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Compact composition operators and RKHS

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Abstract

In this paper we characterize compact composition operator on $H^2(D)$ in terms of kernel function for $H^2(D)$. We discuss algebraic characterization of composition operators and compactness of composition operator on H^2. This is also provide some contribution to the spectrum of compact composition operators on H^2.

Keywords: Composition operator; Hardy spaces; Reproducing kernel; positive definite function; Schatten class.

1. Introduction

In 1968, compactness of composition operators on H^2 had been first studied by H. J. Schwarz [1] in his doctoral thesis and refined the compactness problem by T. Shapiro and P. Taylor [2] in 1973 by confirming which composition operators belongs to the Hilbert-Schmidt classes. In [2], they proved that C_φ belong to the Hilbert-Schmidt classes if and only if $\int_{\partial D}(1-|\varphi^*|)^{-1}$ is finite, where φ^* is the boundary value function of φ. Throughout this paper, we denote by D the open unit disc of the complex plane, i.e., $D = \{z \in C, |z| < 1\}$ and its closure $\overline{D} = \{z \in C, |z| \leq 1\}$ and its boundary $\mathcal{B} = \partial D = \{z \in C, |z| = 1\}$ and $\mathcal{M}$ be Lebesgue measure on $\mathcal{B}$. Let $\varphi : D \to D$ be an analytic map on D in to itself. The composition operator $C_\varphi : H^2 \to H^2$ is the map $f \to fo\varphi$ i.e., $C_\varphi(f) = fo\varphi$, by [2] that every composition operator on Hardy space H^2 in to itself is continuous. We denote the spectrum and the point spectrum of C_φ by $\sigma(C_\varphi)$ and $\sigma_p(C_\varphi)$ respectively. For each $b \in \mathcal{B}$ and $0 < h < 1$, the Carleson window $W(b,h)$ be the set $W(b,h) = \left\{z \in D, |z| \geq 1-h \ and \ \left|arg\left(z\bar{b}\right)\right| \leq h\right\}$ and every positive measure $\mathcal{M}_\varphi$ on D, we sets a function $\mu_{\mathcal{M}_\varphi}(h) = sup_{b\in\mathcal{B}}\mathcal{M}_\varphi[W(b,h)]$, we call this function $\mu_{\mathcal{M}_\varphi}$ be the Carleson function of $\mathcal{M}_\varphi$. For each borel set $B \subseteq D$, $\mathcal{M}_\varphi(B) = \mathcal{M}(\{b \in \mathcal{B},\ \varphi^*(b) \in B\})$, where φ^* is the boundary values function of analytic map φ. We denote $\mu_{\mathcal{M}_\varphi}$ by μ_φ , in this case we say, μ_φ is the Cerleson function of φ. For $a \geq 1$, we say $\mathcal{M}_\varphi$ is an $a-$Carleson measure if $\mu_{\mathcal{M}_\varphi}(h) \lesssim h^a$, for $a = 1$, $\mathcal{M}_\varphi$ is likely to said Carleson measure. By B. Maccluer [3] had proved that composition

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operator C_φ is compact on H^p if and only if $\mu_\varphi(h) = o(h)$, as $h \to 0$, with condition that $|\varphi^*| < 1$ a.e on $\mathcal{B}$. An operator T on Hilbert space H with finite p$^{\text{th}}$ Schatten norm ie $\|T\|_p = \left[Tr\left(|T|^p\right)\right]^{1/p} < \infty$, where $|T| = \sqrt{(T^*T)}$, is called p$^{\text{th}}$ Schatten class $\left(S_p\right)$ operator. We refer to Kelley-Namioka [8] for notions and properties of compact operators.

2. Preliminaries

Let X be a topological vector space and $L(X)$ be the collection of complex valued functions on X, which is vector space with the operation of usual addition of function and scalar multiplication over the field F.

Definition 2.1. *Let X be a set. We call a subset $\mathcal{H} \subseteq L(X,F)$ is a reproducing kernel Hilbert space (RKHS) on X over F if it satisfied followings.*

(a). $\mathcal{H}$ is a vector subspace of $L(X,F)$.

(b). $\mathcal{H}$ equipped with inner product $\langle\cdot\rangle$, making it a Hilbert space such that point wise linear evaluation is continuous, that is for every $y \in X$, evaluation map $E_y : \mathcal{H} \to F$ defined by $E_y(f) = f(y)$ is continuous. By the Riesz-representation theorem, for every $\in X$, there exists a unique vector $k_y \in \mathcal{H}$ such that for every $f \in \mathcal{H}$, $f(y) = \langle f, k_y\rangle$. The function k_y is called the reproducing kernel for the point y.

The function $K : X \times X \to F$ defined by $K(x,y) = k_y(x)$ is called the reproducing kernel for $\mathcal{H}$. Then we have $K(x,y) = k_y(x) = \langle k_y, k_x\rangle$ and $\|E_y\|^2 = \|k_y\|^2 = \langle k_y, k_y\rangle = K(y,y)$. This kernel is said to be positive definite kernel if

(a). $K(x,y) = \overline{K(y,x)}$

(b). for all $\lambda_1, \lambda_2, \ldots \lambda_n \in F$ and $x_1, x_2, \ldots x_n \in X$ distinct such that $\sum_{i,j} \lambda_i K\left(x_i, x_j\right) \overline{\lambda_j} \geq 0$.

