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Correlation theorem for the Fractional Fourier-Laplace Transform and Applications

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Abstract

In this paper, we introduce a novel definition of the cross-correlation operation for the fractional Fourier-Laplace transform (FrFLT) and establish its corresponding correlation theorem. We further explore the application of the FrFLT to generalized wave equations, thereby enriching its theoretical foundation. Our findings not only enhance the mathematical framework underlying the FrFLT but also underscore its promising potential in engineering, mathematical physics, and signal processing.

Keywords: Fractional Fourier transform; Fractional Laplace transform; Fractional Fourier-Laplace transform; wave equation.

2020 Mathematics Subject Classification: 42A38, 44A20,35Q40, 46F12.

1. Introduction

Integral transforms are powerful mathematical tools employed to analyze and solve a wide range of problems in mathematics, physics, engineering, and various scientific fields [1]. By converting functions from one representation to another, these transforms often reveal hidden structures and simplify the complexity of differential equations and other advanced problems. A classical example is the Fourier transform, which decomposes a function into its frequency components, while the Laplace transform is particularly suited to solving differential equations exhibiting exponential behaviour [1, 2, 3].

Over the past century, mathematicians have significantly extended these classical transforms to tackle increasingly sophisticated issues-especially those involving singularities, discontinuities, or mixed physical behaviors [4]. The fractional Fourier transform (FrFT), rigorously introduced by Namias [5] using eigenfunction methods, extends the classical Fourier transform to fractional orders. The FrFT provides a continuous interpolation between time and frequency domains by effectively "rotating" the signal in the time–frequency plane, proving invaluable in optics and nonstationary signal processing [6, 7, 8, 9]. Similarly, the fractional Laplace transform (FrLT), as formulated by Torre [10], generalizes

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the classical Laplace transform to fractional orders and is particularly effective for solving parabolic differential equations and modeling quantum mechanical problems involving repulsive oscillators.

To overcome the limitations inherent in applying these fractional transforms individually, mixed integral transforms have been developed. For instance, the Fourier-Mellin transform (FMT) introduced in the 1970s combines Fourier's frequency analysis with Mellin's scale invariance, proving robust in digital image registration by effectively resolving translation, rotation, and scaling mismatches even in noisy environments [11]. Similarly, the double Laplace–Sumudu transform (DLST) integrates Laplace's exponential damping with the scaling properties of the Sumudu transform, offering a powerful tool for solving partial differential equations with mixed boundary conditions [12]. The triple Laplace–Sumudu transform further extends this concept to multidimensional initial-boundary value problems, demonstrating success in modeling heat flow and wave equations [13].

In our earlier works, we introduced the Fractional Fourier-Laplace Transform (FrFLT), a mixed integral transform that unifies features of both the fractional Fourier transform (FrFT) and the fractional Laplace transform (FrLT). Across these studies, we proved the convolution theorem for the FrFLT [14], product theorem for FrFLT [15], derived its operational transform formulae for FrFLT [16], and extended the framework to the quaternion setting by defining the Quaternion Fractional Fourier-Laplace Transform (QFrFLT) [17]. In this work, we extend the FrFLT to the space $\mathcal{E}'(\mathbb{R}^2)$ of distributions with compact support in $\mathbb{R}^2$. We rigorously prove a correlation theorem for the FrFLT and explore an application of the FrFLT to the generalized wave equation. The remainder of this paper is organized as follows. In Section 2, we present the necessary preliminaries, including the definition of the Fractional Fourier-Laplace Transform (FrFLT) and the testing function space $\mathcal{E}(\mathbb{R}^2)$. In Section 3, we extend the FrFLT to the space $\mathcal{E}'(\mathbb{R}^2)$ by defining the generalized Fractional Fourier-Laplace Transform. Section 4 develops a cross-correlation operation for the FrFLT and proves its correlation theorem. Section 5 provides an application of the FrFLT to the generalized wave equation. Finally, in Section 6, we conclude with a discussion of our findings and their potential applications.

