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Quaternion Fractional Fourier-Laplace Transform and its Convolution Structure

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Abstract

In this paper, we introduce the two-sided Quaternion Fractional Fourier–Laplace Transform (QFrFLT), an extension of the classical Fractional Fourier–Laplace Transform into the quaternion framework. We rigorously define the QFrFLT and establish its reversibility property, as well as develop an associated convolution structure along with a convolution theorem. These results not only advance the theoretical foundation of hypercomplex transforms but also demonstrate the potential of the QFrFLT for applications in multidimensional signal analysis, image processing.

Keywords: Fractional Fourier transform; Fractional Laplace transform; Fractional Fourier-Laplace transform; Quaternion Fractional Fourier-Laplace transform; Image processing.
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1. Introduction

Quaternions were first introduced by Hamilton in 1843 [1] and have since served as a powerful extension of complex numbers for representing multidimensional signals. Over the years, a number of quaternionic integral transforms have been developed, including the quaternion Fourier Transform (QFT) [2, 3] and its fractional counterpart, the fractional quaternion Fourier Transform (FRQFT) [4]. These transforms have proven particularly useful in applications such as color image processing, multidimensional signal analysis, and pattern recognition, due to their ability to encode both amplitude and directional (phase) information in a hypercomplex framework [2, 5, 6].
Traditional quaternionic transforms have primarily focused on extending the Fourier analysis framework to the quaternion domain. However, with the growing need to address signals and systems exhibiting both oscillatory and damping behaviour, a unified transform that incorporates the advantages of both the fractional Fourier transform (FrFT) and the fractional Laplace transforms (FrLT) is highly desirable. In this context, the Fractional Fourier–Laplace Transform (FrFLT) has recently emerged as a versatile tool, unifying features of both FrFT and FrLT [7, 8, 9]. Building on this idea, we introduce the Quaternion Fractional Fourier–Laplace Transform (QFrFLT) in this paper.

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In our work, we define the QFrFLT as a natural extension of the classical FrFLT to the quaternion setting. We rigorously establish its reversibility property (i.e. the inversion formula) and develop a convolution structure for the QFrFLT. These theoretical results extend the quaternionic transform framework and provide a solid mathematical foundation for practical applications in multidimensional and color image processing.
The remainder of this paper is organized as follows. In Section 2 we present the necessary preliminaries, including the definition of the fractional Fourier-Laplace transform (FrFLT), which serves as the basis for our subsequent extension to the quaternion setting. In Section 3, we introduce the definition of the QFrFLT and prove its reversibility. In Section 4, we develop a convolution structure tailored to the QFrFLT and derive the associated convolution theorem. Finally, in the last section, Section 6, we conclude with a discussion of our findings and their potential applications.

2. Preliminaries

2.1 The quaternion algebra

Quaternions were introduced by Hamilton in 1843 as a natural extension of the complex numbers. A quaternion is expressed in the form

$$q = a + bi + cj + dk, \tag{1}$$

where a, b, c and d are real numbers and i, j, k are symbols that can be interpreted as unit-vectors pointing along three spatial axes and hence satisfies

$$i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j.$$

The conjugate of q is given by

$$\bar{q} = a - bi - cj - dk$$

and its norm is defined as

$$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{q\bar{q}}$$

This norm satisfies the multiplicative property $\|pq\| = \|p\| \|q\|$ for any quaternions p and q, and every nonzero quaternion has an inverse given by $q^{-1} = \frac{\bar{q}}{\|q\|^2}$. The set of quaternions denoted by $\mathbb{H}$ form a four-dimensional vector space over $\mathbb{R}$ with the basis $\{1, i, j, k\}$ and obey a multiplication law in which the order of factors is significant, i.e. in general, $pq \neq qp$. Although the multiplication is non-commutative, the associative law holds, so that $(pq)\, r = p\,(qr)$ for all quaternions p, q, and r. These features provide a robust numerical framework for extending classical analysis to higher dimensions and underpin the theoretical development of quaternion-based transforms.