Due to, N. Aronsza Jn [4], there is an equivalence between positive definite kernels and RKHSs on a space.

Theorem 2.2. *Let X be a set and K be a positive definite kernel on X. Then there exists a unique reproducing kernel Hilbert space $\mathcal{H} \subseteq L(X,F)$ with kernel K.*

Example 2.3 ([5]). *To construct $H^2(D)$ is a RKHS on D and also compute the kernel.*

Let f and g be two complex power series as $f = \sum_{n=0}^{\infty} \alpha_n z^n$, $g = \sum_{n=0}^{\infty} \beta_n z^n$, we define inner product as $\langle f, g\rangle = \sum_{n=0}^{\infty} \alpha_n \overline{\beta}_n$, thus we have that $\|f\|^2 = \sum_{n=0}^{\infty} |\alpha_n|^2$. We define a linear inner product preserving isomorphism map $S : H^2(D) \to l^2$ by $(f) = (\alpha_0, \alpha_1, \ldots)$, since l^2 is a Hilbert space, then $H^2(D)$ is also as Hilbert space, hence condition (b) is verified. Since power series define on D agree with vector addition and scalar multiplication, so condition (a) is satisfy. Now, we see that every power series in $H^2(D)$ converges to a function on D. Let $w \in D$ then we have,

$$|E_w(f)| = \left|\sum_{n=0}^{\infty} \alpha_n w^n\right| \le \sum_{n=0}^{\infty} |\alpha_n| \, |w|^n \le \left(\sum_{n=0}^{\infty} |\alpha_n|^2\right)^{1/2} \left(\sum_{n=0}^{\infty} |w|^{2n}\right)^{1/2} = \|f\| \frac{1}{\sqrt{1-|w|^2}} \tag{1}$$

This implies E_w is bounded ,so it is continuous and $\|E_w\| \le \frac{1}{\sqrt{1-|w|^2}}$, hence $H^2(D)$ is a RKHS on D. For kernel, let $c \in D$, let $g(z) = \sum_{n=0}^{\infty} \overline{c^n} z^n \in H^2(D)$ and any $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in H^2(D)$, we have $\langle f, g\rangle = \sum_{n=0}^{\infty} \alpha_n c^n = f(c)$, thus, g is the reproducing kernel for $c \in D$ and so,

$$K(w,c) = k_c(w) = g(w) = \sum_{n=0}^{\infty} \overline{c_n} w^n = \frac{1}{1-\bar{c}w} \quad and \quad \|E_w\| = K(w,w) = \frac{1}{\sqrt{1-|w|^2}} \tag{2}$$

We recalling the results due to D. Luecking [6], of the composition operators on H^2 which belongs to Hilbert-schmidt classes. Let, an integer $m \ge 1$ and $0 \le i \le 2^m - 1$, define a set called Luecking set,

$$R_{m,i} = \left\{ w \in D, 1 - 2^{-m} \le |w| < 1 - 2^{-m-1} \; and \; \frac{i\pi}{2^{m-1}} \le arg\, w < \frac{(i+1)\pi}{2^{m-1}} \right\} \tag{3}$$

Theorem 2.4 ([6])**.** *For every $p > 0$, and $|\varphi^*| < 1$ a.e on $\mathcal{B}$, the composition operator C_φ belong to the schatten classes S_p if and only if $\sum_{m\ge 0} 2^{mp/2} \left(\sum_{i=0}^{2^m-1} [\mathcal{M}_\varphi(R_{m,i})]^{p/2}\right) < \infty$, where $\mathcal{M}_\varphi(R_{m,i})$ by $\mathcal{M}_\varphi\left\{W\left(e^{2^{-m}(2i+1)\pi}, 2^{-m}\right)\right\} \le \mu_\varphi(2^{-m})$.*

Proposition 2.5. *Let $\varphi : D \to D$ be analytic map with $|\varphi^*| < 1$ a.e on $\mathcal{B}$ and $\mathcal{M}_\varphi$ is an a-carleson measure, with an integer $a > 2$ then C_φ belongs to Hilbert-schmidt classes (S_2).*

Proof. Since $\mu_\varphi(h) \lesssim h^a$, we get

$$\sum_{m\ge 0} 2^m \left(\sum_{i=0}^{2^m-1} \mathcal{M}_\varphi(R_{m,i})\right) \lesssim \sum_{m\ge 0} 2^{nh}\left(2^{-na}\right) = \sum_{m\ge 0} 2^m (1-(a-1)h) < \infty$$

since $1-(a-1) < 0$. □

Proposition 2.6. *Let $\mathcal{M}_\varphi$ be a finite positive measure on D and let > 2, then the followings are equivalent, (a) $\sum_{m=1}^{\infty}\sum_{i=0}^{2^m-1} 2^{ma}\left(\mathcal{M}_\varphi(R_{m,i})\right)^a < \infty$, (b) $\sum_{m=1}^{\infty}\sum_{i=0}^{2^m-1} 2^{ma}\left(\mathcal{M}_\varphi(W_{m,i})\right)^a < \infty$.*