2. Preliminaries

2.1 The Fractional Fourier-Laplace Transform

The fractional Fourier-Laplace transform (FrFLT) with angle parameters α and θ of $f(x,t)$ is denoted by $\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}(u,v)$ and is defined as

$$\begin{aligned}\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}(u,v) &= F_{\alpha,\theta}(u,v)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)\, K_{\alpha,\theta}(x,u,t,v)\, dxdt, \qquad (1)\end{aligned}$$

where

$$K_{\alpha,\theta}\left(x,u,t,v\right)=\begin{cases}C\left(\alpha,\theta\right)e^{i\left[a(\alpha)(x^2+u^2)-p(\alpha)xu\right]}e^{\left[b(\theta)(t^2+v^2)-q(\theta)tv\right]}, & \text{if } \alpha,\theta\notin\{k\pi\ : k\in\mathbb{Z}\}\\ C_1\left(\alpha\right)\delta\left(t-v\right)e^{i\left[a(\alpha)(x^2+u^2)-p(\alpha)xu\right]}, & \text{if } \alpha\neq m\pi \text{ and } \theta=2n\pi\\ C_1\left(\alpha\right)\delta\left(t+v\right)e^{i\left[a(\alpha)(x^2+u^2)-p(\alpha)xu\right]}, & \text{if } \alpha\neq m\pi \text{ and } \theta=(2n-1)\pi\\ C_2\left(\theta\right)\delta\left(x-u\right)e^{\left[b(\theta)(t^2+v^2)-q(\theta)tv\right]}, & \text{if } \alpha=2m\pi\text{and } \theta\neq n\pi\\ C_2\left(\theta\right)\delta\left(x+u\right)e^{\left[b(\theta)(t^2+v^2)-q(\theta)tv\right]}, & \text{if } \alpha=(2m-1)\pi \text{ and } \theta\neq n\pi\\ \delta\left(x-u\right)\delta\left(t-v\right), & \text{if } \alpha,\theta\in\{2k\pi\ : k\in\mathbb{Z}\}\\ \delta\left(x-u\right)\delta\left(t+v\right), & \text{if } \alpha=2m\pi\text{and } \theta=(2n-1)\pi\\ \delta\left(x+u\right)\delta\left(t-v\right), & \text{if } \alpha=(2m-1)\pi \text{ and } \theta=2n\pi\\ \delta\left(x+u\right)\delta\left(t+v\right), & \text{if } \alpha,\theta\in\{(2k-1)\pi : k\in\mathbb{Z}\}\end{cases}\tag{2}$$

With

$$C_1\left(\alpha\right)=\sqrt{\frac{1-i\cot\alpha}{2\pi}},\quad C_2\left(\theta\right)=\sqrt{\frac{1-i\cot\theta}{2\pi i}},\quad C\left(\alpha,\theta\right)=C_1\left(\alpha\right)C_2\left(\theta\right),$$

$$a\left(\alpha\right)=\frac{\cot\alpha}{2},\ \ b\left(\theta\right)=\frac{\cot\theta}{2},\ \ p\left(\alpha\right)=\csc\alpha\quad q\left(\theta\right)=\csc\theta\ ,$$

and δ denotes the Dirac delta function. Throughout this paper the constants $C_1(\alpha)$, $C_2\left(\theta\right)$, $C\left(\alpha,\theta\right)$, $a\left(\alpha\right)$, $b\left(\theta\right)$, $p\left(\alpha\right)$ and $q\left(\theta\right)$ will denote these values and for simplicity we may write them as C_1, C_2, C, a, b, p and q respectively. Based on the analysis of FrFLT for different parameter values of α and θ in [16], FrFLT reduces to different well-known transforms (such as identity transform, reflection transforms, Fractional Fourier transform, Fractional Laplace transform and Fourier-Laplace transform if at least one of α, θ is integral multiple of π. Hence, as in [16] we confined our attention to $F_{\alpha,\theta}$ for $\alpha,\theta\notin\{k\pi\ : k\in\mathbb{Z}\}$. In this case the kernel of FrFLT is:

$$K_{\alpha,\theta}\left(x,u,t,v\right)=C\left(\alpha,\theta\right)e^{i\left[a(x^2+u^2)-pxu\right]}e^{\left[b(t^2+v^2)-qtv\right]}\tag{3}$$

that is,

$$K_{\alpha,\theta}\left(x,u,t,v\right)=\sqrt{\frac{1-i\cot\alpha}{2\pi}}\sqrt{\frac{1-i\cot\theta}{2\pi i}}e^{\frac{i}{2\sin\alpha}\left[\left(x^2+u^2\right)\cos\alpha-2xu\right]}e^{\frac{1}{2\sin\theta}\left[\left(t^2+v^2\right)\cos\theta-2tv\right]}\tag{4}$$