2.2 Fractional Fourier-Laplace transform

The fractional Fourier-Laplace transform (FrFLT) with angle parameters α and θ of $f(x,t)$ is denoted by $\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}(u,v)$ and is defined as [9]

$$\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}(u,v) = F_{\alpha,\theta}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)\, K_{\alpha,\theta}(x,u,t,v)\; dx\, dt, \tag{2}$$

where

$$K_{\alpha,\theta}(x,u,t,v) = \begin{cases} C(\alpha,\theta)\, e^{i\left[a(\alpha)(x^2+u^2)-p(\alpha)xu\right]} e^{\left[b(\theta)(t^2+v^2)-q(\theta)tv\right]}, & \text{if } \alpha,\theta \notin \{k\pi\ : k\in\mathbb{Z}\} \\ C_1(\alpha)\,\delta(t-v)\, e^{i\left[a(\alpha)(x^2+u^2)-p(\alpha)xu\right]}, & \text{if } \alpha \neq m\pi \text{ and } \theta = 2n\pi \\ C_1(\alpha)\,\delta(t+v)\, e^{i\left[a(\alpha)(x^2+u^2)-p(\alpha)xu\right]}, & \text{if } \alpha \neq m\pi \text{ and } \theta = (2n-1)\pi \\ C_2(\theta)\,\delta(x-u)\, e^{\left[b(\theta)(t^2+v^2)-q(\theta)tv\right]}, & \text{if } \alpha = 2m\pi \text{ and } \theta \neq n\pi \\ C_2(\theta)\,\delta(x+u)\, e^{\left[b(\theta)(t^2+v^2)-q(\theta)tv\right]}, & \text{if } \alpha = (2m-1)\pi \text{ and } \theta \neq n\pi \\ \delta(x-u)\,\delta(t-v), & \text{if } \alpha,\theta \in \{2k\pi\ : k\in\mathbb{Z}\} \\ \delta(x-u)\,\delta(t+v), & \text{if } \alpha = 2m\pi \text{ and } \theta = (2n-1)\pi \\ \delta(x+u)\,\delta(t-v), & \text{if } \alpha = (2m-1)\,\pi \text{ and } \theta = 2n\pi \\ \delta(x+u)\,\delta(t+v), & \text{if } \alpha,\theta \in \{(2k-1)\,\pi : k\in\mathbb{Z}\} \end{cases} \tag{3}$$

With

$$C_1(\alpha) = \sqrt{\frac{1-i\cot\alpha}{2\pi}},\quad C_2(\theta) = \sqrt{\frac{1-i\cot\theta}{2\pi i}},\quad C(\alpha,\theta) = C_1(\alpha)\,C_2(\theta),$$

$$a(\alpha) = \frac{\cot\alpha}{2},\;\; b(\theta) = \frac{\cot\theta}{2},\;\; p(\alpha) = \csc\alpha\,,\quad q(\theta) = \csc\theta\,,$$

and δ denotes the Dirac delta function. Throughout this paper the constants $C_1(\alpha)$, $C_2(\theta)$, $C(\alpha,\theta)$, $a(\alpha)$, $b(\theta)$, $p(\alpha)$ and $q(\theta)$ will denote these values and for simplicity we may write them as C_1, C_2, C, a, b, p and q respectively. Based on the analysis of FrFLT for different parameter values of α and θ in [9], FrFLT reduces to different well-known transforms (such as identity transform, reflection transforms, Fractional Fourier transform, Fractional Laplace transform and Fourier-Laplace transform if at least one of α, θ is integral multiple of π. Hence, as in [9] we confined our attention to $F_{\alpha,\theta}$ for $\alpha,\theta \notin \{k\pi\ : k\in\mathbb{Z}\}$. In this case the kernel of FrFLT is:

$$K_{\alpha,\theta}(x,u,t,v) = C(\alpha,\theta)\, e^{i\left[a(x^2+u^2)-pxu\right]} e^{\left[b(t^2+v^2)-qtv\right]} \tag{4}$$

that is,

$$K_{\alpha,\theta}(x,u,t,v) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\sqrt{\frac{1-i\cot\theta}{2\pi i}}\, e^{\frac{i}{2\sin\alpha}\left[\left(x^2+u^2\right)\cos\alpha-2xu\right]} e^{\frac{1}{2\sin\theta}\left[\left(t^2+v^2\right)\cos\theta-2tv\right]} \tag{5}$$