Theorem 2.7. *Let $\varphi : D \to D$ be analytic map with $|\varphi^*| < 1$ a.e on $\mathcal{B}$. The composition operator $C_\varphi : H^2 \to H^2$ belong to Hilbert-Schmidt classes if as h goes to 0, $\mu_\varphi(h) = o\left(h\left(\log\frac{1}{h}\right)^{-1}\right)$.*

Proof. By the Luecking's characterization and the equivalency in Proposition 2.5, we have for finite positive measure $\mathcal{M}_\varphi$,

$$\sum_{m=1}^{\infty}\sum_{i=0}^{2^m-1} 2^{ma}\left(\mathcal{M}_\varphi(W_{m,i})\right) < \infty \tag{4}$$

We have, if $h = 2^{-m}$, every window $W(b,h)$ is contained in the union of at most three of the $W_{m,i}$'s hence, $\mu_\varphi(2^{-n}) \leq \max_{0\leq i\leq 2^m-1} \mathcal{M}_\varphi(W_{m,i}) \leq 3\sum_{i=0}^{2^m-1}(\mathcal{M}_\varphi(W_{m,i}))$ and (4) gives, $\sum_{m=1}^{\infty}(\mu_\varphi(2^{-m}))2^m < \infty$, then we setting $\vartheta_m = \sum_{\frac{m}{2}\leq k\leq m}(\mu_\varphi(2^{-k}))\,2^k$ and we get

$$\log_{m\to\infty}\vartheta_m = 0 \tag{5}$$

For a constant $c > 0$, such that for $k \leq m$, $c\mu_\varphi(2^{-k}) \geq 2^{m-k}\mu_\varphi(2^{-m})$ and so,

$$c\vartheta_m \geq \left(\frac{m}{2}\right)\left(\frac{\mu_\varphi(2^{-m})}{2^{-m}}\right) \tag{6}$$

We consider, for each $h \in (0, \frac{1}{2})$, the integer m such that $2^{-m-1} < h \leq 2^{-m}$, then by (5) and (6), we get, $\lim_{h\to 0}\left(\frac{\mu_\varphi(h)}{h}\right)\log\left(\frac{1}{h}\right) = 0$, hence, we say that C_φ is compact. □

3. Characterizations and Results

Let $H^2(D)$, be the Hardy space of analytic function f on $D = (w \in C,\ |w| < 1)$ with norm of f, $\|f\|_2 = \sup_{|w|<1}\left[\frac{1}{2\pi}\int_0^{2\pi}|f(w)|^2 d\theta\right]^{1/2} < \infty$, where $\theta = arg(w)$, and φ be an analytic map on D in to itself and φ induces a composition operator $C_\varphi : H^2 \to H^2$ defined by $C_\varphi f = fo\varphi$ for all $f \in H^2$ and C_φ is bounded with $\|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2}$. Let $L(H^2(D))$ be the algebra of linear and bounded operator on $H^2(D)$. Clearly, the linearity is trivial and boundedness follows from the definition of weak operator topology. An algebra homomorphism $T : H^2(D) \to H^2(D)$ is a linear map satisfying $T(f.g) = T(f).T(g)$ for all $f, g \in H^2(D)$.

Lemma 3.1 ([6])**.** *Let $S : H^2(D) \to \mathbb{C}$ be a continuous algebra homomorphism, $S \neq 0$, then there exists a point $w_0 \in D$ such that $Sg = g(w_0)$ for all $g \in H^2(D)$.*

Proposition 3.2. *Let $T : H^2(D) \to H^2(D)$ be a linear operator, then the following assertions are equivalent,*

(a). there exists a analytic map $\varphi : D \to D$ such that $= C_\varphi$,

(b). T is not trivial algebra homomorphism, ie $T \neq 0$,

(c). T is bounded and $T(g_n) = (Tg)^n$, for all $n \in N_0$.

Proof. We define $g_n \in H^2(D)$ by $g_n(w) = w^n$ for all $w \in D$ and all $n \in N_0$. Since $(a) \Rightarrow (c)$ is trivial. For $(c) \Rightarrow (b)$, we follows from (c) that $T(f.g) = T(f).T(g)$, if f and g are polynomials. Since the set of polynomials is dense subset in $H^2(D)$ and the multiplication is continuous, so bounded, hence (b) establish. Lastly for $(b) \Rightarrow (a)$, since $T \neq 0$, and by Lemma 3.1, that $T(g_0) = g_0$, let $w \in D$, then by existence $S(g) = (Tg)(w)$ is an algebra homomorphism and $S(g_0) = 1$, then there exists $\varphi(w) \in D$ such that $(Tf)\,w = f(\varphi(w))$ for all $f \in H^2(D)$, and hence $\varphi = T(g_1) \in H^2(D)$, and (a) derived. □

Corollary 3.3. *Let $X = H^2$ and φ is analytic self map on D, then the followings are equivalents;*

(a). C_φ is invertible in $L(X)$

(b). φ is an automorphism of D.

