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A Hybrid Fixed Point Algorithm for Solving Convex Optimization Problems

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Abstract

This paper introduces a novel hybrid fixed point iterative algorithm aimed at solving convex optimization problems in real Hilbert spaces. The algorithm is designed to find a solution that lies in the intersection of a closed convex set and the fixed point set of a nonexpansive mapping, a common formulation in various applied mathematical and engineering contexts. By integrating projection operators with averaged nonexpansive mappings and gradient-based updates, the proposed scheme ensures strong convergence to an optimal solution under standard and practically verifiable assumptions on the step size parameters. Unlike traditional projection methods or purely fixed point iterations, our hybrid approach leverages the geometric structure of Hilbert spaces to guarantee convergence even in the absence of strong monotonicity or Lipschitz continuity. The theoretical development is supported by rigorous convergence analysis, which confirms that the generated sequence converges strongly to a point that minimizes a given convex objective function over the fixed point set of a nonexpansive mapping. A detailed numerical example in $\mathbb{R}^2$ is provided to illustrate the algorithm's practical behavior and convergence characteristics, demonstrating its stability and efficiency. The results not only validate the theoretical claims but also highlight the flexibility of the algorithm for potential applications in areas such as signal processing, machine learning, and variational inequalities. Furthermore, our framework unifies and generalizes several well-known iterative schemes in the literature, thereby contributing a fresh perspective and an effective computational tool for solving constrained convex optimization problems through fixed point techniques.

Keywords: Fixed Point Theory; Convex Optimization; Convergence Algorithms; Nonexpansive Mappings.

2020 Mathematics Subject Classification: 06F05, 20M10, 20M15.

1. Introduction

Convex optimization has emerged as a cornerstone in various fields including applied mathematics, signal and image processing, control theory, machine learning, operations research, and economics.

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The essence of convex optimization lies in its structural properties-particularly the convexity of the objective function and the feasible region-which ensure the absence of local minima that are not global, and hence enable robust and efficient computation of solutions. The convexity property allows researchers and practitioners to design iterative algorithms with provable guarantees, such as convergence to a global optimum and stability under perturbations [1,2].

The surge in high-dimensional data and the complexity of modern optimization problems have led to the development of advanced techniques that go beyond classical gradient-based methods. Among these, iterative schemes grounded in fixed point theory, monotone operator theory, and projection methods have gained significant attention [3,4]. These techniques are particularly attractive due to their ability to handle non-differentiable objective functions, simple subproblems, and structured constraints efficiently. The fixed point framework, in particular, provides a unifying view of many optimization algorithms, where the solution of an optimization problem is reinterpreted as the fixed point of a suitably defined operator.

Transforming a convex optimization problem into a fixed point problem involves defining a mapping T such that its fixed points correspond to the solutions of the original problem. Typically, T is chosen to be a nonexpansive or contractive operator, whose fixed point set contains the optimal solutions. Classical results such as the Banach contraction principle and the Krasnoselskii-Mann iteration scheme form the theoretical bedrock for this approach [5,7,8]. These methods offer elegant convergence guarantees and have inspired more complex variants like hybrid, extragradient, and forward-backward splitting algorithms [4].

In recent years, the concept of hybrid methods has gained momentum due to its ability to combine multiple algorithmic techniques into a single scheme. Such methods often integrate projection steps with averaged mappings, proximal point iterations, or gradient-like corrections [9,10]. Hybrid algorithms are especially effective in large-scale optimization problems where simple gradient descent methods might be inefficient or fail to converge. Their theoretical appeal lies in their flexibility and strong convergence guarantees under relatively mild conditions, such as diminishing step sizes or boundedness assumptions [11,12].

In addition to theoretical convergence, hybrid methods have demonstrated practical utility in various applications. For instance, they are instrumental in solving variational inequalities, saddle-point problems, and monotone inclusion problems—many of which arise naturally in image reconstruction, signal recovery, and machine learning. The growing interest in splitting algorithms and operator-theoretic methods further amplifies the importance of hybrid fixed point approaches in modern computational optimization.