3. Quaternion Fractional Fourier-Laplace Transform

For any quaternion signal

$$f\left(x_1,\ t_1,x_2,t_2\right)=f_r\left(x_1,\ t_1,x_2,t_2\right)+if_i\left(x_1,\ t_1,x_2,t_2\right)+jf_j\left(x_1,\ t_1,x_2,t_2\right)+kf_k\left(x_1,t_1,x_2,t_2\right)$$

where f_r, f_i, f_j, and f_k are real valued, the quaternion fractional Fourier-Laplace transform of f is defined as

$$\begin{aligned}&\mathcal{FL}^{i,j}_{\alpha_1,\theta_1,\alpha_2,\theta_2}\left\{f(x_1,t_1,x_2,t_2)\right\}\left(u_1,v_1,u_2,v_2\right)=F^{i,j}_{\alpha_1,\theta_1,\ \alpha_2,\theta_2}\left(u_1,v_1,u_2,v_2\right)\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^i_{\alpha_1,\theta_1}\left(x_1,u_1,t_1,v_1\right)f\left(x_1,\ x_2,t_1,t_2\right)K^j_{\alpha_2,\theta_2}\left(x_2,u_2,t_2,v_2\right)dx_1dt_1dx_2dt_2\end{aligned}\tag{6}$$

where

$$K^i_{\alpha_1,\theta_1}\left(x_1,u_1,t_1,v_1\right)=\sqrt{\frac{1-i\cot\alpha_1}{2\pi}}\sqrt{\frac{1-i\cot\theta_1}{2\pi i}}e^{\frac{i}{2\sin\alpha_1}\ \left[\left(x_1^2+u_1^2\right)\cos\alpha_1\ -2x_1u_1\right]}e^{\frac{1}{2\sin\theta_1}\ \left[\left(t_1^2+v_1^2\right)\cos\theta_1\ -2t_1v_1\right]},$$

$$K^j_{\alpha_2,\theta_2}\left(x_2,u_2,t_2,v_2\right)=\sqrt{\frac{1-j\cot\alpha_1}{2\pi}}\sqrt{\frac{1-j\cot\theta_1}{2\pi j}}e^{\frac{j}{2\sin\alpha_2}\ \left[\left(x_2^2+u_2^2\right)\cos\alpha_2\ -2x_2u_2\right]}e^{\frac{1}{2\sin\theta_2}\ \left[\left(t_2^2+v_2^2\right)\cos\theta_2\ -2t_2v_2\right]}$$

for $\alpha_1,\theta_1,\alpha_2,\theta_2\notin\{k\pi\ :k\in\mathbb{Z}\}$. Further for simplicity, let us denote

$$C^i_{\alpha_1}=\sqrt{\frac{1-i\cot\alpha_1}{2\pi}},\ \ C^i_{\theta_1}=\sqrt{\frac{1-i\cot\theta_1}{2\pi i}},\ \ C^j_{\alpha_2}=\sqrt{\frac{1-j\cot\alpha_2}{2\pi}},\quad C^j_{\theta_2}=\sqrt{\frac{1-j\cot\theta_2}{2\pi j}},$$

$$a_1=\frac{\cot\alpha_1}{2},\ \ b_1=\frac{\cot\theta_1}{2},\ \ a_2=\frac{\cot\alpha_2}{2},\quad b_2=\frac{\cot\theta_2}{2},$$

and

$$p_1=\csc\alpha_1\ ,\ q_1=\csc\theta_1\ ,\ p_2=\csc\alpha_2\ ,\ q_2=\csc\theta_2$$

then $K^i_{\alpha_1,\theta_1}\left(x,u,t,v\right)$ and $K^j_{\alpha_2,\theta_2}\left(x,u,t,v\right)$ takes the form

$$K^i_{\alpha_1,\theta_1}\left(x,u,t,v\right)=C^i_{\alpha_1}C^i_{\theta_1}e^{i\left[a_1(x_1^2+u_1^2)-x_1u_1p_1\right]}e^{\left[b_1\left(t_1^2+v_1^2\right)-t_1v_1q_1\right]},\tag{7}$$

$$K^j_{\alpha_2,\theta_2}\left(x,u,t,v\right)=C^j_{\alpha_2}C^j_{\theta_2}e^{j\left[a_2(x_2^2+u_2^2)-x_2u_2p_2\right]}e^{\left[b_2\left(t_2^2+v_2^2\right)-t_2v_2q_2\right]}\tag{8}$$

Theorem 3.1 (Reversibility Property). *If* $\mathcal{FL}^{i,j}_{\alpha_1,\theta_1,\alpha_2,\theta_2}\left\{f(x_1,t_1,x_2,t_2)\right\}\left(u_1,u_2,v_1,v_2\right)$ $=F^{i,j}_{\alpha_1,\theta_1,\ \alpha_2,\theta_2}\left(u_1,u_2,v_1,v_2\right)$*, then it is possible to reconstruct quaternion signal* $f(x_1,t_1,x_2,t_2)$ *from the quaternion fractional Fourier-Laplace transform* $F^{i,j}_{\alpha_1,\theta_1,\alpha_2,\theta_2}\left(u_1,u_2,v_1,v_2\right)$.

