International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **13(2)**(2025), 183–191
Available Online: http://ijmaa.in

On Some Properties of Legendre Polynomial, Using Fractional Calculus

Pankaj Kumar Shukla[1,*], S. K. Raizada[1]

[1] *Department of Mathematics and Statistics, Dr. Rammanohar Lohia Avadh University, Ayodhya, Uttar Pradesh, India*

Abstract

The main aim of the present paper is to apply the extended Riemann Liouville fractional derivative operator for finding some linear generating relations for Legendre polynomial. Four main results are obtained, which are presented in the form of four theorems.

Keywords: Gamma function; Beta function; Riemann-Liouville fractional derivative; hypergeometric function; Fox H-function; generating functions; Legendre polynomial.

1. Introduction

The subject of fractional calculus, now a days is one of the most rapidly growing subjects of mathematical analysis. The fractional integral operators, involving various Special functions have found significant importance and applications in various sub fields of applicable mathematical analysis. The applications of fractional calculus are also seen in various fields, including turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermal nuclear fusion, non-linear control theory, image processing, nonlinear biological system, astrophysics etc. [1,9,11,13,14,25]. In the last three decades, a number of workers like Love [16], Mc Bride [17], Kalla [18,19], Kalla and Saxena [20], Saigo [21,22], Kilbas [23], have studied the properties, applications & different extensions of various operators of fractional calculus on a number of classical & non classical Special functions & polynomials. A sufficient account of fractional calculus operators along with their properties and applications can be found in the research monographs by Miller and Ross [25], & Kiryakova [24]. The first application of fractional calculus was due to Abel [27] in the solution to the fractional problem. In fractional calculus, the fractional derivatives are defined via fractional integrals. In the recent years, certain extended fractional derivative operators, associated with Special functions have been actively investigated and applied on various Special functions. Authors Agarwal & Choi [12,20], have introduced certain extended fractional derivative operators, and applied them on various Special functions. Motivated by these recent developments in the field of applications of extended

*Corresponding author (shukla.pankaj231991@gmail.com)

fractional derivatives to various Special functions, in the present paper an attempt has been made to obtain some linear generating relations including Legendre Polynomial, using extended Riemann Liouville fractional derivative operator, defined by Choi & Pairs in their very recent paper [1], published in the year 2015.

The extended Gauss hypergeometric function $F_p^{(\alpha,\beta,\kappa,\mu)}(a,b;c;x)$ is defined by Agrawal & Choi [1] as follows:

$$F_p^{(\alpha,\beta,\kappa,\mu)}(a,b;c;x) = \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta,\kappa,\mu)}(b+n,c-b)}{B(b,c-b)}\frac{x^n}{n!}, \tag{1}$$

$|z| < 1$; $\min\{Re(\alpha), Re(\beta), Re(\kappa), Re(\mu)\} > 0$; $Re(c) > Re(b) > 0$; $Re(p) \geq 0$, where, $B(u,v)$ is the familiar Beta function defined as:

$$\begin{aligned} B(u,v) &= \int_0^1 t^{u-1}(1-t)^{v-1}dt, \quad (Re(u) > 0; \quad Re(v) > 0). \\ &= \frac{\Gamma u \Gamma v}{\Gamma v + v}(u,v \in \mathrm{C}), \end{aligned} \tag{2}$$

where Γ denotes the Eulers Gamma function [4]. It is to note here that, for $p = 0$, (1) reduces to the ordinary Gauss hypergeometric function ${}_2F_1(a,b;c;x)$,

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, |z| < 0 \tag{3}$$

Legendre polynomial $P_n(x)$, is defined by

$$P_n(x) = \sum_{m=0}^{\frac{n}{2}} \frac{(-1)^m \left(\frac{1}{2}\right)_{n-m}(2x)^{n-2m}}{m!\,(n-2m)!} \tag{4}$$

The extended beta function $B_p^{(\alpha,\beta,\kappa,\mu)}(x,y)$ is defined by Srivastava [8] as:

$$B_p^{(\alpha,\beta,\kappa,\mu)}(x,y) = \left\{\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^k(1-t)^k}\right)\right\}dt, \tag{5}$$

$k \geq 0$, $\mu \geq 0$, $\min\{Re(\alpha), Re(\beta)\} > 0$; $Re(x) > -Re(\kappa a) > 0$; $Re(y) \geq -Re(\mu a)$ & $Re(\rho)$.

A further extension of the extended Gauss hypergeometric function $F_{p;\kappa,\mu}(a,b;c;z;m)$ is defined by Srivastava as [1]:

$$F_{p;\kappa,\mu}(a,b;c;z;m) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{B_p^{(\alpha,\beta,\kappa,\mu)}(b+n,c-b+m)}{B(b+n,c-b+m)}\frac{z^n}{n!}, \tag{6}$$

where, $(p \geq 0; Re(\kappa) > 0, Re(\mu) > 0; Re(c) > Re(b) > m; Re(p) \geq 0)$.