2.2 Testing function space

The space $\mathcal{E}(\mathbb{R}^2)$ is a space of all complex valued smooth functions on $\mathbb{R}^2$. There is no restriction imposed on growth of any function $\phi\left(x,t\right)\in\mathcal{E}(\mathbb{R}^2)$. It is a multinormed space by assigning to it the following topology: For each compact subset K of $\mathbb{R}^2$ and each non-negative integer $k=(l,m)\in\mathbb{R}^2$, the seminorm is defined by

$$\gamma_{E,k}\left(\phi\right)=\sup_{(x,t)\in K}\left|D_x^l D_t^m\phi(x,t)\right|\tag{5}$$

A function ϕ belongs to $\mathcal{E}(\mathbb{R}^2)$ if for every compact subset K of $\mathbb{R}^2$ and each non-negative integer $k=(l,m)\in\mathbb{R}^2$, the seminorm $\gamma_{E,k}\left(\phi\right)<\infty$. The dual space $\mathcal{E}'(\mathbb{R}^2)$ of $\mathcal{E}(\mathbb{R}^2)$, is the space of all distributions with compact support in $\mathbb{R}^2$ (Zemanian [18]). Moreover, we say that f is a Fractional Fourier-Laplace transformable if it is a member of $\mathcal{E}'$, the dual space of $\mathcal{E}$.

3. Generalized Fractional Fourier-Laplace Transform

The generalized (distributional) fractional Fourier-Laplace transform of $f(x,t) \in \mathcal{E}'(\mathbb{R}^2)$ is defined by

$$\mathcal{FL}_{\alpha,\theta}\{f(x,t)\}(u,v) = F_{\alpha,\theta}(u,v) = \langle f(x,t), K_{\alpha,\theta}(x,u,t,v)\rangle, \tag{6}$$

where $K_{\alpha,\theta}(x,u,t,v)$ is as given in the equation (4). The R.H.S. of equation (6) is meaningful because one can easily show that, $K_{\alpha,\theta}(x,u,t,v) \in \mathcal{E}(\mathbb{R}^2)$ for $\alpha, \theta \notin \{k\pi\ : k \in \mathbb{Z}\}$.

4. Correlation theorem for FrFLT

In this section, we have first defined cross correlation operation $\circledcirc$ for FrFLT and then proved the correlation theorem for FrFLT.

4.1 Cross correlation operation for FrFLT

Definition 4.1. *For any two functions $f(x,t), g(x,t) \in L^1(\mathbb{R}^2)$, the 2D cross-correlation operation $\circledast$ is defined as follows:*

$$f(x,t) \circledast g(x,t) = \overline{f(-x,-t)} * g(x,t),$$

where $*$ denotes the usual convolution operation of two 2-dimensional functions and hence it follows that,

$$(f \circledast g)(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(-r,-s)} g(x-r, t-s)\, drds$$

Letting $r' = -r,\ \ s' = s\ \Rightarrow dr' = -dr,\ ds' = -ds$, so above is equivalent to

$$(f \circledast g)(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(r',s')} g\left(x+r', t+s'\right) dr'ds' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(r,s)} g(x+r,\ t+s)\, drds$$

Definition 4.2. *For any two integrable functions $f(x,t)$ and $g(x,t)$ we define cross-correlation operation $\circledcirc$ for FrFLT as follows:*

$$h(x,t) = (f \circledcirc g)(x,t) = \overline{C} e^{-i\left(ax^2 - ibt^2\right)} (\tilde{f} \circledast \tilde{g})(x,t) \tag{7}$$

Theorem 4.3 (Correlation Theorem). *Let $h(x,t) = (f \circledcirc g)(x,t)$ and $F_{\alpha,\theta}, G_{\alpha,\theta}$ and $H_{\alpha,\theta}$ denotes the fractional Fourier-Laplace transform (FrFLT) of f, g and h respectively. Then*

$$H_{\alpha,\theta}(u,v) = e^{i\left(au^2 + ibv^2\right)} \overline{F_{\alpha,\theta}(u,-v)} G_{\alpha,\theta}(u,v) \tag{8}$$