Proof. By definition of quaternion fractional Fourier-Laplace transform we have,

$$\begin{aligned}&\mathcal{FL}^{i,j}_{-\alpha_1,-\theta_1,-\alpha_2,-\theta_2}\left\{F^{i,j}_{\alpha_1,\theta_1,\ \alpha_2,\theta_2}\left(u_1,v_1,u_2,v_2\right)\right\}\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^i_{-\alpha_1,-\theta_1}\left(u_1,y_1,v_1,z_1\right)F^{i,j}_{\alpha_1,\theta_1,\alpha_2,\theta_2}\left(u_1,v_1,u_2,v_2\right)\end{aligned}$$

$$\times K^{j}_{-\alpha_2,-\theta_2}\left(u_2,y_2,v_2,z_2\right)du_1dv_1du_2dv_2$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^{i}_{-\alpha_1,-\theta_1}\left(u_1,y_1,v_1,z_1\right)\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^{i}_{\alpha_1,\theta_1}\left(x_1,u_1,t_1,v_1\right)\right.$$

$$\left.f\left(x_1,\ t_1,x_2,t_2\right)K^{j}_{\alpha_2,\theta_2}\left(x_2,u_2,t_2,v_2\right)dx_1dt_1dx_2dt_2\right\}K^{j}_{-\alpha_2,-\theta_2}\left(u_2,y_2,v_2,z_2\right)du_1dv_1du_2dv_2$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^{i}_{-\alpha_1,-\theta_1}\left(u_1,y_1,v_1,z_1\right)\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^{i}_{\alpha_1,\theta_1}\left(x_1,u_1,t_1,v_1\right)\left[f_r\left(x_1,\ t_1,x_2,t_2\right)\right.\right.$$

$$\left.\left.+if_i\left(x_1,\ t_1,x_2,t_2\right)+jf_j\left(x_1,\ t_1,x_2,t_2\right)+kf_k\left(x_1,\ t_1,x_2,t_2\right)\right]\ K^{j}_{\alpha_2,\theta_2}\left(x_2,u_2,t_2,v_2\right)dx_1\ dt_1\ dx_2\ dt_2\right\}$$

$$\times\ K^{j}_{-\alpha_2,-\theta_2}\left(u_2,y_2,v_2,z_2\right)du_1dv_1du_2dv_2$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_r\left(x_1,\ t_1,x_2,t_2\right)\left\{\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^{i}_{-\alpha_1,-\theta_1}\left(u_1,y_1,v_1,z_1\right)K^{i}_{\alpha_1,\theta_1}\left(x_1,u_1,t_1,v_1\right)\ du_1\ dv_1\right]\right.$$

$$\left.\times\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^{j}_{-\alpha_2,-\theta_2}\left(u_2,y_2,v_2,z_2\right)K^{j}_{\alpha_2,\theta_2}\left(x_2,u_2,t_2,v_2\right)du_2dv_2\right]\right\}dx_1dt_1dx_2dt_2$$

$$+i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_i\left(x_1,\ t_1,x_2,t_2\right)\left\{\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^{i}_{-\alpha_1,-\theta_1}\left(u_1,y_1,v_1,z_1\right)K^{i}_{\alpha_1,\theta_1}\left(x_1,u_1,t_1,v_1\right)\ du_1\ dv_1\right]\right.$$

$$\left.\times\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^{j}_{-\alpha_2,-\theta_2}\left(u_2,y_2,v_2,z_2\right)K^{j}_{\alpha_2,\theta_2}\left(x_2,u_2,t_2,v_2\right)du_2dv_2\right]\right\}dx_1dt_1dx_2dt_2$$

$$+j\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_j\left(x_1,t_1,x_2,t_2\right)\left\{\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^{i}_{-\alpha_1,-\theta_1}\left(u_1,y_1,v_1,z_1\right)K^{i}_{\alpha_1,\theta_1}\left(x_1,u_1,t_1,v_1\right)\ du_1\ dv_1\right]\right.$$

$$\left.\times\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^{j}_{-\alpha_2,-\theta_2}\left(u_2,y_2,v_2,z_2\right)K^{j}_{\alpha_2,\theta_2}\left(x_2,u_2,t_2,v_2\right)du_2dv_2\right]\right\}dx_1dt_1dx_2dt_2$$

$$+k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_k\left(x_1,t_1,x_2,t_2\right)\left\{\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^{i}_{-\alpha_1,-\theta_1}\left(u_1,y_1,v_1,z_1\right)K^{i}_{\alpha_1,\theta_1}\left(x_1,u_1,t_1,v_1\right)\ du_1\ dv_1\right]\right.$$

$$\left.\times\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^{j}_{-\alpha_2,-\theta_2}\left(u_2,y_2,v_2,z_2\right)K^{j}_{\alpha_2,\theta_2}\left(x_2,u_2,t_2,v_2\right)du_2dv_2\right]\right\}dx_1\ dt_1\ dx_2\ dt_2$$

