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Random Normed Stability of General Quartic Functional Equation

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Abstract

The goal of this paper is to study the Generalized Hyers-Ulam-Rassias (HUR) stability of quartic functional equation (Q.F.E.)

$$f_q(kr_1+(k-1)r_2)+f_q(kr_1-(k-1)r_2)=2k^4f_q(r_1)+2(k-1)^4f_q(r_2)$$
$$+6k^2(k-1)^2[f_q(r_1+r_2)+f_q(r_1-r_2)]-12k^2(k-1)^2[f_q(r_1)+f_q(r_2)]$$

in Random Normed Spaces (RN-Spaces) using direct method.

Keywords: Generalised Hyers-Ulam-Rassias (HUR) Stability; Random Normed Spaces; Quartic Functional Equation.

1. Introduction

To begin, let us recall the chronicle in the stability theory for functional equations (FEs). The stability problem for the FEs about the stability of group homomorphisms was started by Ulam [15]. The Ulam′s question was to an extent solved by Hyers [3]. Subsequently, Hyers' result was extended by several mathematicians like Aoki [1], Th. M. Rassias [12], Găvruta [2] and more. The Q.F.E. was first introduced by Rassias [11], who addressed its Ulam stability problem. Later, Lee [6] remodified Rassias' Q.F.E. and obtained its general solution. Numerous mathematicians have extensively studied the stability problems of various Q.F.E.s in a variety of spaces, including random normed spaces, intuitionistic fuzzy normed spaces, non-Archimedean fuzzy normed spaces, Modular spaces, Banach spaces, orthogonal spaces and many other (see [4,5,9,10,13]).

2. Preliminaries

In [14], A. N. Sherstnev instituted random normed spaces, which can be considered as a generalisation of probabilistic metric space instituted by K. Menger [7,8].

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Definition 2.1 (Random Normed Space [14]). *A Random Normed Space (RN-space) is a triplet $(N_r, \zeta, \top)$, where N_r is a vector space, $\top$ is a continuous t-norm and $\zeta : N_r \to D^+$ is a mapping such that the following conditions hold:*

RN1: $\zeta_x(t) = \epsilon_0(t)$ *for all* $t > 0$ *if and only if* $x = 0$,

RN2: $\zeta_{dx}(t) = \zeta_x(t/|d|)$ *for all* $d \in R$, $d \neq 0$, $x \in N_r$ *and* $t > 0$,

RN3: $\zeta_{x+y}(t+s) \geq \top(\zeta_x(t), \zeta_y(s))$, $\forall x, y \in N_r$ *and* $t, s \geq 0$, *where* ζ_x *denotes the value of* ζ *at a point* $x \in N_r$.

Example 2.2.

- *We can label* $(\chi, \varrho, \mathfrak{T}_M)$ *as a RN-space, for every normed space* $(\chi, ||.||)$, $\forall\, t > 0$, *where*

$$\varrho_y(\kappa) = \frac{\kappa}{\kappa + ||y||}.$$

  This space is labeled as the induced RN-space.

- *A countable normed space is also an example of RN-space.*

Theorem 2.3 ([14]). *Let* $(N_r, \zeta, \top)$ *be an RN-space.*

(1). A sequence $\{x_n\}$ *in* N_r *is labeled to be convergent to* $x \in N_r$ *if* $\forall\, \epsilon > 0$ *and* $\zeta > 0$, *there exists* $N \in \mathbb{Z}^+$ *in a manner that* $\zeta_{x_n - x}(\epsilon) > 1 - \zeta$ *whenever* $n \geq N$.

(2). A sequence $\{x_n\}$ *in* N_r *is labeled as a Cauchy sequence in* N_r *if* $\forall\, \epsilon > 0$ *and* $\zeta > 0$, $\exists\, N \in \mathbb{Z}^+$ *in a manner that* $\zeta_{x_n - x_m}(\epsilon) > 1 - \zeta$, *whenever* $n \geq m \geq N$.

(3). The RN-space $(N_r, \zeta, \top)$ *is labeled to be complete if every Cauchy sequence in* N_r *is convergent.*

(4). If $(N_r, \zeta, \top)$ *is RN-space and* $\{x_n\}$ *is a sequence in a manner that* $x_n \to x$, *then* $\lim_{n\to\infty} \zeta_{x_n}(t) = \zeta_x(t)$.