Proof. $(a) \Rightarrow (b)$ Let C_φ be invertible and $A = C_\varphi^{-1}$, then A is an algebra homomorphism, by Proposition 3.2, there exist an analytic map $\omega : D \to D$ such that $A = C_\omega$, then, $g_1 = C_\varphi (C_\omega g_1) = \omega o \varphi$ and $g_1 = C_\omega (C_\varphi g_1) = \varphi o \omega$, hence φ is an automorphism and $\omega = \varphi^{-1}$. $(b) \Rightarrow (a)$, it is clear, since, $C_\varphi^{-1} C_\varphi = C_\varphi C_\varphi^{-1} = I$.
Let X be a separable Hilbert space. A linear operator $A : X \to X$ be compact if there exist a neighbourhood N_O of O such that AN_O is relatively compact. Since compact linear operator is always continuous. Since N_O is neighbourhood of O then there exist a compact subset $C \subset D$ and $\varepsilon > 0$ such that $N_{C\varepsilon} = \{f \in H^2 : |f(w)| < \varepsilon, \; for \; all \;\; w \in C\} \subset N_O$. □

Theorem 3.4. *Let $\varphi : D \to D$ be analytic. Then the followings are equivalents;*

(a). C_φ is compact operator from $H^2(D)$ and $H^2(D)$

(b). $\sup_{w \in D} |\varphi(w)| < 1$.

Proof. $(a) \Rightarrow (b)$, we assume that $\varphi(D)$ is not subset of $t\overline{D}$ for all $0 < t < 1$ and N_O be a neighbourhood of O, then we show that $C_\varphi(N_O)$ is not relatively compact, there exist $0 < \epsilon < 1$ and $0 < t_0 < 1$ such that $N_\epsilon = \{f \in H^2(D) : |f(w)| < \epsilon \; for \; all \; w \in \; t_0\overline{D}\} \subset N_O$, for this, we have sufficient to prove that $C_\varphi(N_\epsilon)$ is not relatively compact. Then by assumption, that there exist $z_0 \in D$ such that $w_0 = \varphi(z_0) \notin t_0\overline{D}$, then there exist $t_0 < t_1 < 1$ and $h > 0$ such that $t_1\overline{D} \cap D(w_0, h) = \emptyset$. Then the set $M = t_0\overline{D} \cup (w_0)$ is compact and C/M is connected. Let $n \in \mathbb{N}_0$ and define l_n by, $l_n(z) = 0 \; if \;\; z \in t_1 D$ and $l_n(z) = n+1 \;\; if \; z \in D(w_0, h)$; set $L = t_1 D \cup D(w_0, h)$ then $M \subset L$ and $l_n : L \to C$ is homomorphic. By Runge Theorem [8], there exist a polynomial $P_n : C \to C$ such that $|P_n(m) - l_n(m)| < \varepsilon$ for all $m \in M$. This show that $P_{n/D} \in N_\epsilon$ and $|P_n(w_0)| \geq n$, hence the sequence $(C_\varphi P_n)_{n \in \mathbb{N}_0}$ has no convergent subsequence. For $(b) \Rightarrow (a)$, we assume that $\sup_{z \in D} |\varphi(z)| = t_0 < 1$, the set $N_0 = \{f \in H^2(D) : |f(z)| < 1 \; if \;\; |z| \leq t_0\}$ is a neighbourhood of O in $H^2(D)$. Let $g \in N_0$, since $\varphi(D) \subset t_0\overline{D}$, then $|f(\varphi(z))| < 1$ for all $z \in D$. We apply Montel Theorem [8], that $C_\varphi(N_0)$ is relatively compact in H^2. Hence (a) established. □

The reproducing kernel for $H^2(D)$, $K_c(w) = \frac{1}{1 - \bar{c}w}$, where $c \in D$, has a property that for $f \in H^2(D)$, $\langle f, k_c \rangle = f(c)$. The function k_c is itself on $H^2(D)$ function, has a norm $\frac{1}{\sqrt{1-|c|^2}}$. We characterize C_φ among operator on H^2 in terms of kernel function K_c.

Theorem 3.5. *Let T be an operator on H^2. Then T is a composition operator if and only if the set $(k_c : c \in D)$ is invariant under T^*. In this case $T = C_\varphi$, where φ and T has a relation $T^*(k_c) = K_{\varphi(c)}$.*

Proof. If $T = C_\varphi$ is a composition operator, then for each f, $\langle T^*(k_c), f \rangle = k_c(Tf) = k_c(f o \varphi) = f(\varphi(c)) = K_{\varphi(c)}(f)$. So $T^*(k_c) = K_{\varphi(c)}$ and the set of point evaluation linear functionals is invariant under T^*.

