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A Note on a Sandwich Type Inequality

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Abstract

In 2016 Andreescu and Saul introduced a sandwich type inequality in their book. In this paper we provided an alternative proof of this inequality by generalizing a problem proposed by Ecker in the journal *MathAMATYC Educator* in 2024. We also introduced the geometry of this inequality.

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1. Introduction

Inequality is an important topic in Mathematics. Since Polya first systematically introduced this topic in [4], there are more and more people studying this topic. Some scholars introduced their studies as a textbook, like [4] and [5], proving each inequality and then applying them in examples. Some scholars gathered many problems involving inequalities, and emphasize on techniques of solving them, like [1] and [2]. In our study, we found one example in [1]. It states:

Example 1.1. *Suppose we have n fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \cdots, \frac{a_n}{b_n}$. If m and M are the smallest and largest of these fractions respectively, show that*

$$m \leq \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq M.$$

The proof provided in the book is very neat and take good advantage of the index.

Proof. (by Andreesue and Saul) Since m and M are the smallest and largest of these fractions, we have $m \leq \frac{a_i}{b_i} \leq M$, or equivalently $mb_i \leq a_i \leq Mb_i$ for every i. Summing these inequalities from $i = 1$ to $i = n$, we have

$$m\left(b_1 + b_2 + \cdots + b_n\right) \leq a_1 + a_2 + \cdots + a_n \leq M\left(b_1 + b_2 + \cdots + b_n\right),$$

which is equivalent to $m \leq \frac{a_1+a_2+\cdots+a_n}{b_1+b_2+\cdots+b_n} \leq M$, the claimed inequality. □

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In 2024, Ecker proposed a problem in [3].

Example 1.2. *(Problem M-3) Let each of $q, s, u > 0$.*

(a). For two fractions $\frac{p}{q} \leq \frac{r}{s}$, prove that $\frac{p}{q} \leq \frac{p+r}{q+s} \leq \frac{r}{s}$.

(b). For three fractions $\frac{p}{q} \leq \frac{r}{s} \leq \frac{t}{u}$, prove that $\frac{p}{q} \leq \frac{p+r}{q+s} \leq \frac{p+r+t}{q+s+u} \leq \frac{r+t}{s+u} \leq \frac{t}{u}$.

By solving this problem, we realize that we can generalize our proof to solve the problem proposed by Andreesue and Saul. Therefore, in the next section, we will first introduce our proof to the Problem M-3 proposed by Ecker. Then we will show our proof to the problem proposed by Andreesue and Saul. In section 3, we will discuss the geometry of these fractions, and the geometric meaning of the resulting inequalities.

2. An Alternative Proof

We will start with our proof to Example 1.2.

Proof. Since $\frac{p}{q} \leq \frac{r}{s}$, we have $ps \leq qr$. Adding pq to both sides and factoring out common factors, we get $p(q+s) \leq q(p+r)$, which is equivalent to $\frac{p}{q} \leq \frac{p+r}{q+s}$. Instead of pq, this time we add rs to both sides of $ps \leq qr$ and factor, we then get $s(p+r) \leq r(q+s)$, which is equivalent to $\frac{p+r}{q+s} \leq \frac{r}{s}$. Combining the two new inequalities, we then proved part (a).
To prove part (b), we apply the inequality in part (a). Since $\frac{p}{q} \leq \frac{r}{s} \leq \frac{t}{u}$, we immediately get $\frac{p}{q} \leq \frac{p+r}{q+s} \leq \frac{r}{s} \leq \frac{r+t}{s+u} \leq \frac{t}{u}$. Considering $\frac{p}{q} \leq \frac{r+t}{s+u}$, we have $\frac{p}{q} \leq \frac{p+r+t}{q+s+u} \leq \frac{r+t}{s+u}$. Similarly, considering $\frac{p+r}{q+s} \leq \frac{t}{u}$, we have $\frac{p+r}{q+s} \leq \frac{p+r+t}{q+s+u} \leq \frac{t}{u}$. Combining all the above, we get $\frac{p}{q} \leq \frac{p+r}{q+s} \leq \frac{p+r+t}{q+s+u} \leq \frac{r+t}{s+u} \leq \frac{t}{u}$, hence proving part (b). □

The next theorem is a generalization of Example 1.2.