(5). Let $(N_r, \mu, \top_M)$ *be an RN-space and define* $A_{\lambda,\mu} : X \to \mathbb{R}^+ \cup \{0\}$ *as* $A_{\lambda,\mu}(x) = inf\{t > 0; \mu_x(t) > 1 - \lambda\}$, *for all* $\lambda \in (0,1)$ *and* $x \in N_r$. *Then* $A_{\lambda,\mu}(x_1 - x_n) \leq A_{\lambda,\mu}(x_1 - x_2) + ... + A_{\lambda,\mu}(x_{n-1} - x_n)$, *for all* $x_1, x_2, ..., x_n \in N_r$ *and the sequence* $\{x_n\}$ *is convergent to* x *w.r.t. random norm* μ *iff* $A_{\lambda,\mu}(x_n - x) \to 0$ *as* $n \to \infty$.

3. RN Stability of Q.F.E.

In present segment, we will study the following Q.F.E.

$$\begin{aligned} f_q(kr_1 + (k-1)r_2) + f_q(kr_1 - (k-1)r_2) &= 2k^4 f_q(r_1) + 2(k-1)^4 f_q(r_2) \\ &+ 6k^2(k-1)^2[f_q(r_1 + r_2) + f_q(r_1 - r_2)] - 12k^2(k-1)^2[f_q(r_1) + f_q(r_2)] \end{aligned} \tag{1}$$

in RN-Spaces using direct method. It is easy to see that for any real constant t, $f_q(r_1) = tr_1^4$ is solution of (1), thus it is natural to label it as a Q.F.E. Throuhout the section, we label $\mathcal{V}$ as a real linear space and define a difference operator $\hat{D}_{qf}$ as follows:

$$\hat{D}f_q(r_1, r_2) = f_q(kr_1 + (k-1)r_2) + f_q(kr_1 - (k-1)r_2) - 2k^4 f_q(r_1) - 2(k-1)^4 f_q(r_2)$$
$$- 6k^2(k-1)^2[f_q(r_1 + r_2) + f_q(r_1 - r_2)] + 12k^2(k-1)^2[f_q(r_1) + f_q(r_2)]$$

for all $r_1, r_2 \in \mathcal{V}$.

Theorem 3.1. *Let $(N_r, \zeta, \top)$ be a complete RN-space and $f_q : \mathcal{V} \to N_r$ be a mapping which maps zero to zero. Further, let $\Phi : \mathcal{V}^2 \to D^+$ be a mapping such that*

$$\zeta_{\hat{D}_{qf}(r_1,r_2)}(t) \geq \Phi_{r_1,r_2}(t). \tag{2}$$

$\forall\ r_1, r_2 \in \mathcal{V}$ and all $t > 0$, where $\Phi(r_1, r_2)$ is denoted by Φ_{r_1,r_2}. If

$$\lim_{n\to\infty} \top_{m=1}^{\infty}\Big(\Phi_{k^{(m+n-1)}r_1,0}(2k^{(3m+4n)}t)\Big) = 1 \tag{3}$$

$$\lim_{n\to\infty} \Phi_{k^n r_1, k^n r_2}(k^{4n}t) = 1, \tag{4}$$

$\forall\ r_1, r_2 \in \mathcal{V}$ and all $t > 0$, then $Q_{f_q}(r_1) = \lim\limits_{n\to\infty} \frac{f_q(k^n r_1)}{4n}$ exists $\forall\ r_1 \in \mathcal{V}$ and defines a unique quartic mapping $Q_{f_q} : \mathcal{V} \to N_r$ such that

$$\zeta_{Q_{f_q}(r_1) - f_q(r_1)}(t) \geq \top_{m=1}^{\infty}\Big(\Phi_{k^{(m-1)}r_1,0}(2k^{3m}t))\Big), \tag{5}$$

$\forall\ r_1, r_2 \in \mathcal{V}$ and all $t > 0$.