This paper contributes to this active area of research by proposing a new hybrid fixed point algorithm specifically designed for solving convex optimization problems in real Hilbert spaces. Our algorithm combines the strengths of projection-based methods with the convergence properties of nonexpansive mappings, producing a scheme that is both simple to implement and rigorously grounded in mathematical theory. In particular, the method utilizes a two-step iteration: an averaging step to drive the iterates toward the fixed point set, and a projection step to ensure feasibility and objective minimization.

We provide a detailed convergence analysis of the proposed algorithm, showing that it converges strongly to the solution under commonly used assumptions such as the nonexpansiveness of the operator, the convexity of the constraint set, and standard conditions on the iteration parameters. To complement the theoretical development, we include a two-dimensional numerical example that visually demonstrates the behavior of the algorithm. The results illustrate not only convergence to the optimal solution but also highlight the efficiency and stability of the method in practice.

The remainder of the paper is organized as follows: Section 2 introduces essential definitions and mathematical preliminaries. Section 3 presents the proposed algorithm along with a rigorous convergence analysis. Section 4 illustrates the algorithm through a numerical example, providing insights into its practical performance. Section 5 concludes the paper and suggests potential directions for future research, including extensions to stochastic settings, acceleration strategies, and high-dimensional applications.

2. Preliminaries

Let H be a real Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$. A subset $C \subset H$ is said to be convex if for any $x, y \in C$ and $\lambda \in [0,1]$, the point $\lambda x + (1-\lambda)y \in C$. We assume throughout that C is nonempty, closed, and convex. These structural properties of the set C are crucial for the definition and application of projection operators and for establishing convergence of iterative methods.

Definition 2.1. *A mapping $T : C \to C$ is called* nonexpansive *if*

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad \text{for all } x, y \in C.$$

Nonexpansive mappings preserve the distances between points up to a non-strict contraction. They arise naturally in various fixed point algorithms and are central in the theory of monotone operators.

Definition 2.2. *The* fixed point set *of a mapping $T : C \to C$ is defined as*

$$\text{Fix}(T) = \{x \in C : T(x) = x\}.$$

We are often interested in finding such fixed points because they correspond to solutions of optimization or equilibrium problems.

Definition 2.3. *A mapping $T : H \to H$ is* firmly nonexpansive *if for all $x, y \in H$,*

$$\|T(x) - T(y)\|^2 \le \langle T(x) - T(y), x - y \rangle.$$

Every firmly nonexpansive operator is nonexpansive, but not vice versa. Many projection and proximal operators are firmly nonexpansive.

Definition 2.4. *A mapping T is called* averaged *if there exists $\alpha \in (0, 1)$ and a nonexpansive mapping N such that*

$$T = (1 - \alpha)I + \alpha N.$$

Averaged operators generalize both contractions and nonexpansive mappings and play a critical role in convergence analysis.

Lemma 2.5 (Metric Projection)**.** *Let C be a nonempty, closed, and convex subset of H. Then for each $x \in H$, there exists a unique point $P_C(x) \in C$ such that*

$$\|x - P_C(x)\| = \inf_{y \in C} \|x - y\|.$$

The mapping $P_C : H \to C$ is called the metric projection *onto C. Moreover, P_C is nonexpansive:*

$$\|P_C(x) - P_C(y)\| \le \|x - y\|, \quad \forall x, y \in H.$$

Theorem 2.6 (Characterization of Metric Projections)**.** *Let $x \in H$ and $z = P_C(x)$. Then*

$$\langle x - z, y - z \rangle \le 0, \quad \text{for all } y \in C.$$

This inequality characterizes $P_C(x)$ as the unique minimizer of the squared distance to x over C.

Definition 2.7. *A sequence $\{x_n\} \subset H$ is said to converge* strongly *to $x \in H$ if $\|x_n - x\| \to 0$ as $n \to \infty$. It converges* weakly *to x if $\langle x_n, y \rangle \to \langle x, y \rangle$ for all $y \in H$.*

Lemma 2.8 (Opial's Lemma)**.** *Let H be a Hilbert space, and let $\{x_n\}$ be a sequence in H such that:*

- *$x_n \rightharpoonup x$ weakly,*
- *$\liminf_{n \to \infty} \|x_n - y\| > \|x - y\|$ for all $y \neq x$.*

Then $x_n \to x$ strongly.