Using properties of kernel of FrFLT, right hand side of above expression becomes

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_r\left(x_1,t_1,x_2,t_2\right)\left\{\delta\left(x_1-y_1\right)\delta\left(t_1-z_1\right)\delta\left(x_2-y_2\right)\ \delta\left(t_2-z_2\right)\right\}dx_1\ dt_1\ dx_2\ dt_2$$

$$+i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_i\left(x_1,t_1,x_2,t_2\right)\left\{\delta\left(x_1-y_1\right)\delta\left(t_1-z_1\right)\delta\left(x_2-y_2\right)\ \delta\left(t_2-z_2\right)\right\}dx_1\ dt_1\ dx_2\ dt_2$$

$$+j\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_j\left(x_1,t_1,x_2,t_2\right)\left\{\delta\left(x_1-y_1\right)\delta\left(t_1-z_1\right)\delta\left(x_2-y_2\right)\ \delta\left(t_2-z_2\right)\right\}dx_1\ dt_1\ dx_2\ dt_2$$

$$+k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_k\left(x_1,t_1,x_2,t_2\right)\left\{\delta\left(x_1-y_1\right)\delta\left(t_1-z_1\right)\delta\left(x_2-y_2\right)\ \delta\left(t_2-z_2\right)\right\}dx_1\ dt_1\ dx_2\ dt_2$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[f_r\left(x_1,\ t_1,x_2,t_2\right)+if_i\left(x_1,\ t_1,x_2,t_2\right)+jf_j\left(x_1,\ t_1,x_2,t_2\right)kf_k\left(x_1,\ t_1,x_2,t_2\right)\right]$$

$$\times\left\{\delta\left(x_1-y_1\right)\delta\left(t_1-z_1\right)\delta\left(x_2-y_2\right)\ \delta\left(t_2-z_2\right)\right\}dx_1dt_1dx_2dt_2$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f\left(x_1,\ t_1,x_2,t_2\right)\delta\left(x_1-y_1,\ t_1-z_1,x_2-y_2,t_2-z_2\right)dx_1dt_1dx_2dt_2$$

$= f(x_1,t_1,x_2,t_2)$ (By using the Sifting property of Dirac delta)

□

4. Convolution Structure of QFrFLT

To obtain the convolution structure of quaternion fractional Fourier-Laplace transform (QFrFLT), we have introduced the following definitions:

Definition 4.1. *The convolution of two, four-dimensional functions* $f(x_1, t_1, x_2, t_2)$ *and* $g(x_1, t_1, x_2, t_2)$ *is defined by*

$$(f * g)(x_1, t_1, x_2, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r_1, s_1, r_2, s_2)\, g(x_1 - r_1, t_1 - s_1, x_2 - r_2, t_2 - s_2) dr_1 ds_1 dr_2 ds_2 \tag{9}$$

Definition 4.2. *For any quaternion function* $f(x_1, t_1, x_2, t_2), g(x_1, t_1, x_2, t_2)$ *we define* $\tilde{f}(x_1, t_1, x_2, t_2)$ *and* $\tilde{g}(x_1, t_1, x_2, t_2)$ *by*

$$\tilde{f}(x_1, t_1, x_2, t_2) = f(x_1, t_1, x_2, t_2)\, e^{i\left(a_1 x_1^2 - i b_1 t_1^2\right)} e^{j\left(a_2 x_2^2 - j b_2 t_2^2\right)},$$
$$\tilde{g}(x_1, t_1, x_2, t_2) = g(x_1, t_1, x_2, t_2)\, e^{i\left(a_1 x_1^2 - i b_1 t_1^2\right)} e^{j\left(a_2 x_2^2 - j b_2 t_2^2\right)},$$