Proof. **Existence:** Assuming $r_2 = 0$ in (2) we get, $\zeta_{2f_q(kr_1) - 2k^4 f_q(r_1)}(t) \geq \Phi_{r_1,0}(t)$, for all $r_1 \in \mathcal{V}$ and all $t > 0$. Or we can say,

$$\zeta_{\frac{1}{k^4}f_q(kr_1) - f_q(r_1)}(t) \geq \Phi_{r_1,0}(2k^4 t), \tag{6}$$

for all $r_1 \in \mathcal{V}$ and all $t > 0$. After replacing r_1 by $k^n r_1$ in (6) we get,

$$\zeta_{\frac{1}{k^{4(n+1)}}f_q(k^{(n+1)}r_1) - \frac{1}{k^{4n}}f_q(k^n r_1)}(t) \geq \Phi_{k^n r_1,0}(2k^{4(n+1)}t),$$

for all $r_1 \in \mathcal{V}$ and all $t > 0$. That is

$$\zeta_{\frac{1}{k^{4(n+1)}}f_q(k^{(n+1)}r_1) - \frac{1}{k^{4n}}f_q(k^n r_1)}\left(\frac{t}{k^{(n+1)}}\right) \geq \Phi_{k^n r_1,0}(2k^{3(n+1)}t), \tag{7}$$

for all $r_1 \in \mathcal{V}$ and all $t > 0$. Since,

$$\frac{1}{k^{4n}} f_q(k^n r_1) - f_q(r_1) = \sum_{m=0}^{n-1} \left(\frac{1}{k^{4(m+1)}} f_q(k^{(m+1)} r_1) - \frac{1}{k^{4m}} f_q(k^m r_1) \right) \tag{8}$$

and $1 > \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \frac{1}{k^4} + ... + \frac{1}{k^n}$, thus we attain,

$$\begin{aligned} \zeta_{\frac{1}{k^{4n}} f_q(k^n r_1) - f_q(r_1)}(t) &\geq \top_{m=0}^{n-1} \left(\zeta_{\frac{1}{k^{4(m+1)}} f_q(k^{(m+1)} r_1) - \frac{1}{k^{4m}} f_q(k^m r_1)} \left(\frac{t}{k^{(n+1)}} \right) \right) \\ &\geq \top_{m=0}^{n-1} (\Phi_{k^m r_1, 0}(2k^{3(m+1)} t)) \\ &= \top_{m=1}^{n} (\Phi_{k^{(m-1)} r_1, 0}(2k^{3m} t)) \end{aligned} \tag{9}$$

$\forall\, n \in \mathbb{N}$ and all $t > 0$. By putting $k^p r_1$ in place of r_1 in (9), we acquire

$$\zeta_{\frac{1}{k^{4(n+p)}} f_q(k^{(n+p)} r_1) - \frac{1}{k^{4p}} f_q(k^p r_1)}(t) \geq \top_{m=1}^{n} (\Phi_{k^{(m+p-1)} r_1, 0}(2k^{(3m+4p)} t)). \tag{10}$$

Since the right hand side of (10) tends to 1 as n and p tends to ∞, therefore, sequence $\left\{ \frac{1}{k^{4n}} f_q(k^n r_1) \right\}$ is Cauchy in $(N_r, \zeta, \top)$, which converges due to completeness of $(N_r, \zeta, \top)$, so $\exists$ some $Q_{f_q}(r_1) \in N_r$ such that

$$\lim_{n \to \infty} \frac{1}{k^{4n}} f_q(k^n r_1) = Q_{f_q}(r_1),$$

$\forall\, r_1 \in \mathcal{V}$. Taking $k^n r_1$ in place of r_1 and $k^n r_2$ in place of r_2 in (2), we get

$$\zeta_{\frac{\hat{D}_{qf}(k^n r_1, k^n r_2)(t)}{k^{4n}}} \geq \Phi_{k^n r_1, k^n r_2}(k^{4n} t), \tag{11}$$

for all $r_1, r_2 \in \mathcal{V}$ and all $t > 0$. Thus with $n \to \infty$ in (11), we find that Q_{f_q} satisfies (11), $\forall\, r_1, r_2 \in \mathcal{V}$. Thus, Q_{f_q} is a quartic mapping. By taking limit $n \to \infty$ in (9), we get (5).

Uniqueness: Suppose that $Q'_{f_q} : \mathcal{V} \to N_r$ is another mapping which follows (5). In view of the fact that f_q is a quartic mapping thus, Q'_{f_q} and Q_{f_q} are also quartic. Therefore $\forall\, n \in \mathbb{N}$ and every $r_1 \in \mathcal{V}$,

$$Q'_{f_q}(k^n r_1) = k^{4n} Q'_{f_q}(r_1) \qquad \textit{and} \qquad Q_{f_q}(k^n r_1) = k^{4n} Q_{f_q}(r_1).$$

