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Fibonacci Divisor Cordial Labeling in the Context of Graph Operations on Grötzsch

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Abstract

Let $G = (V, E)$ be a (p, q)-graph. A Fibonacci divisor cordial labeling of a graph G with vertex set V is a bijection $f : V \to \{F_1, F_2, F_3, \ldots, F_p\}$, where F_i is the i^{th} Fibonacci number such that if each edge uv is assigned the label 1 if $f(u)$ divides $f(v)$ or $f(v)$ divides $f(u)$ and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If a graph has a Fibonacci divisor cordial labeling, then it is called Fibonacci divisor cordial graph. In this research paper, we investigate the Fibancci divisor cordial labeling bahevior for Grötzsch graph, fusion of any two vertices in Grötzsch graph, duplication of an arbitrary vertex in Grötzsch graph, duplication of an arbitrary vertex by an edge in Grötzsch graph, switching of an arbitrary vertex of degree four in Grötzsch graph, switching of an arbitrary vertex of degree three in Grötzsch graph and path union of two copies of Grötzsch.

Keywords: Fibonacci divisor cordial labeling; fusion; duplication; switching; path union.
2020 Mathematics Subject Classification: 05A05, 05A17, 11B25.

1. Introduction

Let $G = (V, E)$ be a simple, finite, undirected and non-trivial graph with the vertex set V. The number of elements of V, denoted as $|V(G)|$ is called the order of G while the number of elements of E, denoted as $|E(G)|$ is called the size of G. More detail of graph labeling results and its applications can be found in Gallian [1]. We provide brief summary of definitions and other related information which are useful for the further investigations. The present work is aimed to discuss one such labeling known as Fibonacci divisor cordial labeling.

Definition 1.1. *A Grötzsch graph G_z is a is a triangle-free bipartite undirected graph with 11 vertices and 20 edges, chromatic number 4, and crossing number 5.*

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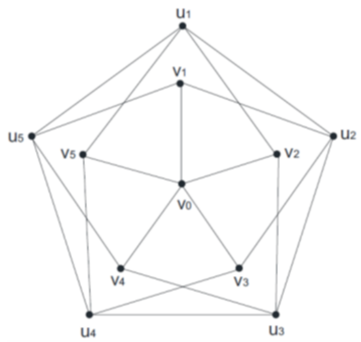

Figure 1:

In this research paper, we always fix the position of vertices $v_1, v_2, v_3, v_4, v_5, u_1, u_2, u_3, u_4, u_5$ as mentioned in Figure A, unless or otherwise specified.

2. Main Results

Theorem 2.1. *The graph G_z is a Fibonacci divisor cordial graph.*

Proof. Let G_z be Grötzsch graph and let v_0 be the central vertex and $v_1, v_2, v_3, v_4, v_5, u_1, u_2, u_3, u_4, u_5$ be the remaining vertices of the G_z. Then $|V(G_z)| = 11$ and $|E(G_z)| = 20$. We define labeling function $f : V(G_z) \to \{F_1, F_2, \ldots, F_{11}\}$ as follows:

$$\begin{aligned} f(v_0) &= F_1, & f(u_1) &= F_3, \\ f(v_1) &= F_5, & f(u_2) &= F_6, \\ f(v_2) &= F_7, & f(u_3) &= F_8, \\ f(v_3) &= F_{10}, & f(u_4) &= F_4, \\ f(v_4) &= F_2, & f(u_5) &= F_9. \\ f(v_5) &= F_{11}, \end{aligned}$$

Hence, we observe that $|e_f(0) - e_f(1)| = 1$. So, G_z is a Fibonacci divisor cordial graph. □

Example 2.2. *A Fibonacci divisor cordial labeling of G_z is shown in Figure 2.*

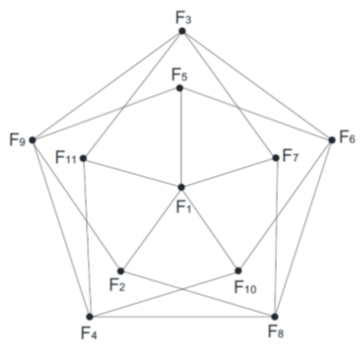

Figure 2:

Theorem 2.3. *The fusion of any two adjacent vertices of degree 4 in the Grotzsch graph is a Fibonacci Divisor cordial graph.*