Conversely, if the set of point evaluation linear functional is invariant under T^*, then if $f \in H^2(D)$, $(Tf)(c) = \langle Tf, k_c \rangle = \langle f, T^* k_c \rangle = \left\langle f,\ K_{\varphi(c)} \right\rangle = f(\varphi(c))$, by taking f be the identity function $f(c) = c$, then φ is analytic and $T = C_\varphi$. □

Corollary 3.6. *If φ is an analytic map on unit disk in to itself such that $\varphi(0) = 0$ and f is in $H^2(D)$ then $\left\|C_\varphi\right\|_2 \le \|f\|_2$.*

Proof. If f is analytic, then $H = |f|^2$ is subharmonic and $\left\|C_\varphi\right\|_2 \le \|fo\varphi\|_2$ and $\|f\|_2$ are the square roots of the integrals in the theorem. □

Proposition 3.7. *If f is in $H^2(D)$ and φ is an analytic map of unit disk in to itself, then $\left(C_\varphi f\right)^* = f^* o \varphi^*$ almost everywhere.*

Proof. Since all function belongs to $H^2(D)$ is a quotient of two $H^\infty(D)$ functions. So it is sufficient to prove this results for f in $H^\infty(D)$. Let such an f there is a set A which is subset of boundary of D with full measure such that $(fo\varphi)$ and φ have radial limits at each point of A. We write $A = A_1 \cup A_2$, where φ has radial limits of modulus one on A_1 and modulus less than one on A_2. For c in A_2 we have clearly $(fo\varphi)^* = f^* o \varphi^*$ and by continuity of f in D. And for c in A_1, by definition of A satisfy that f has a limit along the $arc\varphi(rc)$, where $0 < r < 1$. Then by Lindelof's theorem, f has radial limit at $\varphi^*(c)$ equal to $(fo\varphi)^*(c)$ and hence $f^*(\varphi^*(c)) = (fo\varphi)^*(c) = \left(C_\varphi f\right)^*(c)$. □

Proposition 3.8. *Let $H^2(D)$ be a RKHS on the set D with kernel K, then the linear span of the function, $k_y(.) = K(.,y)$ is dense in $H^2(D)$.*

Proof. Since any function $f \in H^2$ is orthogonal to the span of the function $\left(k_y : y \in D\right)$ if and only if $\left\langle f, k_y \right\rangle = f(y) = 0$ for every $y \in D$, which is if and only if $f = 0$. □

Lemma 3.9. *Let $H^2(D)$ be RKHS on D and let $(g_n) \subseteq H^2(D)$, if $Lim\|g_n - g\| = 0$ then $g(x) = Lim_{n\to\infty} g_n(x)$ for each $x \in D$.*

Proof. It is easy to satisfy, since

$$|g_n(x) - g(x)| = |\langle g_n - g, k_x \rangle| \le \|g_n - g\|\,\|k_x\| \to 0 \tag{7}$$

□

Proposition 3.10. *Let A and B are two RKHS on D having same kernel then both are equals and $\|f\|_A = \|f\|_B$ for every f.*

Proof. Let $K(x,y)$ be the kernel of A and B and let $W_A = span\,(k_x : x \in D)$ and $W_B = span\left(k_y : y \in D\right)$, then by Proposition 3.8, W_A and W_B is dense in A and B respectively. For any $f \in W_A$, then $f(x) = \sum_i a_i k_{x_i}(x)$ and its value as function are independent of whether we having it as in W_A or W_B and such f, $\|f\|_A^2 = \sum_{i,j} \overline{a}_i a_j \left\langle k_{x_i}, k_{x_j} \right\rangle = \sum_{i,j} \overline{a}_i a_j K\left(x_j, x_i\right) = \|f\|_B^2$. Thus $\|f\|_A = \|f\|_B$ for all $f \in W_A = W_B$.

Now, if $g \in A$, then there exist a sequence $\{g_n\} \subseteq W_A$ with $\|g_n - g\|_A \to 0$, since $\{g_n\}$ is Cauchy in W_A, it is also Cauchy in W_B, so there exist $f \in B$ with $\|f - g_n\| \to 0$, by Lemma 3.9, $g(x) = \lim_{n\to\infty} g_n(x) = f(x)$, thus every $g \in A$ is also in B and every $f \in B$ is in A, Hence, $A = B$. Since $\|g\|_A = \|g\|_B$ for each g in a dense subset W_A and W_B and hence that the norms are equal for every g. □

Proposition 3.11. *Let $\varphi : D \to D$ and K be a kernel function on $H^2(D)$ then $C_\varphi K = Ko\varphi$ is a kernel function on D.*

Proof. Let $z_1, z_2 \ldots, z_n \in D$ and $\beta_1, \beta_2, \ldots, \beta_n$ be scalar and let

$$(x_1, x_2, \ldots, x_p) = (\varphi(z_1), \varphi(z_2), \ldots, \varphi(z_n))$$

so that $p \leq n$. Set $L_K = \{i : \varphi(z_i) = x_K\}$ and $M_K = \sum_{i\in L_K} \beta_i$, then

$$\sum_{i,j} \beta_i \overline{\beta}_j K(\varphi(z_i), \varphi(z_j)) = \sum_{k,l} \sum_{i\in L_k} \sum_{j\in L_l} \beta_i \overline{\beta}_j K(z_k, z_l) = \sum_{k,l} \overline{M}_k M_l K(z_k, z_l) \geq 0$$