Proof. By definition of FrFLT we have,

$$\begin{aligned}
H_{\alpha,\theta}(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,t)\; K_{\alpha,\theta}(x,u,t,v)\,dxdt \\
&= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (f\circledcirc g)(x,t)\, e^{i\left[a(x^2+u^2)-pxu\right]} e^{\left[b(t^2+v^2)-qtv\right]} dxdt \\
&= C\overline{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\left(ax^2-ibt^2\right)} \left(\tilde{f}\circledast\tilde{g}\right)(x,t)\, e^{i\left[a(x^2+u^2)-pxu\right]} e^{\left[b(t^2+v^2)-qtv\right]} dxdt \\
&= C\overline{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\left(ax^2-ibt^2\right)} \left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\tilde{f}(r,s)}\tilde{g}(x+r,\ t+s)\,drds\right\} \\
&\times e^{i\left[a(x^2+u^2)-pxu\right]} e^{\left[b(t^2+v^2)-qtv\right]} dxdt \\
&= C\overline{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\left(ax^2-ibt^2\right)} \\
&\times \left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(r,s)\, e^{i(ar^2-ibs^2)}} g(x+r,t+s)\, e^{i(a(x+r)^2-ib(s+t)^2} drds\right\} \\
&\times e^{i\left[a(x^2+u^2)-pxu\right]} e^{\left[b(t^2+v^2)-qtv\right]} dxdt
\end{aligned}$$

Letting $x+r=m, s+t=n \Rightarrow dx=dm, dt=dn$ above becomes,

$$\begin{aligned}
H_{\alpha,\theta}(u,v) &= C\overline{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(r,s)} g(m,n) e^{-i\left(a(m-r)^2-ib(n-s)^2\right)} e^{-iar^2+bs^2} e^{i(am^2-ibn^2)} \\
&\times e^{i\left[a((m-r)^2+u^2)-p(m-r)u\right]} e^{\left[b((n-s)^2+v^2)-q(n-s)v\right]} drdsdmdn \\
&= \overline{C}e^{iau^2}e^{-bv^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(r,s)} e^{-i\left[a(r^2+u^2)-pru\right]} e^{\left[b(s^2+(-v)^2)-qs(-v)\right]}\, drds \\
&\times C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(m,n)\, e^{i\left[a(m^2+u^2)-pmu\right]} e^{\left[b\left(n^2+v^2\right)-qnv\right]} dmdn \\
&= \overline{C}e^{iau^2}e^{-bv^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f(r,s)}\,\overline{e^{i[a(r^2+u^2)-pru]}}\,\overline{e^{\left[b(s^2+(-v)^2)-qs(-v)\right]}} drds \\
&\times C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(m,n)\, e^{i\left[a(m^2+u^2)-pmu\right]} e^{\left[b\left(n^2+v^2\right)-qnv\right]} dmdn \\
&= e^{i\left(au^2+ibv^2\right)}\overline{\mathcal{FL}_{\alpha,\theta}\{f(r,s)\}(u,-v)}\mathcal{FL}_{\alpha,\theta}\{g(m,n)\}(u,v) \\
&= e^{i(au^2+ibv^2)}\overline{F_{\alpha,\theta}(u,-v)}G_{\alpha,\theta}(u,v)
\end{aligned}$$

This completes the proof. □

5. Application of FrFLT to Generalized Wave Equation

Using the fractional Fourier-Laplace transform we investigate the solution of the generalized wave equation associated with FrFLT.

$$\frac{\partial^2\phi(x,t,\tau)}{\partial\tau^2} = \left[\left(\Delta'_x\right)^2 + \left(\Delta'_x\right)^2\right]\phi(x,t,\tau), (x,t)\in\mathbb{R}^2, \tau>0 \tag{9}$$

where

$$\Delta'_x = -\left(\frac{d}{dx} + ix\cot\alpha\right) \quad and \quad \Delta'_t = i\left(\frac{d}{dt} - t\cot\theta\right)$$

with initial conditions

$$\phi\left(x,t,0\right) = f\left(x,t\right), -\infty < x < \infty, -\infty < t < \infty \tag{10}$$

and

$$\left[\frac{\partial\phi\left(x,t,\tau\right)}{\partial\tau}\right]_{\tau=0} = g\left(x,t\right), -\infty < x < \infty, -\infty < t < \infty \tag{11}$$