Proof. Let G be the graph made from G_z by fusion of any two vertices in G_z. Then $|V(G_z)| = 10$ and $|E(G_z)| = 19$. We define labeling function $f : V(G_z) \rightarrow \{F_1, F_2, \ldots, F_{10}\}$ as follows:

Case 1: Without loss of generality, we may assume that the vertices u_1 & u_2 are fussed to the new vertex u and $u = u_1u_2$.

$$\begin{aligned} f(v_0) &= F_7, & f(u) &= F_1, \\ f(v_1) &= F_3, & f(u_3) &= F_{10}, \\ f(v_2) &= F_4, & f(u_4) &= F_2 \\ f(v_3) &= F_5, & f(u_5) &= F_8. \\ f(v_4) &= F_6, \\ f(v_5) &= F_9, \end{aligned}$$

Case 2: Without loss of generality, we may assume that the vertices u_2 & u_3 are fussed to the new vertex u and $u = u_2u_3$.

$$\begin{aligned} f(v_0) &= F_7, & f(u) &= F_1, \\ f(v_1) &= F_9, & f(u_1) &= F_8, \\ f(v_2) &= F_3, & f(u_4) &= F_{10}, \\ f(v_3) &= F_4, & f(u_5) &= F_2. \\ f(v_4) &= F_5, \\ f(v_5) &= F_6, \end{aligned}$$

Case 3: Without loss of generality, we may assume that the vertices u_3 & u_4 are fussed to the new vertex u and $u = u_3u_4$.

$$\begin{aligned} f(v_0) &= F_7, & f(u) &= F_1, \\ f(v_1) &= F_6, & f(u_1) &= F_2, \\ f(v_2) &= F_9, & f(u_2) &= F_8, \\ f(v_3) &= F_3, & f(u_5) &= F_{10}. \\ f(v_4) &= F_4, \\ f(v_5) &= F_5, \end{aligned}$$

Case 4: Without loss of generality, we may assume that the vertices u_4 & u_5 are fussed to the new vertex u and $u = u_4u_5$.

$$f(v_0) = F_7, \quad f(u) = F_1,$$
$$f(v_1) = F_5, \quad f(u_1) = F_{10},$$
$$f(v_2) = F_6, \quad f(u_2) = F_2,$$
$$f(v_3) = F_9, \quad f(u_3) = F_8.$$
$$f(v_4) = F_3,$$
$$f(v_5) = F_4,$$

Case 5: Without loss of generality, we may assume that the vertices u_1 & u_5 are fussed to the new vertex u and $u = u_1u_5$.

$$f(v_0) = F_7, \quad f(u) = F_1,$$
$$f(v_1) = F_4, \quad f(u_2) = F_8,$$
$$f(v_2) = F_5, \quad f(u_3) = F_{10},$$
$$f(v_3) = F_6, \quad f(u_4) = F_2.$$
$$f(v_4) = F_9,$$
$$f(v_5) = F_3,$$

From above all the cases, we observe that $|e_f(0) - e_f(1)| = 1$. So, G is a Fibonacci divisor cordial graph. □

Example 2.4. *The graph made from fusion of two vertices u_1 and u_2 in G_z is a Fibonacci divisor cordial graph as shown in Figure 3.*

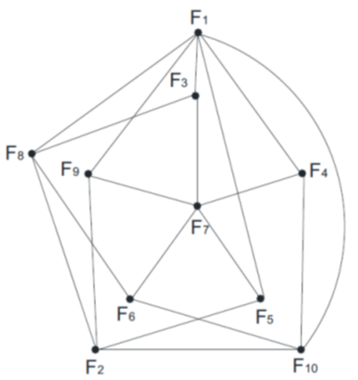

Figure 3:

Theorem 2.5. *The graph made from duplication of an arbitrary vertex in Gz is a Fibonacci divisor cordial graph.*

Proof. Let G_z be the Grötzsch graph with $|V(G_z)| = 11$ and $|E(G_z)| = 20$. Let G be the graph made by duplication of an arbitrary vertex w in G_z. Then $|V(G_z)| = 12$ and $|E(G_z)| = 23$. We define labeling function $f : V(G_z) \to \{F_1, F_2, \ldots, F_{12}\}$ as follows:

Case 1: Without loss of generality, we may take the vertex $w = u_1$ to be the duplicating vertex and let u_1' be the duplication vertex of u_1.