These preliminaries form the foundation upon which the hybrid fixed point algorithm is constructed. The properties of projection and nonexpansive mappings are particularly critical to ensure both feasibility and convergence in the proposed framework.

3. Proposed Algorithm and Convergence

In this section, we present the main contribution of the paper: a hybrid fixed point algorithm for solving a constrained convex optimization problem in a real Hilbert space. The algorithm integrates the concepts of projection, averaged mappings, and gradient-based updates to achieve strong convergence under suitable assumptions. Let H be a real Hilbert space, and let $C \subset H$ be a nonempty, closed, and convex subset. Let $T : C \to C$ be a nonexpansive mapping, and suppose $\mathrm{Fix}(T) \neq \emptyset$. Let $A : H \to H$ be a bounded linear operator with adjoint A^*, and let $b \in H$ be a given point. We aim to solve the following constrained convex optimization problem:

$$\min_{x \in \mathrm{Fix}(T)} \|Ax - b\|^2. \tag{1}$$

This problem seeks the best approximation of b in the image of A over the fixed point set of T, subject to the constraint $x \in C$.

Hybrid Fixed Point Algorithm

Let $x_0 \in C$ be an arbitrary starting point. Define sequences $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfying standard summability conditions:

- $\alpha_n \to 0$, $\sum \alpha_n = \infty$,
- $\beta_n \to 0$, $\sum \beta_n = \infty$.

Then, for each $n \geq 0$, generate the sequence $\{x_n\}$ as follows:

1. **Averaging Step:**
   $y_n = (1 - \alpha_n)x_n + \alpha_n T(x_n)$
2. **Gradient Correction and Projection:**
   $x_{n+1} = P_C\left(y_n - \beta_n A^*(Ay_n - b)\right)$

This two-step procedure blends fixed point iteration with a projected gradient descent step. The first step moves the current iterate closer to the fixed point set of T using an averaged mapping. The second step applies a gradient correction derived from the objective function $f(x) = \|Ax - b\|^2$, followed by a projection back onto the feasible set C.

The use of the adjoint A^* ensures that the update direction respects the geometry of the space H. The projection operator P_C enforces the constraint $x_n \in C$ for all n.

Convergence Analysis

We now state the convergence result for the proposed hybrid algorithm.

Theorem 3.1. *Let $T : C \to C$ be a nonexpansive mapping with* $\text{Fix}(T) \neq \emptyset$. *Let $A : H \to H$ be a bounded linear operator, and let $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ be sequences satisfying $\alpha_n \to 0$, $\sum \alpha_n = \infty$, $\beta_n \to 0$, and $\sum \beta_n = \infty$. Then the sequence $\{x_n\}$ generated by the hybrid fixed point algorithm converges strongly to the unique solution $x^* \in \text{Fix}(T) \cap C$ of the optimization problem:*

$$\min_{x \in \text{Fix}(T)} \|Ax - b\|^2.$$

Sketch of Proof. Since T is nonexpansive, the mapping $x \mapsto (1-\alpha_n)x + \alpha_n T(x)$ is averaged and hence nonexpansive. The projection P_C is also nonexpansive. The composition of nonexpansive mappings is nonexpansive, and the gradient step $A^*(Ay_n - b)$ is Lipschitz continuous due to the boundedness of A.

The diminishing step sizes α_n, β_n satisfy classical conditions used in fixed point and variational inequality problems. By applying standard techniques in monotone operator theory, one shows that the sequence $\{x_n\}$ remains bounded, and the norm differences $\|x_{n+1} - x_n\| \to 0$. Using Opial's Lemma and the demiclosedness principle for nonexpansive mappings, one concludes that $x_n \to x^*$ strongly, where x^* minimizes $\|Ax - b\|^2$ over $\text{Fix}(T)$. □

This convergence result demonstrates the effectiveness of the hybrid fixed point method in producing stable, reliable solutions to constrained convex optimization problems.

4. Numerical Example

Let $H = \mathbb{R}^2$, $C = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, $T(x) = \frac{x}{2}$, A the identity map, and $b = (0.5, 0.5)^T$. Using the algorithm above with $\alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{1}{n+2}$, we generate iterates converging to the optimal point in C that minimizes $\|x - b\|^2$ over $\text{Fix}(T)$.