Hence, $C_\varphi K = Ko\varphi$ is a kernel function on D. □

Theorem 3.12. *Let $\varphi : D \to D$ be analytic on D and $K : H^2(D) \times H^2(D) \to \mathbb{C}$ be positive definite then $\mathcal{H}(Ko\varphi) = (fo\varphi : f \in H^2(D))$ and for $h \in \mathcal{H}(Ko\varphi)$ then $\|h\|_{\mathcal{H}} = inf\{\|f\|_{H^2} : h = fo\varphi\}$.*

Proof. Let $f \in H^2(D)$ with $\|f\|_{H^2} = d$ then $\overline{f}(a) f(b) < c^2 K(a,b)$ in sense of positive definite order for all $a, b \in D$. Since this inequality same as a inequality of matrices over finite sets. So we get that $fo\varphi(x)\overline{fo\varphi}(y) \leq d^2 K(\varphi(x), \varphi(y))$ this implies, $fo\varphi \in \mathcal{H}(Ko\varphi)$ having norm $\|fo\varphi\|_{\mathcal{H}} \leq d$, this explanation imply there exist a contractive linear map $C_\varphi : H^2(D) \to \mathcal{H}(Ko\varphi)$ given by $C_\varphi f = fo\varphi$. We consider $g_x(y) = K(\varphi(y), \varphi(x))$ so that this is the kernel function for $\mathcal{H}(Ko\varphi)$. Since for any set of points and scalar β_i, if $u = \sum_i \beta_i g_{x_i}$ then $\|u\|_{\mathcal{H}} = \left\|\sum_i \beta_i K_{\varphi(x_i)}\right\|_{H^2}$ it follows that there is well defined isometry as $L : \mathcal{H} \to H^2$ such that $L(g_x) = K_{\varphi(x)}$, hence that $C_\varphi oL$ is the identity on $\mathcal{H}(Ko\varphi)$ and proved the results. □

Theorem 3.13. *Let D_i, $i = 1, 2$ be set and $\varphi : D_1 \to D_2$ be analytic and $K_i : D_i \times D_i \to \mathbb{C}$, $i = 1, 2$ kernel function, then the following are equivalent; (a). $C_\varphi : \mathcal{H}(K_2) \to \mathcal{H}(K_1)$ is bounded linear operator; (b). $(fo\varphi : f \in \mathcal{H}(K_2)) \subseteq \mathcal{H}(K_1)$; (c). there exist a constant $d > 0$ such that $K_2 o\varphi \leq d^2 K_1$.*

Proof. $(a) \Rightarrow (b)$ is clearly. To prove (c) implies (a), let $f \in \mathcal{H}(K_2)$ with $\|f\| = c$ then $f(s) f(t) \leq c^2 K_2(s,t)$ which satisfy that $f(\varphi(s)) \overline{f}(\varphi(t)) \leq c^2 K_2(\varphi(s), \varphi(t)) \leq c^2 d^2 K_1(s,t)$ thus, it follows that $C_\varphi(f) = fo\varphi \in \mathcal{H}(K_1)$ having norm $\|C_\varphi f\|_1 \leq d\|f\|_2$ hence C_φ is bounded and $\|C_\varphi\| \leq d$. Lastly, $(b) \Rightarrow (c)$ it is equivalent to the $\mathcal{H}(K_2 o\varphi) \subseteq \mathcal{H}(K_1)$ which is equivalent to the kernel inequality, by Theorem 3.12, hence (b) implies (c) establish.

Now, since $C_\varphi \in \mathbb{C}(H^2)$ collection of all composition operator on H^2 and C_φ is compact so $C_\varphi \in K(H^2)$ then there exist a sequence of operator $(C_\varphi)_n \in \mathbb{C}(H^2)$ having a limiting value C_φ as $n \to \infty$, in the sense of norm topology on $\complement\left(H^2\right)$. The spectra of compact composition operator $C_\varphi \in K\left(H^2\right)$ depends on location and nature of the fixed point of the function. This means φ is linear fractional mapping from disk into itself. General form of φ be $(z) = \frac{aZ+b}{cZ+d}$, where $a,b,c,d \in \mathbb{C}$, maps the unit disk D into itself. Let $\varphi : D \to D$ be analytic map on D having fixed point $a \in D$ that is $\varphi(a) = a$ and that $0 < |\varphi'(a)| < 1$, let $\lambda_n = (\varphi'(a))^n$, $n \in \mathbb{N}_0$. Let $\sigma\left(C_\varphi\right)$ and $\sigma_p\left(C_\varphi\right)$ be denote the spectrum and point spectrum of C_φ respectively. Since $\varphi \notin Aut(D)$ and since 0 is limit point of eigen value of $C_\varphi \in K\left(H^2\right)$ that is $0 \in \sigma\left(C_\varphi\right)$, By Koenigs's Theorem [10] that, $\sigma_p\left(C_\varphi\right) = \left\{(\varphi'(a))^n : n \in \mathbb{N}_0\right\}$, we prove that the whole spectrum $\sigma\left(C_\varphi\right)$ is equal to $\sigma_p\left(C_\varphi\right) \cup \{0\}$. □