$$\begin{aligned} f(v_0) &= F_1, & f(u_1') &= F_2, \\ f(v_1) &= F_5, & f(u_1) &= F_3, \\ f(v_2) &= F_7, & f(u_2) &= F_6, \\ f(v_3) &= F_{10}, & f(u_3) &= F_4, \\ f(v_4) &= F_{12}, & f(u_4) &= F_8, \\ f(v_5) &= F_{11}, & f(u_5) &= F_9. \end{aligned}$$

Case 2: Without loss of generality, we may take the vertex $w = v_1$ to be the duplicating vertex and let v_1' be the duplication vertex of v_1.

$$\begin{aligned} f(v_0) &= F_1, & f(u_1) &= F_3, \\ f(v_1) &= F_5, & f(u_2) &= F_6, \\ f(v_2) &= F_7, & f(u_3) &= F_8, \\ f(v_3) &= F_{10}, & f(u_4) &= F_4, \\ f(v_4) &= F_2, & f(u_5) &= F_9. \\ f(v_5) &= F_{11}, \\ f(v_0') &= F_{12}, \end{aligned}$$

Case 3: Without loss of generality, we may take the vertex $w = v_0$ to be the duplicating vertex and let v_0' be the duplication vertex of v_0.

$$\begin{aligned} f(v_0) &= F_1, & f(u_1) &= F_3, \\ f(v_1) &= F_5, & f(u_2) &= F_6, \\ f(v_2) &= F_7, & f(u_3) &= F_4, \\ f(v_3) &= F_{10}, & f(u_4) &= F_8, \\ f(v_4) &= F_{12}, & f(u_5) &= F_9. \\ f(v_5) &= F_{11}, \\ f(v_0') &= F_2, \end{aligned}$$

From above all the cases, we observe that $\left|e_f(0) - e_f(1)\right| = 1$. So, G is a Fibonacci divisor cordial graph. □

Example 2.6. *The Fibonacci divisor cordial labelling of the graph obtained by duplication of a vertex u_1 in G is Shown in Figure 4.*

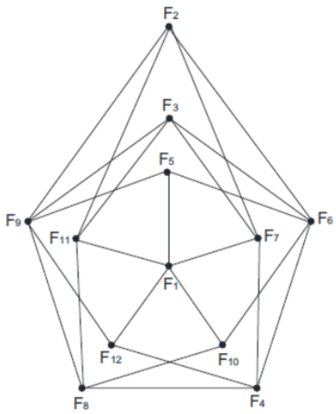

Figure 4:

Example 2.7. *The Fibonacci divisor cordial labelling of the graph obtained by duplication of a vertex v_1 in G is Shown in Figure 5.*

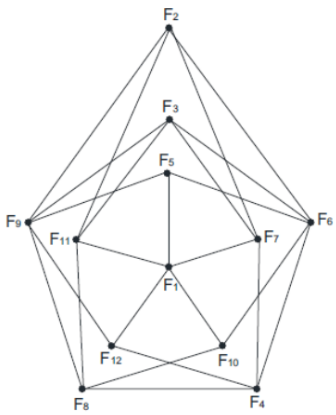

Figure 5:

Theorem 2.8. *The graph made from duplication of an arbitrary vertex by an edge in G_z is a Fibonacci divisor cordial graph.*

Proof. Let G_z be a Grötzsch graph and let v_0 be the central vertex and $v_1, v_2, v_3, v_4, v_5, u_1, u_2, u_3, u_4, u_5$ be the remaining vertices of the G_z. Let G be the graph made from duplicating an arbitrary vertex w by an edge e in G_z. We define labeling function $f : V(G_z) \rightarrow \{F_1, F_2, \ldots, F_{13}\}$ as follows:

Case 1: Without loss of generality, we may take the duplication of a central vertex $w = v_0$ by an edge $e = v_0'v_0''$ in G_z.

$$f(v_0) = F_1, \qquad f(u_1) = F_3,$$
$$f(v_1) = F_5, \qquad f(u_2) = F_6,$$

$$f(v_2) = F_7, \qquad f(u_3) = F_8,$$
$$f(v_3) = F_{10}, \qquad f(u_4) = F_4,$$
$$f(v_4) = F_2, \qquad f(u_5) = F_9.$$
$$f(v_5) = F_{11},$$
$$f(v_0') = F_{12},$$
$$f(v_0'') = F_{13},$$