where $a_1 = a_1(\alpha)$, $a_1 = a_2(\alpha)$, $b_1 = b_1(\theta)$, *and* $b_2 = b_2(\theta)$ *are as given in the definition of kernel of QFrFLT. Then we define the convolution operation* $\star$ *for QFrFLT by*

$$\begin{aligned} h(x_1, t_1, x_2, t_2) &= (f \star g)(x_1, t_1, x_2, t_2) \\ &= C^{i,j} e^{-i\left(a_1 x_1^2 - i b_1 t_1^2\right)} e^{-j\left(a_2 x_2^2 - j b_2 t_2^2\right)} \left(\tilde{f} * \tilde{g}\right)(x_1, t_1, x_2, t_2), \end{aligned} \tag{10}$$

where $*$ *is the convolution operation given in (9) and*

$$C^{i,j} = C^{i,j}(\alpha_1, \theta_1, \alpha_2, \theta_2) = C^i_{\alpha_1} C^i_{\theta_1} C^j_{\alpha_2} C^j_{\theta_2},$$

where the individual constants $C^i_{\alpha_1}$, $C^i_{\theta_1}$, $C^j_{\alpha_2}$, *and* $C^j_{\theta_2}$ *comes from the definitions of the kernel of QFrFLT. Thus, equation (10) gives the convolution structure of the QFrFLT.*

Note: Since, f, g, and h are quaternions and hence they can be written as

$$h(x_1, t_1, x_2, t_2) = h_r(x_1, t_1, x_2, t_2) + i h_i(x_1,\ t_1, x_2, t_2) + j h_j(x_1, t_1, x_2, t_2) + k h_k(x_1, t_1, x_2, t_2)$$

where h_r, h_i, h_j, and h_k are real valued functions. Similarly, $f(x_1, t_1, x_2, t_2)$ and $g(x_1, t_1, x_2, t_2)$ can also be written in above form. Thus, equation (10) gives

$$\begin{aligned} h_l(x_1, t_1, x_2, t_2) &= C^{i,j} e^{-i\left(a_1 x_1^2 - i b_1 t_1^2\right)} e^{-j\left(a_2 x_2^2 - j b_2 t_2^2\right)} \left(\widetilde{f_l} * \widetilde{g_l}\right)(x_1, t_1, x_2, t_2) \\ &= C^{i,j} e^{-i\left(a_1 x_1^2 - i b_1 t_1^2\right)} e^{-j\left(a_2 x_2^2 - j b_2 t_2^2\right)} \\ &\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{f_l}(r_1, s_1, r_2, s_2)\, \widetilde{g_l}(x_1 - r_1, t_1 - s_1, x_2 - r_2, t_2 - s_2)\, dr_1 ds_1 dr_2 ds_2 \end{aligned}$$

$$= C^{i,j} e^{-i\left(a_1 x_1^2 - i b_1 t_1^2\right)} e^{-j\left(a_2 x_2^2 - j b_2 t_2^2\right)}$$
$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_l\left(r_1, s_1, r_2, s_2\right) e^{i\left(a_1 r_1^2 - i b_1 s_1^2\right)} e^{j\left(a_2 r_2^2 - j b_2 s_2^2\right)} g_l\left(x_1 - r_1, t_1 - s_1, x_2 - r_2, t_2 - s_2\right))$$
$$\times e^{i\left(a_1 (x_1 - r_1)^2 - i b_1 (t_1 - s_1)^2\right)} e^{j\left(a_2 (x_2 - r_2)^2 - j b_2 (t_2 - s_2)^2\right)} dr_1 ds_1 dr_2 ds_2 \tag{11}$$

where $l \in \{r, i, j, k\}$.

Theorem 4.3 (Convolution theorem). *Let* $h\left(x_1, t_1, x_2, t_2\right) = (f \star g)\left(x_1, t_1, x_2, t_2\right)$ *and* $F^{i,j}_{\alpha_1,\theta_1,\ \alpha_2,\theta_2}\left(u_1, v_1, u_2, v_2\right)$, $G^{i,j}_{\alpha_1,\theta_1,\alpha_2,\theta_2}\left(u_1, v_1, u_2, v_2\right)$ *and* $H^{i,j}_{\alpha_1,\theta_1,\ \alpha_2,\theta_2}\left(u_1, v_1, u_2, v_2\right)$ *denotes the quaternion fractional Fourier-Laplace transforms (QFrFLT) of* f, g *and* h *respectively. Then*