Theorem 2.1. *For fractions $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \cdots \leq \frac{a_n}{b_n}$ with denominators all positive, we have*

$$\frac{a_1}{b_1} \leq \frac{a_1+a_2}{b_1+b_2} \leq \cdots \leq \frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} \leq \cdots \leq \frac{a_{n-1}+a_n}{b_{n-1}+b_n} \leq \frac{a_n}{b_n}.$$

Proof. We prove this theorem by induction. The cases of $n = 2$ and $n = 3$ are part (a) and part (b) in Example 1.2, which have been proved earlier. Assume that the inequality is true when $n = k$, a fixed random positive integer. That means, for fractions $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \cdots \leq \frac{a_k}{b_k}$, we have

$$\frac{a_1}{b_1} \leq \frac{a_1+a_2}{b_1+b_2} \leq \cdots \leq \frac{a_1+\cdots+a_k}{b_1+\cdots+b_k} \leq \cdots \leq \frac{a_{k-1}+a_k}{b_{k-1}+b_k} \leq \frac{a_k}{b_k}.$$

Since $\frac{a_1+\cdots+a_k}{b_1+\cdots+b_k} \leq \frac{a_k}{b_k} \leq \frac{a_{k+1}}{b_{k+1}}$, applying part (a) in Example 1.2 on $\frac{a_1+\cdots+a_k}{b_1+\cdots+b_k} \leq \frac{a_{k+1}}{b_{k+1}}$ we have $\frac{a_1+\cdots+a_k}{b_1+\cdots+b_k} \leq \frac{a_1+\cdots+a_{k+1}}{b_1+\cdots+b_{k+1}} \leq \frac{a_{k+1}}{b_{k+1}}$.

According to the assumption again, for fractions $\frac{a_2}{b_2} \le \frac{a_3}{b_3} \le \cdots \le \frac{a_{k+1}}{b_{k+1}}$, we have

$$\frac{a_2}{b_2} \le \frac{a_2 + a_3}{b_2 + b_3} \le \cdots \le \frac{a_2 + \cdots + a_{k+1}}{b_2 + \cdots + b_{k+1}} \le \cdots \le \frac{a_k + a_{k+1}}{b_k + b_{k+1}} \le \frac{a_{k+1}}{b_{k+1}}.$$

Since $\frac{a_1}{b_1} \le \frac{a_2}{b_2} \le \frac{a_2+\cdots+a_{k+1}}{b_2+\cdots+b_{k+1}}$, applying part (a) in Example 1.2 on $\frac{a_1}{b_1} \le \frac{a_2+\cdots+a_{k+1}}{b_2+\cdots+b_{k+1}}$ we have $\frac{a_1}{b_1} \le \frac{a_1+a_2+\cdots+a_{k+1}}{b_1+b_2+\cdots+b_{k+1}} \le \frac{a_2+\cdots+a_{k+1}}{b_2+\cdots+b_{k+1}}$. Summing all the above we therefore have

$$\frac{a_1}{b_1} \le \frac{a_1 + a_2}{b_1 + b_2} \le \cdots \le \frac{a_1 + \cdots + a_k}{b_1 + \cdots + b_k} \le \frac{a_1 + \cdots + a_{k+1}}{b_1 + \cdots + b_{k+1}} \le \frac{a_2 + \cdots + a_{k+1}}{b_2 + \cdots + b_{k+1}} \le \cdots \le \frac{a_k + a_{k+1}}{b_k + b_{k+1}} \le \frac{a_{k+1}}{b_{k+1}}.$$

This shows that the case of $n = k + 1$ is also true. Therefore, according to the principle of mathematical induction, the inequality is true for any integer $n \ge 2$. □

Apparently this theorem implies Example 1.1. It also shows more detail between the minimum, the fraction $\frac{a_1+a_2+\cdots+a_n}{b_1+b_2+\cdots+b_n}$, and the maximum.

3. Geometry of the Inequality

While Andreesue, Saul, and we all use algebraic method to prove this inequality, the geometry of these fractions and the geometric meaning of the inequality is quite interesting.

In a Cartesian plane, consider a random point at the right half plane (not including the y-axis) with coordinates (x_0, y_0). Its position vector is $\langle x_0, y_0 \rangle$, and the slope of this position vector is $\frac{y_0}{x_0}$. Please be aware that, with this setting, x_0 will never be 0 since y-axis is not considered. Also, different points may relate to a same slope, like $(1, 2)$ and $(2, 4)$, whose associated slopes are all 2. Now we consider two points (x_1, y_1) and (x_2, y_2), with position vectors $\langle x_1, y_1 \rangle$ and $\langle x_2, y_2 \rangle$. Apparently, the sum of these two vectors, $\langle x_1 + x_2, y_1 + y_2 \rangle$, will fall in between of $\langle x_1, y_1 \rangle$ and $\langle x_2, y_2 \rangle$. Therefore the slope of $\langle x_1 + x_2, y_1 + y_2 \rangle$ will be between of the two slopes formed by $\langle x_1, y_1 \rangle$ and $\langle x_2, y_2 \rangle$. This provides the geometric explanation of part (a) in Example 1.2.

If we have n points, namely $(x_1, y_1), \cdots, (x_n, y_n)$, still at the right half plane, their centroid would be $\left(\frac{x_1+\cdots+x_n}{n}, \frac{y_1+\cdots+y_n}{n}\right)$. Then, the slope of $\left\langle \frac{x_1+\cdots+x_n}{n}, \frac{y_1+\cdots+y_n}{n} \right\rangle$ is $\frac{y_1+\cdots+y_n}{x_1+\cdots+x_n}$, which will apparently fall between the largest slope and the smallest slope associated to these points. This also provides a geometric explanation of Example 1.1.

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