_k \xi_k| = \sum_{k=n_p+1}^{n_{p+1}} \frac{k|a_k|}{2^{p+1}} > 1 \quad \text{for all } p \geq 1$$

This implies $\sum_k |a_k \xi_k| = \infty$, a contradiction. □

Theorem 3.2.

$$[C^0(\Delta)]^{\alpha\alpha} = \left\{ a = (a_k) : \sup_k k^{-1}|a_k| < \infty \right\} = D_2.$$

Proof. Taking $m = 1$ and $X = c$ in Theorem 2.13 of [6], we have

$$[C^0(\Delta)]^{\alpha\alpha} = \left\{ a = (a_k) : \sup_k k^{-1}|a_k| < \infty \right\}$$

and so $[c(\Delta)]^{\alpha\alpha} = [C^0(\Delta)]^{\alpha\alpha}$ □

Corollary 3.3. *$C^0(\Delta)$ is not perfect.*

Proof. The proof follows at once when we observe that the sequence $(k) \in [C^0(\Delta)]^{\alpha\alpha}$ but does not belong to $C^0(\Delta)$. □

Theorem 3.4.

$$[C^0(\Delta)]^{\beta} = \left\{ a = (a_k) : \sum_k k|a_k| < \infty \right\} = D_3.$$

Proof. Let $a = (a_k) \in D_3$ and $\xi = (\xi_k) \in C^0(\Delta)$. Then $\left(\frac{1}{k}\sum_{i=1}^{k} \Delta\xi_i\right) \in c_0$. For $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} a_k\xi_k = -\sum_{k=2}^{n}(k-1)a_k\left(\frac{1}{k-1}\sum_{i=1}^{k-1}\Delta\xi_i\right) + \xi_1\sum_{k=1}^{n} a_k.$$

Because $(a_k) \in D_3$ so (a_k), $((k-1)a_k) \in \ell_1$. We define $\eta = (\eta_k)$ by $\eta_1 = 0$ and $\eta_k = \frac{1}{k-1}\sum_{i=1}^{k-1}\Delta\xi_i$ for all $k \geq 2$. Then $\eta \in c_0$ and since $c_0^\alpha = \ell_1$, the series $\sum_{k=2}^{\infty}(k-1)a_k\left(\frac{1}{k-1}\sum_{i=1}^{k-1}\Delta\xi_i\right)$ converges absolutely.
Conversely, if $a = (a_k) \in [C^0(\Delta)]^\beta$, then $\sum_k a_k\xi_k$ converges for all $\xi = (\xi_k) \in C^0(\Delta)$. In particular, taking $\xi_k = 1$ for all k, we have $\sum_k a_k$ converges and so $\sum_{k=2}^{\infty}(k-1)a_k\left(\frac{1}{k-1}\sum_{i=1}^{k-1}\Delta\xi_i\right)$ converges for all $\xi = (\xi_k) \in c_0(\Delta)$. Since $\xi = (\xi_k) \in C^0(\Delta)$ if and only if $\eta = \left(\frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i\right) \in c_0$, we have $((k-1)a_k) \in [c_0]^\alpha = \ell_1$. □

Theorem 3.5. *$A \in (C^0(\Delta), \ell_\infty)$ if and only if $\sup_n \sum_{k=2}^{\infty}(k-1)|a_{nk}| < \infty$.*

Proof. Suppose that $\sup_n \sum_{k=2}^{\infty}(k-1)|a_{nk}| < \infty$ and $\xi = (\xi_k) \in C^0(\Delta)$. Proceeding as in Theorem 3.4, we have $\sum_{k=2}^{\infty} | a_{nk} \sum_{i=1}^{k-1}\Delta\xi_i| < \infty$. For $m \in \mathbb{N}$,

$$\sum_{k=1}^{m} a_{nk}\xi_k = -\sum_{k=1}^{m} a_{nk}(\sum_{i=1}^{k-1}\Delta\xi_i) + \xi_1\sum_{k=1}^{m} a_{nk}$$