Thus, we have for all $r_1 \in \mathcal{V}$ and all $t > 0$,

$$\begin{aligned} \zeta_{Q'_{f_q}(r_1) - Q_{f_q}(r_1)}(2t) &= \zeta_{\frac{Q'_{f_q}(k^n r_1)}{k^{4n}} - \frac{Q_{f_q}(k^n r_1)}{k^{4n}}}(2t) = \zeta_{Q'_{f_q}(k^n r_1) - Q_{f_q}(k^n r_1)}(2k^{4n} t) \\ &\geq \top(\zeta_{Q'_{f_q}(k^n r_1) - f_q(k^n r_1)}(k^{4n} t), \zeta_{f_q(k^n r_1) - Q_{f_q}(k^n r_1)}(k^{4n} t)) \\ &\geq \top\left(\top_{m=1}^{\infty} \left(\Phi_{k^{m+n-1} r_1, 0}(2k^{3m+4n} t) \right), \top_{m=1}^{\infty} \left(\Phi_{k^{m+n-1} r_1, 0}(2k^{3m+4n} t) \right) \right). \end{aligned}$$

By taking $n \to \infty$, we get $Q'_{f_q}(r_1) = Q_{f_q}(r_1)$ for all $r_1 \in \mathcal{V}$. Hence, the result is ratified. □

Theorem 3.2. *Let $(\d{N}, \Phi, \top_M)$ be an RN-space and $\phi : \mathcal{V}^2 \to \d{N}$ be a function such that for $0 < \mu < k^4$, we have*

$$\Phi_{\phi(kr_1,0)}(t) \geq \Phi_{\mu\phi(r_1,0)}(t) \tag{12}$$

$$\text{and } \lim_{n\to\infty} \Phi_{\phi(k^n r_1, k^n r_2)}(k^{4n}t) = 1, \tag{13}$$

$\forall\, r_1, r_2 \in \mathcal{V}$ and all $t > 0$, where $\top_M(r_1, r_2) = min(r_1, r_2)$. Further, assume $(N_r, \zeta, \top_M)$ as complete RN-space with $f_q : \mathcal{V} \to N_r$ such that,

$$\zeta_{\hat{D}f_q(r_1,r_2)}(t) \geq \Phi_{\phi(r_1,r_2)}(t) \tag{14}$$

$\forall\, r_1, r_2 \in \mathcal{V}$ and $t > 0$ with $f_q(0) = 0$. Then the limit,

$$Q_{f_q}(r_1) = lim_{n\to\infty} \frac{f_q(k^n r_1)}{k^{4n}}$$

exists for all $r_1 \in \mathcal{V}$ and defines a unique quartic mapping $Q_{f_q} : \mathcal{V} \to N_r$ such that

$$\zeta_{Q_{f_q}(r_1) - f_q(r_1)}(t) \geq \Phi_{\phi(r_1,0)}(2(k^4 - \mu)t), \tag{15}$$

$\forall\, r_1, r_2 \in \mathcal{V}$.

Corollary 3.3. *Let $(\d{N}, \Phi, \top_M)$ and $(N_r, \zeta, \top_M)$ be RN-space and complete RN-space respectively. Also, let $p \in (0, 1)$, $z_0 \in \d{N}$ with $f_q : \mathcal{V} \to N_r$ be a mapping satisfying*

$$\zeta_{\hat{D}_f(r_1,r_2)}(t) \;\geq\; \begin{cases} \Phi_{(||r_1||^p + ||r_2||^p)z_0}(t), \\ \Phi_{\delta z_0}(t) \end{cases} \tag{16}$$

and $f_q(0) = 0$, for each $r_1, r_2 \in \mathcal{V}$ and all $t > 0$. Then, there exists a unique $Q_{f_q} : \mathcal{V} \to N_r$ such that

$$\zeta_{Q_{f_q}(r_1) - f_q(r_1)}(t) \;\geq\; \begin{cases} \Phi_{||r_1||^p z_0}(2(k^4 - k^{4p})t) \\ \Phi_{\delta z_0}(2(k^4 - 1)t), \end{cases} \tag{17}$$

for all $r_1 \in \mathcal{V}$ and all $t > 0$, which is quartic in nature.