Case 2: Without loss of generality, we may take the duplication of a central vertex $w = u_1$ by an edge $e = u_1'u_1''$ in G_z.

$$f(v_0) = F_1, \qquad f(u_1') = F_{12},$$
$$f(v_1) = F_5, \qquad f(u_1'') = F_{13},$$
$$f(v_2) = F_7, \qquad f(u_1) = F_3,$$
$$f(v_3) = F_{10}, \qquad f(u_2) = F_6,$$
$$f(v_4) = F_2, \qquad f(u_3) = F_8,$$
$$f(v_5) = F_{11}, \qquad f(u_4) = F_4,$$
$$f(u_5) = F_9.$$

Case 3: Without loss of generality, we may take the duplication of a central vertex $w = v_1$ by an edge $e = v_1'v_1''$ in G_z.

$$f(v_0) = F_1, \qquad f(u_1) = F_5,$$
$$f(v_1) = F_3, \qquad f(u_2) = F_6,$$
$$f(v_2) = F_7, \qquad f(u_3) = F_8,$$
$$f(v_3) = F_{10}, \qquad f(u_4) = F_4,$$
$$f(v_4) = F_2, \qquad f(u_5) = F_9.$$
$$f(v_5) = F_{11},$$
$$f(v_1') = F_{12},$$
$$f(v_1'') = F_{13},$$

From above all the cases, we observe that $|e_f(0) - e_f(1)| = 1$. So, G is a Fibonacci divisor cordial graph. □

Example 2.9. *The graph made from duplicating of vertex v_0 by an edge $e = v_0'v_0''$ in G_z is a Fibonacci cordial graph as shown in Figure 6.*

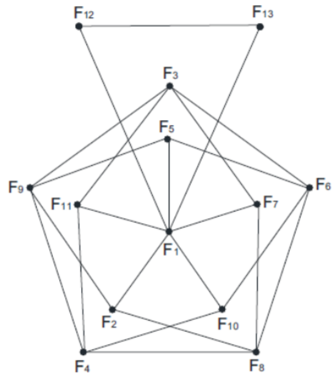

Figure 6:

Theorem 2.10. *The graph made from switching of an arbitrary vertex of degree four in G_z is a Fibonacci divisor cordial graph.*

Proof. Let G_z be a Grötzsch graph and let v_0 be the central vertex and $v_1, v_2, v_3, v_4, v_5, u_1, u_2, u_3, u_4, u_5$ be the remaining vertices of the G_z. Let G be the graph made from switching an arbitrary vertex of degree four in G. We define labeling function $f : V(G_z) \rightarrow \{F_1, F_2, \ldots, F_{11}\}$ as follows:
Without loss of generality, we may take the switching of a vertex u_1 in G.

$$f(v_0) = F_1, \quad f(u_1) = F_3,$$
$$f(v_1) = F_7, \quad f(u_2) = F_5,$$
$$f(v_2) = F_6, \quad f(u_3) = F_8,$$
$$f(v_3) = F_{10}, \quad f(u_4) = F_4,$$
$$f(v_4) = F_2, \quad f(u_5) = F_9.$$
$$f(v_5) = F_{11},$$

From the above labeling pattern, we have $\left|e_f(0) - e_f(1)\right| = 1$. So, G is a Fibonacci divisor cordial graph. □

Example 2.11. *The graph made from switching of vertex u_1 in G_z is a Fibonacci divisor cordial graph as shown in Figure 7.*

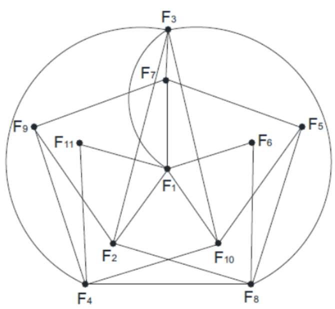

Figure 7:

Theorem 2.12. *The graph made from switching of an arbitrary vertex of degree three in G_z is a Fibonacci divisor cordial graph.*

Proof. Let G_z be a Grötzsch graph and let v_0 be the central vertex and $v_1, v_2, v_3, v_4, v_5, u_1, u_2, u_3, u_4, u_5$ be the remaining vertices of the G_z. Let G be the graph made from switching an arbitrary vertex of degree three in G. We define labeling function $f : V(G_z) \to \{F_1, F_2, \ldots, F_{11}\}$ as follows:
Without loss of generality, we may take the switching of a vertex u_1 in G.