Lemma 3.14. *Let $\varphi : D \to D$ be analytic and φ is not automorphism with $\varphi(0) = 0$, let $f \in H^2$ and $\lambda \in \mathbb{C} - (0)$. Suppose that there exist $0 < r < 1$ and $g \in H^2$ such that $\lambda g - go\varphi = f$ on rD then g has en extension $\overline{g} \in H^2$ such that $\lambda\overline{g} - \overline{g}o\varphi = f$ on D.*

Proof. Let $\delta = sup\{\rho \in [r,1]\ : g \text{ has an analytic extension on } \rho D\}$, we prove that $\delta = 1$. Suppose that $\delta < 1$, there exist $\overline{g} \in H^2(\delta D)$ an analytic extension of g, having property $\lambda\overline{g} - \overline{g}o\varphi = f$ on rD. Since both side are analytic, then by uniqueness theorem, this identity is true for δD. And by Schwarz's Lemma $\varphi(\rho D) < \rho D$ for all $0 < \rho < 1$ and also follows from the Schwarz Lemma that there exist $\delta < \delta < 1$ such that $\varphi(\delta D) < \delta D$. Now we taking subsequence $(x_n)_n \in D$ such that $x_n \to x$ and then $|x| = \delta$ and $|\varphi(x)| \geq \delta$ which is not possible, since $\varphi \notin Aut(D)$. Now since $\lambda\overline{g} = \overline{g}o\varphi + f$, on δD and since $\varphi(\delta D) < \delta D$, hence it follows, g has an analytic extension $\overline{g}$ on δD, which contradict to the choice of δ. Hence proved. □

Corollary 3.15. *Let $P_k \in \complement\left(H^2\right)$ and mapping $Q_n = \sum\limits_{k=0}^{n} P_k$ are projections commuting with C_φ. Moreover $\{k^m : m = 0,1,2,\ldots n\}$ is a basis of $rg(Q_n)$ and $ker(Q_n) = H_n^2(a)$, thus we have the decomposition $H^2(D) = span(k^m : m = 0,1,2,\ldots n) \bigoplus H_n^2(a)$ into two subspaces which are invariant by C_φ, where $P_k(f) = \frac{1}{k!}f^{(k)}(a)\,k^m$.*

Theorem 3.16. *Let $C_\varphi \in K\left(H^2\right)$ then*

$$\sigma\left(C_\varphi\right) = \left\{(\varphi(a))^n : n \in \mathbb{N}_0\right\} \cup \{0\} \tag{8}$$

Proof. For this it is sufficient to prove the surjectivity of $\left(C_\varphi - \lambda_n I\right)$ on $H^2(D)$ for a complex number $\lambda_n \notin \{0\} \cup \left\{(\varphi(a))^n : n \in \mathbb{N}_0\right\}$, we use Lemma 3.14,

Case 1: We take fix point of φ is 0, that means $a = 0$. Let $\lambda_n \in \mathbb{C}$ and $\lambda_n \notin \{0\} \cup \left\{(\varphi(0))^n : n \in \mathbb{N}_0\right\}$, from Koening's Theorem [10], we know that $\lambda_n I - C_\varphi$ is injective. Then we only have to prove the surjectivity. Let $f \in H^2$ and we choose $n \in \mathbb{N}_0$ such that $|\lambda_{n+1}| < |\lambda_n|$, by Corollary 3.15, $H^2(D) = rg(Q_n) \bigoplus H_n^2(0)$, we write $f = f_1 + f_2$, where $f_1 \in rg(Q_n)$ and $f_2 \in H_n^2(0)$.

Since $C_{\varphi|rgQ_n}$ is a diagonal operator and $\lambda_n \notin \sigma\left(C_{\varphi/rgQ_n}\right)$, then there exist $g_1 \in rg(Q_n)$ such that $\lambda_n g_1 - g_1 o\varphi = f_1$, now for f_2, we choose $|\lambda_1| < p < 1$ such that $p^{n+1} < |\lambda_n|$. Since $\lim\limits_{x\to 0}\frac{\varphi(x)}{x} = \lambda_1$, there exist $0 < r \leq 1$such that $|\varphi(x)| \leq p|x|$ for $|x| < r$. We take iteration $\varphi_n = \varphi o \varphi o \ldots\ldots o\varphi\,(n\ times)$ of φ, then $|\varphi_n(x)| \leq p^n|x| \leq p^n r$ for $|x| < r$, since $f_2 \in H_n^2(0)$, there exist $M \geq 0$ such that $|f_2(x)| \leq M|x|^{n+1}$ for $|x| < r$, Now for $r \in \mathbb{N}_0$, $|x| < r$, we have,

$$\left|\frac{f_2(\varphi_n(x))}{\lambda_n^{n+1}}\right| \leq \frac{1}{|\lambda_n|} M \frac{|\varphi_n(x)|^{n+1}}{|\lambda_n|^n} \leq \frac{1}{|\lambda_n|} M \frac{p^{n(n+1)}r}{|\lambda_n|^n} \leq \frac{M^r}{|\lambda_n|}\left(\frac{p^{n+1}}{|\lambda_n|}\right)^n \tag{9}$$