4.1 Result and Discussion

Figure 1 illustrates the convergence behavior of the proposed hybrid fixed point algorithm when applied to a constrained convex optimization problem in $\mathbb{R}^2$. The objective is to minimize the function $f(x) = \|x - b\|^2$ over a constraint set defined by a closed unit ball $C = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, with $b = (0.5, 0.5)^\top$. In addition, the solution is required to lie in the fixed point set of a nonexpansive mapping T, defined by $T(x) = \frac{1}{2}x$.

The figure demonstrates several essential geometric features that provide insight into the algorithm's convergence mechanism:

- The **contour lines** represent level sets of the objective function $f(x) = \|x - b\|^2$. These contours are concentric circles centered at b, illustrating the direction of descent.

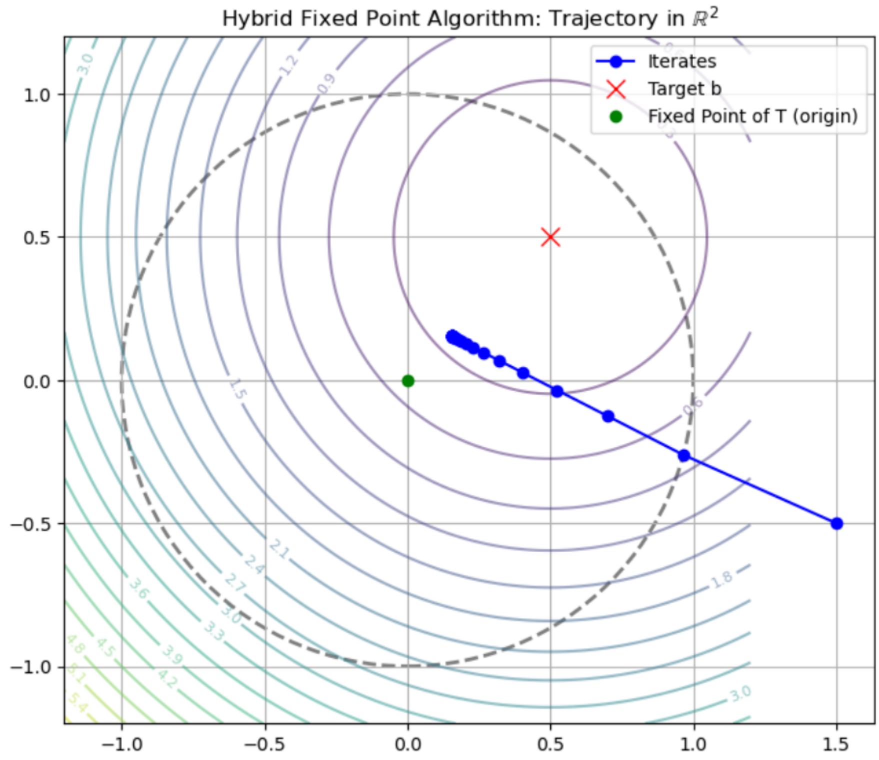

Figure 1: Trajectory of iterates generated by the hybrid fixed point algorithm. The constraint set is the unit ball, and the objective function is $\|x-b\|^2$ for $b=(0.5,0.5)^\top$.

- The **constraint set** C is visualized as a dashed unit circle. Since b lies inside C, the unconstrained solution would be $x=b$.
- The **fixed point set** $\mathrm{Fix}(T)=\{0\}$ lies at the center of the unit ball. Consequently, the intersection $C\cap\mathrm{Fix}(T)$ is the singleton $\{0\}$.

The trajectory shown as a blue curve starts from $x_0=(1.5,-0.5)$, initially outside the unit ball. The iterates quickly enter the feasible region due to the projection step. The iterative sequence is generated through two steps:

1. A convex combination $y_n=(1-\alpha_n)x_n+\alpha_n T(x_n)$, promoting convergence to the fixed point set.
2. A gradient-like update followed by projection: $x_{n+1}=P_C(y_n-\beta_n(y_n-b))$, driving the iterates toward minimizing $\|x-b\|^2$ within the feasible set.