$$f(v_0) = F_1, \qquad f(u_1) = F_6,$$
$$f(v_1) = F_2, \qquad f(u_2) = F_4,$$
$$f(v_2) = F_3, \qquad f(u_3) = F_5,$$
$$f(v_3) = F_7, \qquad f(u_4) = F_{11},$$
$$f(v_4) = F_8, \qquad f(u_5) = F_9.$$
$$f(v_5) = F_{10},$$

From the above labeling pattern, we have $|e_f(0) - e_f(1)| = 1$. So, G is a Fibonacci divisor cordial graph. □

Example 2.13. *A graph made from switching of an arbitrary vertex v_1 in G_z is a Fibonacci Divisor cordial graph as shown in Figure 8.*

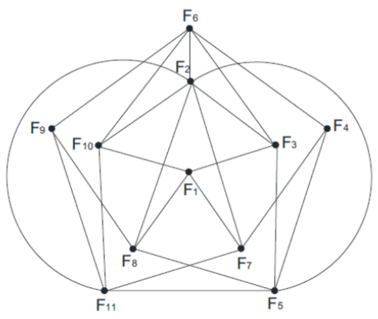

Figure 8:

Theorem 2.14. *The graph made from path union of two copies of G_z graph is a Fibonacci divisor cordial graph.*

Proof. Consider two copies of Grötzsch graph G_z and G'_z respectively. Let $V(G_z) = \{v_0, v_i, u_i : 1 \leq i \leq 5\}$ and $V(G'_z) = \{v'_0, v'_i, u'_i : 1 \leq i \leq 5\}$. Then $|V(G_z)| = 11$ and $|E(G_z)| = 20$ and $|V(G'_z)| = 11$ and $|E(G'_z)| = 20$. Let G be the graph made from the path union of two copies of Grötzsch graph G_z and G'_z. Then $V(G) = V(G_z) \cup V(G'_z)$ and $E(G) = E(G_z) \cup E(G'_z) \cup \{u_1\ u'_1\}$. Note that G has 22 vertices and 41 edges.
We define labeling function $f : V(G_z) \to \{F_1, F_2, \ldots, F_{22}\}$ as follows:

$$f(v_0) = F_1, \qquad f(u_1) = F_7,$$

$$
\begin{aligned}
f(v_1) &= F_2, & f(u_2) &= F_6,\\
f(v_2) &= F_{14}, & f(u_3) &= F_{11},\\
f(v_3) &= F_{17}, & f(u_4) &= F_{13},\\
f(v_4) &= F_{19}, & f(u_5) &= F_{16},\\
f(v_5) &= F_{22}, & f(u'_1) &= F_5,\\
f(v'_0) &= F_3, & f(u'_2) &= F_{10},\\
f(v'_1) &= F_9, & f(u'_3) &= F_4,\\
f(v'_2) &= F_{12}, & f(u'_4) &= F_8,\\
f(v'_3) &= F_{15}, & f(u'_5) &= F_{20}.\\
f(v'_4) &= F_{18}, & &\\
f(v'_5) &= F_{21}, & &
\end{aligned}
$$

From the above labeling pattern, we have $|e_f(0) - e_f(1)| = 1$. So, G is a Fibonacci divisor cordial graph. □

Example 2.15. *The graph made from path union of two copies of Grötzsch graph G_z is a Fibonacci divisor cordial graph as shown in Figure 9.*

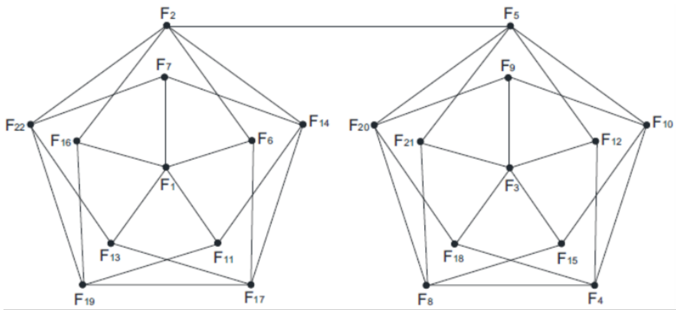

Figure 9:

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