$$H^{i,j}_{\alpha_1,\theta_1,\alpha_2,\theta_2}\left(u_1, v_1, u_2, v_2\right) = e^{-i\left(a_1 u_1^2 - i b_1 v_1^2\right)} e^{-j\left(a_2 u_2^2 - j b_2 u_2^2\right)}$$
$$\times \left[F^{i,j}_{\alpha_1,\theta_1,\ \alpha_2,\theta_2}\left(u_1, v_1, u_2, v_2\right) \odot G^{i,j}_{\alpha_1,\theta_1,\ \alpha_2,\theta_2}\left(u_1, v_1, u_2, v_2\right)\right],$$

where $\odot$ *is the componentise product.*

Proof. By the definition of QFrFLT we have,

$$H^{i,j}_{\alpha_1,\theta_1,\alpha_2,\theta_2}\left(u_1, v_1, u_2, v_2\right)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^i_{\alpha_1,\theta_1}\left(x_1, u_1, t_1, v_1\right) h\left(x_1, x_2, t_1, t_2\right) K^j_{\alpha_2,\theta_2}\left(x_2, u_2, t_2, v_2\right) dx_1 dt_1 dx_2 dt_2$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^i_{\alpha_1,\theta_1}\left(x_1, u_1, t_1, v_1\right) h_r\left(x_1, x_2, t_1, t_2\right) K^j_{\alpha_2,\theta_2}\left(x_2, u_2, t_2, v_2\right) dx_1 dt_1 dx_2 dt_2$$
$$+ i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^i_{\alpha_1,\theta_1}\left(x_1, u_1, t_1, v_1\right) h_i\left(x_1, x_2, t_1, t_2\right) K^j_{\alpha_2,\theta_2}\left(x_2, u_2, t_2, v_2\right) dx_1 dt_1 dx_2 dt_2$$
$$+ j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^i_{\alpha_1,\theta_1}\left(x_1, u_1, t_1, v_1\right) h_j\left(x_1, x_2, t_1, t_2\right) K^j_{\alpha_2,\theta_2}\left(x_2, u_2, t_2, v_2\right) dx_1 dt_1 dx_2 dt_2$$
$$+ k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^i_{\alpha_1,\theta_1}\left(x_1, u_1, t_1, v_1\right) h_k\left(x_1, x_2, t_1, t_2\right) K^j_{\alpha_2,\theta_2}\left(x_2, u_2, t_2, v_2\right) dx_1 dt_1 dx_2 dt_2$$

By using equation (11) we have, for each $l \in \{r, i, j, k\}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^i_{\alpha_1,\theta_1}\left(x_1, u_1, t_1, v_1\right) h_l\left(x_1, x_2, t_1, t_2\right) K^j_{\alpha_2,\theta_2}\left(x_2, u_2, t_2, v_2\right) dx_1 dt_1 dx_2 dt_2$$
$$= C^{i,j} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^i_{\alpha_1,\theta_1}\left(x_1, u_1, t_1, v_1\right) \left\{ e^{-i\left(a_1 x_1^2 - i b_1 t_1^2\right)} e^{-j\left(a_2 x_2^2 - j b_2 t_2^2\right)}\right.$$
$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_l\left(r_1, s_1, r_2, s_2\right) e^{i\left(a_1 r_1^2 - i b_1 s_1^2\right)} e^{j\left(a_2 r_2^2 - j b_2 s_2^2\right)}$$
$$g_l\left(x_1 - r_1, t_1 - s_1, x_2 - r_2, t_2 - s_2\right) e^{i\left(a_1 (x_1 - r_1)^2 - i b_1 (t_1 - s_1)^2\right)} e^{j\left(a_2 (x_2 - r_2)^2 - j b_2 (t_2 - s_2)^2\right)}$$
$$\left. dr_1 ds_1 dr_2 ds_2 \right\} K^j_{\alpha_2,\theta_2}\left(x_2, u_2, t_2, v_2\right) dx_1 dt_1 dx_2 dt_2$$

Substituting $x_1 - r_1 = m_1,\ \ t_1 - s_1 = n_1,\ \ x_2 - r_2 = m_2,\ t_2 - s_2 = n_2 \Rightarrow x_1 = m_1 + r_1, t_1 = n_1 + s_1 x_2 = m_2 + r_2, t_2 = n_2 + s_2$ in above and after simplifying we can write right hand side as