Applying the FrFLT to both sides of (9), we get

$$\frac{\partial^2\Phi\left(u,v,\tau\right)}{\partial\tau^2} + \left[u^2\csc^2\alpha + v^2\csc^2\theta\right]\Phi\left(u,v,\tau\right) = 0, \tag{12}$$

where

$$\Phi\left(u,v,\tau\right) = \mathcal{FL}_{\alpha,\theta}\left\{\phi\left(x,t,\tau\right)\right\}$$

The equation (12) is a second-order ordinary differential equation (ODE) in τ for each fixed (u,v). It can be written as

$$\frac{\partial^2\Phi\left(u,v,\tau\right)}{\partial\tau^2} + \gamma^2\Phi\left(u,v,\tau\right) = 0,$$

where

$$\gamma^2 = u^2\csc^2\alpha + v^2\csc^2\theta$$

Hence, the general solution of this ODE is of form

$$\Phi\left(u,v,\tau\right) = A\left(u,v\right)\cos\left(\gamma\tau\right) + B(u,v)\sin\left(\gamma\tau\right) \tag{13}$$

Now, to determine the constants $A\left(u,v\right)$ and $B(u,v)$, we use the initial conditions at $\tau = 0$ we have,

$$\begin{aligned}\Phi\left(u,v,0\right) &= \mathcal{FL}_{\alpha,\theta}\left\{\phi\left(x,t,0\right)\right\}\\ &= \mathcal{FL}_{\alpha,\theta}\left\{f\left(x,t\right)\right\}\left(u,v\right)\end{aligned}$$

and equation (13) at $\tau = 0$ gives

$$\begin{aligned}&\Phi\left(u,v,0\right) = A\left(u,v\right)\\ &\Rightarrow A\left(u,v\right) = \mathcal{FL}_{\alpha,\theta}\left\{f\left(x,t\right)\right\}\left(u,v\right)\end{aligned}$$

Now, by the initial condition (11), we have

$$\left[\frac{\partial\Phi\left(u,v,\tau\right)}{\partial\tau}\right]_{\tau=0} = \mathcal{FL}_{\alpha,\theta}\left\{\frac{\partial\phi\left(x,t,\tau\right)}{\partial\tau}\right\}\left(u,v\right) = \mathcal{FL}_{\alpha,\theta}\left\{g\left(x,t\right)\right\}\left(u,v\right) \tag{14}$$

and from (13), we have

$$\frac{\partial\Phi\left(u,v,\tau\right)}{\partial\tau} = -\gamma A\left(u,v\right)\sin\left(\gamma\tau\right) + \gamma B(u,v)\cos\left(\gamma\tau\right)$$

Setting $\tau = 0$ above gives

$$\left[\frac{\partial \Phi(u, v, \tau)}{\partial \tau}\right]_{\tau=0} = \gamma B(u, v)$$

Therefore using (14) above gives

$$\gamma B(u, v) = \mathcal{F}\mathcal{L}_{\alpha,\theta}\{g(x, t)\}(u, v)$$
$$\Rightarrow B(u, v) = \frac{1}{\gamma}\mathcal{F}\mathcal{L}_{\alpha,\theta}\{g(x, t)\}(u, v)$$

Thus, the complete transform–domain solution is

$$\Phi(u, v, \tau) = \mathcal{F}\mathcal{L}_{\alpha,\theta}\{f(x, t)\}(u, v)\cos(\gamma\tau) + \frac{1}{\gamma}\mathcal{F}\mathcal{L}_{\alpha,\theta}\{g(x, t)\}(u, v)\sin(\gamma\tau) \tag{15}$$