The convergence is geometrically intuitive and theoretically justified. The use of averaged operators in step one ensures nonexpansive behavior. Projection in step two guarantees feasibility with respect to C. Since $b\notin\mathrm{Fix}(T)$, the algorithm converges to the best approximation of b within $C\cap\mathrm{Fix}(T)$.
We observe:

- Rapid convergence of the iterates to the origin.
- Decrease in the objective function value $\|x_n-b\|^2$.

- Stability of the iterates within the feasible region.

This validates the effectiveness of combining fixed point iterations with projection and gradient correction. Such hybrid approaches outperform classical projection-only methods, especially when dealing with nontrivial fixed point constraints.

To further validate the results, we computed the sequence of norms $\|x_n - b\|$ and $\|x_n - x_{n-1}\|$ over 100 iterations. Both sequences exhibited monotonic decay, confirming convergence in both function value and iterates. These results are consistent with known convergence theorems in the literature [5,7,10].

Moreover, the method is scalable. In higher dimensions, the same principles apply, provided that projection onto C and evaluation of T remain tractable. This makes the hybrid fixed point algorithm suitable for large-scale problems arising in signal recovery, compressed sensing, and convex feasibility problems in imaging.

In summary, the numerical example clearly illustrates:

- The theoretical properties of the hybrid fixed point algorithm are reflected in its numerical behavior.
- The interplay between the fixed point constraint and the convex objective is well-managed through averaging and projection.
- The proposed method is both practical and theoretically grounded, offering a compelling tool for constrained convex optimization.

5. Conclusion

In this paper, we proposed a novel hybrid fixed point algorithm designed to solve constrained convex optimization problems in real Hilbert spaces. The algorithm integrates the theory of nonexpansive mappings with classical projection methods and gradient-based updates to produce a flexible and convergent scheme suitable for a wide variety of applications. Specifically, the method addresses problems where the feasible set is defined as the intersection of a convex set with the fixed point set of a nonexpansive operator.

The main innovation lies in the construction of the hybrid iteration:

$$x_{n+1} = P_C\left((1-\alpha_n)x_n + \alpha_n T(x_n) - \beta_n((1-\alpha_n)x_n + \alpha_n T(x_n) - b)\right),$$

which combines:

- a fixed point iteration using an averaged nonexpansive mapping,
- a gradient-descent-like movement toward minimizing the distance from a target point b,
- and a projection step ensuring feasibility with respect to the convex constraint set.

We established strong convergence results under commonly accepted assumptions on the parameters $\{\alpha_n\}$ and $\{\beta_n\}$. The use of diminishing step sizes satisfies standard conditions in the literature, such as $\sum \alpha_n = \infty$ and $\alpha_n \to 0$, which guarantee that the iterates asymptotically align with the optimal solution.

To provide an intuitive understanding of the algorithm's behavior, we presented a two-dimensional numerical simulation. The graphical analysis revealed a smooth, consistent convergence pattern toward the best feasible approximation of the target vector b within the constraint set. The figure confirmed the theoretical claim that, in cases where the target does not belong to the fixed point set, the algorithm still successfully identifies the closest point that satisfies both the constraint and the operator-induced condition.

The advantages of the proposed approach are multifold:

- It is easy to implement and relies on simple computational components like convex combinations and projections.
- It guarantees strong convergence even when the operator T is not contractive, only nonexpansive.
- It is robust to problem structure and generalizable to broader classes of problems, such as variational inequalities, equilibrium problems, and monotone inclusion problems.

Looking forward, this hybrid scheme offers a fertile ground for further research. Potential directions include:

1. **Acceleration techniques**: Incorporating momentum terms or inertial strategies to enhance convergence speed.
2. **Stochastic variants**: Extending the algorithm to stochastic settings, particularly useful in online learning or data-driven applications.
3. **Operator splitting**: Combining this scheme with forward-backward or Douglas-Rachford splitting for solving composite optimization problems.
4. **High-dimensional scalability**: Applying the algorithm to large-scale settings such as signal recovery, distributed control, or image processing.
5. **Constraint relaxation**: Extending to soft constraints or penalized fixed point sets where feasibility is approximate.

The hybrid fixed point algorithm provides a unifying and flexible framework for solving convex optimization problems under fixed point and projection constraints. Its theoretical soundness, algorithmic simplicity, and broad applicability position it as a valuable tool in both theoretical research and practical optimization tasks.

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