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^i_{\alpha_1,\theta_1}\left(r_1, u_1, s_1, v_1\right) f_l\left(r_1, s_1, r_2, s_2\right) K^j_{\alpha_2,\theta_2}\left(r_2, u_2, s_2, v_2\right) dr_1 ds_1 dr_2 ds_2$$

$$\times e^{-ia_1u_1^2}e^{-ja_2u_2^2}e^{-b_1v_1^2}e^{-b_2v_2^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K^i_{\alpha_1,\theta_1}(m_1,u_1,n_1,v_1)$$
$$\times g_l(m_1,n_1,m_2,n_2)\,K^j_{\alpha_2,\theta_2}(m_2,u_2,n_2,v_2)\,dm_1dn_1dm_2dn_2$$
$$= e^{-i\left(a_1u_1^2-ib_1v_1^2\right)}e^{-j\left(a_2u_2^2-jb_2u_2^2\right)}F_l(u_1,v_1,u_2,v_2)\,G_l(u_1,v_1,u_2,v_2) \quad (12)$$

where

$$F_l(u_1,v_1,u_2,v_2) = \mathcal{FL}^{i,j}_{\alpha_1,\theta_1,\ \alpha_2,\theta_2}\{f_l\}(u_1,v_1,u_2,v_2),$$
$$G_l(u_1,v_1,u_2,v_2) = \mathcal{FL}^{i,j}_{\alpha_1,\theta_1,\ \alpha_2,\theta_2}\{g_l\}(u_1,v_1,u_2,v_2)$$

i.e. F_l and G_l are the QFrFLT of f_l and g_l respectively. Since equation (12) holds for each $l \in \{r,i,j,k\}$. Hence, by using equation (12) we can write equation (12) as

$$\begin{aligned}
H^{i,j}_{\alpha_1,\theta_1,\alpha_2,\theta_2}(u_1,v_1,u_2,v_2) &= e^{-i\left(a_1u_1^2-ib_1v_1^2\right)}e^{-j\left(a_2u_2^2-jb_2u_2^2\right)}F_r(u_1,v_1,u_2,v_2)\,G_r(u_1,v_1,u_2,v_2)\\
&+ ie^{-i\left(a_1u_1^2-ib_1v_1^2\right)}e^{-j\left(a_2u_2^2-jb_2u_2^2\right)}F_i(u_1,v_1,u_2,v_2)\,G_i(u_1,v_1,u_2,v_2)\\
&+ je^{-i\left(a_1u_1^2-ib_1v_1^2\right)}e^{-j\left(a_2u_2^2-jb_2u_2^2\right)}F_j(u_1,v_1,u_2,v_2)\,G_j(u_1,v_1,u_2,v_2)\\
&+ ke^{-i\left(a_1u_1^2-ib_1v_1^2\right)}e^{-j\left(a_2u_2^2-jb_2u_2^2\right)}F_k(u_1,v_1,u_2,v_2)\ G_k(u_1,v_1,u_2,v_2)\\
&= e^{-i\left(a_1u_1^2-ib_1v_1^2\right)}e^{-j\left(a_2u_2^2-jb_2u_2^2\right)}\\
&\times\{F_r(u_1,v_1,u_2,v_2)\ G_r(u_1,v_1,u_2,v_2) + iF_i(u_1,v_1,u_2,v_2)\ G_i(u_1,v_1,u_2,v_2)\\
&+ jF_j(u_1,v_1,u_2,v_2)\ G_j(u_1,v_1,u_2,v_2) + kF_k(u_1,v_1,u_2,v_2)\ G_k(u_1,v_1,u_2,v_2)\}\\
&= e^{-i\left(a_1u_1^2-ib_1v_1^2\right)}e^{-j\left(a_2u_2^2-jb_2u_2^2\right)}\\
&\times\left[F^{i,j}_{\alpha_1,\theta_1,\ \alpha_2,\theta_2}(u_1,v_1,u_2,v_2)\odot G^{i,j}_{\alpha_1,\theta_1,\ \alpha_2,\theta_2}(u_1,v_1,u_2,v_2)\right]
\end{aligned}$$

This completes the proof. □

5. Conclusion

This paper presents a comprehensive development of the two-sided Quaternion Fractional Fourier–Laplace Transform (QFrFLT). We rigorously defined the QFrFLT, established its inversion formula, and proved its convolution theorem. Our results demonstrate that the QFrFLT effectively generalizes the fractional Fourier-Laplace transform to the quaternion setting. The theoretical findings offer a solid foundation for further research in hypercomplex signal processing, and the QFrFLT shows significant potential for practical applications in image processing, optics, and fractional dynamics. Future work will focus on algorithmic developments and the exploration of additional applications in engineering and mathematical physics.

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