To recover the solution $\phi(x, t, \tau)$ in the physical domain, we now apply the inverse FrFLT:

$$\phi(x, t, \tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\mathcal{F}\mathcal{L}_{\alpha,\theta}\{f(x, t)\}(u, v)\cos(\gamma\tau) + \frac{1}{\gamma}\mathcal{F}\mathcal{L}_{\alpha,\theta}\{g(x, t)\}(u, v)\sin(\gamma\tau)\right\} \tilde{K}_{\alpha,\theta}(x, u, t, v)\, dudv \tag{16}$$

Example 5.1. *Solve*

$$\frac{\partial^2 \phi(x, t, \tau)}{\partial \tau^2} = \left[\left(\Delta'_x\right)^2 + \left(\Delta'_x\right)^2\right] \phi(x, t, \tau), (x, t) \in \mathbb{R}^2, \tau > 0 \tag{17}$$

where

$$\Delta'_x = -\left(\frac{d}{dx} + ix\cot\alpha\right) \quad and \quad \Delta'_t = i\left(\frac{d}{dt} - t\cot\theta\right)$$

with initial conditions

$$\phi(x, t, 0) = f(x, t) = e^{-x^2-t^2}, -\infty < x < \infty, -\infty < t < \infty \tag{18}$$

and

$$\left[\frac{\partial \phi(x, t, \tau)}{\partial \tau}\right]_{\tau=0} = g(x, t) = 0, -\infty < x < \infty, -\infty < t < \infty \tag{19}$$

In this case (15) *takes the form*

$$\Phi(u, v, \tau) = \mathcal{F}\mathcal{L}_{\alpha,\theta}\left\{e^{-x^2-t^2}\right\}(u, v)\cos\left(\sqrt{u^2\csc^2\alpha + v^2\csc^2\theta\tau}\right)$$

and thus (16) *takes the form*

$$\phi(x, t, \tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\mathcal{F}\mathcal{L}_{\alpha,\theta}\left\{e^{-x^2-t^2}\right\}(u, v)\right\} \cos\left(\sqrt{u^2\csc^2\alpha + v^2\csc^2\theta\tau}\right)\Big\}\, \tilde{K}_{\alpha,\theta}(x, u, t, v)\, dudv$$

For $\alpha = \theta = \pi$, this gives

$$\begin{aligned}\phi(x,t,\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{e^{-u^2-v^2}\cos\tau\right\}\delta(x+u)\,\delta(t+v)\,dudv\\ &= \cos\tau\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-u^2-v^2}\delta(x+u)\,\delta(t+v)\,dudv\\ &= e^{-x^2-t^2}\cos\tau\end{aligned}$$

Following Figure 1, shows the plot of $\phi(x,t,\tau)$ for $\tau = 2$.

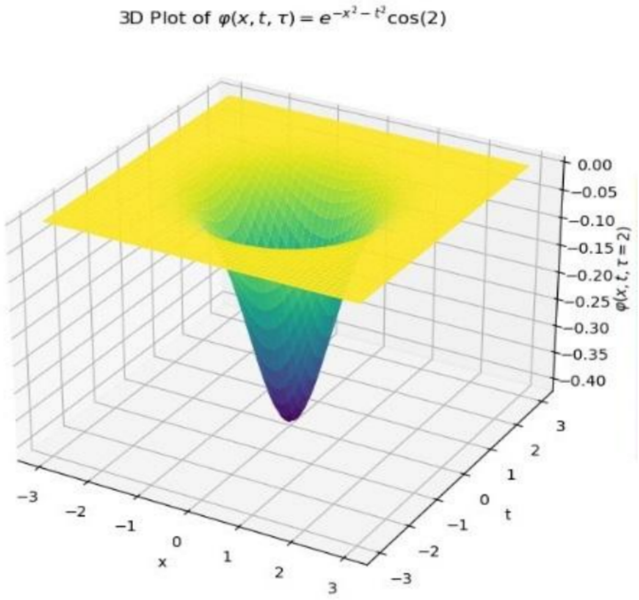

Figure 1:

6. Conclusion

In conclusion, this work extends the theory of the Fractional Fourier-Laplace Transform (FrFLT) by applying it to distributions and proving a cross-correlation theorem. The use of FrFLT in generalized wave equations shows its practical value in solving complex problems. These results lay a clear foundation for future research in this area. Further studies can build on these findings to explore additional applications and deepen our understanding of the transform.

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