Proof. Let $\phi : \mathcal{V}^2 \to \d{N}$ as $\phi(r_1, r_2) \;=\; \begin{cases} (||r_1||^p + ||r_2||^p)z_0, \\ \delta z_0. \end{cases}$. Then, proof follows from Theorem 3.2 by plugging $\mu \;=\; \begin{cases} k^{4p} \\ 1 \end{cases}$. □

Theorem 3.4. *Let $(N_r, \zeta, \top)$ be a complete RN-space and $f_q : \mathcal{V} \to N_r$ be a function obeying the equation (2)* *along with* $f_q(0) = 0 \; \forall \, r_1, r_2 \in \mathcal{V}$ *and all* $t > 0$. *If*

$$\lim_{n\to\infty} \top_{m=0}^{\infty}\Big(\zeta_{\frac{r_1}{k^{m+n+1}},0}\Big(\frac{t}{k^{4(m+n)}}\Big)\Big) = 1 \tag{18}$$

$$\lim_{n\to\infty} \zeta_{\frac{r_1}{k^n},\frac{r_2}{k^n}}\Big(\frac{t}{k^{4n}}\Big) = 1, \tag{19}$$

$\forall \, r_1, r_2 \in \mathcal{V}$ *and all* $t > 0$, *then* $D_{f_q}(r_1) = \lim_{n\to\infty} k^{4n} f_q\big(\frac{r_1}{k^n}\big)$ *exists for all* $r_1 \in \mathcal{V}$ *and defines a unique quartic mapping* $D_{f_q} : \mathcal{V} \to N_r$ *which obeys*

$$\zeta_{D_{f_q}(r_1)-f_q(r_1)}(t) \geq \top_{m=0}^{\infty}\Big(\zeta_{\frac{r_1}{k^{m+1}},0}\Big(\frac{t}{k^{4m}}\Big)\Big). \tag{20}$$

$\forall \, r_1, r_2 \in \mathcal{V}$ *and all* $t > 0$.

Proof. **Existence:** Setting $r_2 = 0$ in (2), we claim

$$\zeta_{2f_q(kr_1)-2k^4 f_q(r_1)}(t) \geq \zeta_{r_1,0}(t), \tag{21}$$

$\forall \, r_1 \in \mathcal{V}$ and all $t > 0$. Replacing r_1 by $\frac{r_1}{k^{(n+1)}}$, we get

$$\zeta_{k^{4n} f_q\left(\frac{r_1}{k^n}\right) - k^{4n+4} f_q\left(\frac{r_1}{k^{n+1}}\right)}(t) \;\; \geq \;\; \zeta_{\frac{r_1}{k^{(n+1)}},0}\Big(\frac{t}{2k^{4n}}\Big) \tag{22}$$

$\forall \, r_1 \in \mathcal{V}$ and all $t > 0$. Since,

$$k^{4n} f_q\Big(\frac{r_1}{k^n}\Big) - f_q(r_1) = \sum_{m=0}^{n-1}\Big(k^{4m+4} f_q\Big(\frac{r_1}{k^{m+1}}\Big) - k^{4m} f_q\Big(\frac{r_1}{k^m}\Big)\Big), \tag{23}$$

and $k^n > k^{(n-1)} + k^{(n-2)} + ... + 1$, thus we claim

$$\begin{aligned} \zeta_{k^{4n} f_q\left(\frac{r_1}{k^n}\right) - f_q(r_1)}(2k^{4m}t) \;\; &\geq \;\; \top_{m=0}^{n-1}\Big(\zeta_{k^{4m+4} f_q(\frac{r_1}{k^{m+1}}) - k^{4m} f_q(\frac{r_1}{k^m})}(2k^{4m}t)\Big) \\ &\geq \;\; \top_{m=0}^{n-1}\Big(\zeta_{\frac{r_1}{k^{m+1}},0}(t)\Big). \end{aligned} \tag{24}$$

Hence,

$$\zeta_{k^{4n} f_q\left(\frac{r_1}{k^n}\right) - f_q(r_1)}(t) \geq \top_{m=0}^{n-1}\Big(\zeta_{\frac{r_1}{k^{m+1}},0}\Big(\frac{t}{2k^{4m}}\Big)\Big). \tag{25}$$

Substituting r_1 by $\frac{r_1}{k^p}$ in (25), we attain

$$\zeta_{k^{4(n+p)} f_q\left(\frac{r_1}{k^{n+p}}\right) - k^{4p} f_q\left(\frac{r_1}{k^p}\right)}(t) \geq \top_{m=0}^{n-1}\Big(\zeta_{\frac{r_1}{k^{m+p+1}},0}\Big(\frac{t}{2k^{4(m+p)}}\Big)\Big). \tag{26}$$