This yields $\sum_k |a_{nk}\xi_k| < \infty$, for each $n \in \mathbb{N}$ and finally we have,

$$\left|\sum_k a_{nk}\xi_k\right| \leq \left(\sup_{k\geq 2}\left|\frac{1}{k-1}\sum_{i=1}^{k-1}\Delta\xi_i\right|\right)\left(\sup_n \sum_{k=2}^{\infty}(k-1)\,|a_{nk}|\right) + \xi_1 \sup_n \sum_k (k-1)\,|a_{nk}|$$
$$< \infty \text{ for all } n \in \mathbb{N}.$$

which yields $(A_n\xi) = A\xi \in \ell_\infty$. Thus $A \in (C^0(\Delta), \ell_\infty)$.
Conversely, let $A \in (C^0(\Delta), \ell_\infty)$. For all $\xi = (\xi_k) \in C^0(\Delta)$, we have

$$\left|\sum_k a_{nk}\xi_k\right| = |A_n(\xi)| \leq \sup_n |A_n(\xi)| = \|A\xi\|_\infty \leq \|A\|\|\xi\|_\Delta, \tag{1}$$

for each $n \in \mathbb{N}$ and $\xi = (\xi_k) \in C^0(\Delta)$.
Choose any $n \in \mathbb{N}$ and any $r \in \mathbb{N}$ and define

$$\xi_k = \begin{cases} (k-1)\, sgn\, a_{nk}, & \text{if } 1 < k \leq r; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\xi = (\xi_k) \in c_0 \subset C^0(\Delta)$ with $\|\xi\|_\Delta = 1$. Inserting this value of $\xi = (\xi_k)$ in (1) , we have

$$\sum_{k=2}^{r}(k-1)|a_{nk}| \leq \|A\|. \tag{2}$$

Letting $r \to \infty$ and noting that (2) holds for every $n \in \mathbb{N}$, we are through. □

Theorem 3.6. *$A \in (C^0(\Delta), c_0)$ if and only if*

(i) $\sup_n \sum_{k=2}^{\infty}(k-1)|a_{nk}| < \infty$,

(ii) $\lim_n a_{nk} = 0$ for each k,

(iii) $\lim_n \sum_k a_{nk} = 0$.

Proof. Let the conditions (i)-(iii) hold and suppose that $\xi = (\xi_k) \in C^0(\Delta)$ with $\lim_k \frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i = 0$. It is implicit in (i) that, for each $n \in \mathbb{N}$, $\sum_k (k-1)|a_{nk}|$ converges. It follows that $\sum_{k=2}^{\infty}(k-1)a_{nk}\left(\frac{1}{k-1}\sum_{i=1}^{k-1}\Delta\xi_i\right)$ converges, hence

$$\sum_k a_{nk}\xi_k = -\sum_{k=2}^{\infty}(k-1)a_{nk}\left(\frac{1}{k-1}\sum_{i=1}^{k-1}\Delta\xi_i -\right) + \xi_1\sum_k a_{nk}. \tag{3}$$

Let $\sigma_k = \frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i$, $H = \sup_k|\sigma_k|$ and $M = \sup_n \sum_k (k-1)|a_{nk}|$. Then for any $p \in \mathbb{N}$, we have

$$\begin{aligned}\left|\sum_{k=2}^{\infty}(k-1)a_{nk}\left(\frac{1}{k-1}\sum_{i=1}^{k-1}\Delta\xi_i\right)\right| &= \left|\sum_{k\geq 2}(k-1)a_{nk}\sigma_{k-1}\right| \\ &\leq \sum_{k=2}^{p}(k-1)|a_{nk}|\,|\sigma_{k-1}| + \sum_{k=p+1}^{\infty}(k-1)|a_{nk}|\,|\sigma_{k-1}| \\ &\leq H\sum_{k=2}^{p}(k-1)|a_{nk}| + M\sup_{k>p}|\sigma_{k-1}|\end{aligned}$$

and hence

$$\limsup_n\left|\sum_{k=2}^{\infty}(k-1)a_{nk}\left(\frac{1}{k-1}\sum_{i=1}^{k-1}\Delta\xi_i\right)\right| \leq M\sup_{k>p}|\sigma_{k-1}|.$$

Letting $p \to \infty$, we have $\sum_{k=2}^{\infty}(k-1)a_{nk}\left(\frac{1}{k-1}\sum_{i=1}^{k-1}\Delta\xi_i\right) \to 0$ as $n \to \infty$. Making use of this and (iii) in (3) we get the result.

Conversely, let $A \in (C^0(\Delta), c_0)$. Then $\left(\sum_k a_{nk}\xi_k\right)_{n\in\mathbb{N}} \in c_0$ for all $x = (\xi_k) \in C^0(\Delta)$. By the same argument as in Theorem 3.5, we have $\sup_n \sum_{k=2}^{\infty}(k-1)|a_{nk}| < \infty$. Taking $x = e_k \in C^0(\Delta)$, we get $(a_{nk})_{n\in\mathbb{N}} \in c_0$ with $\lim_n a_{nk} = 0$ for each k. Finally $x = (1,1,1,\ldots) \in C^0(\Delta)$ yields $\lim_n \sum_k a_{nk} = 0$. □

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