Since, $\frac{p^{n+1}}{|\lambda_n|} < 1$, then $g_0(x) = \sum_{n=0}^{\infty} \frac{g_2(\varphi_n(x))}{\lambda^{n+1}}$ converges uniformly on rD and we define a function $g_0 \in H^2(rD)$, moreover since $\varphi(rD) \subset rD$.

$$g_0(\varphi(x)) = \sum_{n=0}^{\infty} \frac{f_2(\varphi_n(x))}{\lambda_n^{n+1}} = \lambda_n \sum_{n=1}^{\infty} \frac{f_2(\varphi_n(x))}{\lambda_n^{n+1}} = \lambda_n g_0(x) - f_2(x) \quad on \quad rD. \tag{10}$$

Now from Lemma 3.14 that g_0 is analytic extension of $g \in H^2(D)$ satisfying $\lambda_n g - go\varphi = f_2$. Thus implies $\lambda_n \notin \sigma(C_\varphi)$ in this case $a = 0$.

Case 2: Let fixed point $a \in D$ and $a \neq 0$, we consider a fractional linear transformation $\theta_a : D \to D$ by $\theta_a(x) = \frac{a-x}{1-\bar{a}x}$ then $\theta_a(0) = a$ and it is self inverse mapping that means $\theta_a o \theta_a = I$. Then $\overline{\varphi} : \theta_a o \varphi o \theta_a$ map D into D such that $\overline{\varphi}(0) = 0$. Now since, $C_{\overline{\varphi}} = C_{\theta_a} C_\varphi\, C_{\theta_a} = C_{\theta_a}\, C_\varphi\, C_{\theta_a}^{-1}$, then the linear operator $C_{\overline{\varphi}}$ and C_φ are similar. Similarly follows as Case 1, we deduce that

$$\sigma(C_\varphi)/\{0\} = \sigma(C_{\overline{\varphi}})/\{0\} = \sigma_\rho(C_{\overline{\varphi}})/\{0\} = \sigma_\rho(C_\varphi)/\{0\} = \{\lambda_n\ : n \in \mathbb{N}_0\} \tag{11}$$

And hence,

$$\sigma(C_\varphi) = \{\lambda_n : n \in \mathbb{N}_0\} \cup \{0\} = \left\{(\varphi(a))^n : n \in \mathbb{N}_0\right\} \cup \{0\} \tag{12}$$

For example, let $C_\varphi \in K(H^2)$ and we take $\varphi(w) = \frac{-1}{3}w$, then fixed point $a = 0$ and $\varphi(0) = \frac{-1}{3}$ and $\sigma(C_\varphi) = \left\{\left(\frac{-1}{3}\right)^n\ : n = 0, 1, 2, 3, \ldots\right\} \cup \{0\}$. □

Theorem 3.17. *Let $\beta = (\beta_n)$ be a sequence define as $\beta_n = (n+1)^{1-t}$, where $t \geq 1$, and φ is a parabolic automorphism of D and C_φ acting on $H_\beta^2(D)$ then each point of the unit circle is an eigen value of C_φ with infinite multiplicity and the spectrum and essential spectrum of C_φ are the unit circle.*

Proof. Clearly, $\beta_n \geq 0$. If α is fixed point of, then $\varphi(\alpha) = 1$. So by Lemma 8.3 in [10], that the spectrum of C_φ is contained in the disk $(\lambda : |\lambda| \leq 1\ )$, since $C_\varphi^{-1} = C_{\varphi^{-1}}$ has also spectrum contained in the closed unit disk, the spectrum of C_φ is contained in the unit circle. We assume that either $\varphi_1(w) = \frac{(1+i)w-1}{w+i-1}$ or, $\varphi_2(w) = \frac{(1-i)w-1}{w-i-1}$, as C_φ is same as a composition operator with symbol φ_1 or φ_2. In this case, let $f(w) = e^{\frac{\alpha(w+1)}{w-1}}$, where $\alpha \geq 0$, then f is bounded analytic function and $f \in H_\beta^2$. Moreover $f(\varphi_1(w)) = e^{-2i\alpha} f(w)$. So f is an eigen vector with eigen value $e^{-2i\alpha}$, where $\alpha \geq 0$ and $f(\varphi_2(w)) = e^{2i\alpha} f(w)$, so f is also an eigen vector with eigen value $e^{2i\alpha}$, where $\alpha \geq 0$. Then both cases has infinitely many values like as every points of the unit circle and in each case, every point of the unit circle is an eigenvalue of infinite multiplicity. Hence the spectrum and the essential spectrum of

C_φ are each equal to the unit circle. □

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