Since right hand side of (26) tends to 1 as n and p tends to ∞, thus we get, sequence $\Big\{k^{4n} f_q\Big(\frac{r_1}{k^n}\Big)\Big\}$ as a

Cauchy, so there exists some point $Q_{f_q}(r_1) \in N_r$ such that

$$\lim_{n\to\infty} \left\{k^{4n} f_q\left(\frac{r_1}{k^n}\right)\right\} = Q_{f_q}(r_1),$$

for all $r_1 \in \mathcal{V}$. By replacing r_1 by $\frac{r_1}{k^n}$ and r_2 by $\frac{r_2}{k^n}$ in (2), we get

$$\zeta_{k^{4n}\hat{D}_f\left(\frac{r_1}{k^n},\frac{r_2}{k^n}\right)}(t) \geq \zeta_{\frac{r_1}{k^n},\frac{r_2}{k^n}}\left(\frac{t}{k^{4n}}\right), \tag{27}$$

$\forall\, r_1, r_2 \in \mathcal{V},\ t > 0$. Thus, by taking $n \to \infty$ in (27) and using (19), we have,

$$\begin{aligned} Q_{f_q}(kr_1 + (k-1)r_2) + Q_{f_q}(kr_1 - (k-1)r_2) &= 2k^4 Q_{f_q}(r_1) + 2(k-1)^4 Q_{f_q}(r_2) \\ &\quad + 6k^2(k-1)^2[Q_{f_q}(r_1+r_2) + Q_{f_q}(r_1-r_2)] - 12k^2(k-1)^2[Q_{f_q}(r_1) + Q_{f_q}(r_2)] \end{aligned}$$

The justification of uniqueness of $Q_{f_q}(r_1)$ can be easily generated from proof of Theorem 3.1. □

Theorem 3.5. *Let $(N, \Phi, \top_M)$ be an RN-space and $\phi : \mathcal{V}^2 \to N$ be a function such that, for some, we have $0 < \mu < \frac{1}{k^4}$*

$$\Phi_{\phi(\frac{r_1}{k},0)}(t) \geq \Phi_{\mu\phi(r_1,0)}(t) \tag{28}$$

$$\lim_{n\to\infty} \Phi_{\phi(\frac{r_1}{k^n},\frac{r_2}{k^n})}\left(\frac{t}{k^{4n}}\right) = 1, \tag{29}$$

for all $r_1, r_2 \in \mathcal{V}$, $t > 0$. Further assume, $(N_r, \zeta, \top_M)$ is some complete RN-space and $f_q : \mathcal{V} \to N_r$ is a mapping satisfying (14) along with $f_q(0) = 0$, $\forall\, r_1, r_2 \in \mathcal{V}$ and all $t > 0$. Then, the limit

$$Q_{f_q}(r_1) = \lim_{n\to\infty} k^{4n} f_q\left(\frac{r_1}{k^n}\right)$$

exists and defines a unique quartic mapping $C_{f_q} : \mathcal{V} \to N_r$ in a manner that

$$\zeta_{Q_{f_q}(r_1) - f_q(r_1)}(t) \geq \Phi_{\phi(r_1,0)}\left(\frac{(1-\mu k^4)}{2\mu} t\right). \tag{30}$$

Corollary 3.6. *Let $p > 1$ and $z_0 \in N$ and if $f_q : \mathcal{V} \to N_r$ is mapping satisfying (16) along with $f_q(0) = 0$. Thus, we get a unique quartic mapping $Q_{f_q} : \mathcal{V} \to N_r$ such that*

$$\zeta_{Q_{f_q}(r_1) - f_q(r_1)}(t) \ \geq \ \begin{cases} \Phi_{||r_1||^p z_0}((k^{4p} - k^4)/2t), \\ \Phi_{\delta z_0}(k^4(k-1)/2t), \end{cases} \tag{31}$$

where, $Q_{f_q}(r_1) = \lim\limits_{n\to\infty} k^{4n} f_q\left(\frac{r_1}{k^n}\right)$, $\forall\, r_1 \in \mathcal{V}$ and all $t > 0$.

Proof. Let $\phi : \mathcal{V}^2 \to \mathrm{N}$ be defined as

$$\phi(r_1, r_2) = \begin{cases} (||r_1||^p + ||r_2||^p)z_0, \\ \delta z_0, \end{cases}$$

and to get the result take $\mu = \begin{cases} k^{-4p}, \\ k^{-4} \end{cases}$, in Theorem 3.5. □

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