The extended Riemann-Liouville fractional derivative of $F(z)$ of order v is defined by Agarwal, Choi & Pairs [1] by the following relations:

$$D_z^{v,p,k,\mu}f(z) = \frac{1}{\Gamma - v}\int_0^z (z-t)^{-v-1}f(t)\,dt \;\; {}_1F_1\left(\alpha;\beta;\frac{pz^{k+\mu}}{t^k(z-t)^{\mu}}\right)dt, \tag{7}$$

and

$$D_z^{v,p,k,\mu} f(z) = \frac{d^m}{dz^m} D_z^{v-m;p;k,\mu} f(z)$$
$$= \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma m - v}\int_0^z (z-t)^{m-v-1} f(t)\, dt \times {}_1F_1\left(\alpha;\beta;-\frac{pz^{k+\mu}}{t^k(z-t)^{\mu}}\right) dt\right\}, \tag{8}$$

where $(m-1 \leq Re(v) < m), (Re(v) < 0; Re(p) > 0; Re(k) > 0; Re(\mu) > 0)$. From (7) & (8), it may easily be seen that for $p = 0$, we obtain the classical-Riemann Liouville fractional derivative. In the present paper, we have obtained certain linear generating relations, involving Legendre polynomial (4), using operators (7) & (8).

2. Preliminaries

While proving the main results, the following well-known identities & results will be used:
The elementary identity [24]:

$$[(1-x)-t]^{-\alpha} = (1-t)^{-\alpha}\left(1 - \frac{x}{1-t}\right)^{-\alpha} \tag{9}$$

The identity [7]:

$$[(1-x)-t]^{-\alpha} = (1-t)^{-\alpha}\left(1 + \frac{xt}{1-t}\right)^{-\alpha} \tag{10}$$

The result [1]:

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} F_{p,k,\mu}(\alpha+n,\lambda,v;z;m)t^n = (1-t)^{-\alpha} F_{p,k,\mu}\left(\alpha,\lambda;v;\frac{x}{(1-t)};m\right) \tag{11}$$

The generalized binomial theorem [1]:

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!}, (\alpha \in C) \tag{12}$$

$$(z)_n = \frac{\Gamma z + n}{\Gamma z}, \quad [4] \tag{13}$$

The result [1]:

$$D_z^{\lambda-v,p,k,\mu}\{z^{\lambda-1}(1-z)^{-\alpha}\} = \frac{\Gamma(\lambda)\, z^{(v-1)}}{\Gamma v} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\lambda)_n}{(v)_n} \frac{B_p^{\alpha,\beta,k,\mu}(\lambda+n, v-\lambda+m)}{B(\lambda+n, v-\lambda+m)} \frac{z^n}{n!},$$
$$= \frac{\Gamma(\lambda)\, z^{(v-1)}}{\Gamma v} F_{p,\kappa,\mu}(\alpha,\lambda;v;z;m), \tag{14}$$

where, $m-1 \leq Re(v) < m$, for some $m \in N$ & $Re(v) < Re(\lambda)$.

The result [1]:

$$D_z^{\lambda-v,p,k,\mu}\{(1-az)^{-\alpha}(1-az)^{-\beta}\} = \frac{\Gamma(\lambda)\, z^{(v-1)}}{\Gamma v} F_{1,p,k,\mu}(\alpha,\beta,\lambda;v;az;bz;m), \tag{15}$$

where, $m-1 \le Re(v) < m$, for some $m \in N$ & $Re(v) < Re(\lambda)$.

The result [1]:

$$D_z^{\lambda-v,p,k,\mu}\left\{(1-z)^{-\alpha} z^{\lambda-1} F_{p,k,\mu}\left(\alpha,\lambda;v;\frac{x}{(1-t)};m\right)\right\} = \frac{\Gamma(\lambda)\, z^{(v-1)}}{\Gamma v} F_{2,p,k,\mu}(\alpha,\beta,\lambda;v;x;z;m), \tag{16}$$

where, $m-1 \le Re(v) < m$, for some $m \in N$, & $Re(v) < Re(\lambda)$.

3. Linear Generating Relations for the Legendre Polynomial $P_n(x)$

We use extended fractional derivatives, defined in (8), for establishing some linear generating relations for the Legendre polynomial:

Theorem 3.1. *The following linear generating relation holds*

$$D_x^{v,\rho;\kappa,\mu}\{P_n(x)\} = \sum_{m=0}^{\frac{n}{2}} \frac{(-1)^m \left(\frac{1}{2}\right)_{n-m} (2)^{n-2m}}{m!\,(n-2m)!} \frac{\Gamma(n-2m+1)\, \beta_p^{\alpha,\beta;\kappa,\mu}(n-2m+1, m-v)}{\Gamma(n-2m-v+1), \beta(n-2m+1, m-v)} x^{n-2m-v} \tag{17}$$

where, $\{|x| < \min(1, |1-t|)\}$, $\left(\alpha \in C, |x| < 1; |\frac{t}{1-y}| < 1\right)$, *&* $m-1 \le Re(\beta - x) < m < Re(\beta)$, *for some* $m \in N$, *&* $Re(x) < Re(v)$.

Proof. Applying (8) in (17) to the function x^{n-2m}, we obtain:

$$D_x^{v,\rho;\kappa,\mu}\{P_n(x)\} = \sum_{m=0}^{\frac{n}{2}} \frac{(-1)^m \left(\frac{1}{2}\right)_{n-m} (2)^{n-2m}}{m!\,(n-2m)!} \frac{d^m}{dx^m}\left\{\frac{1}{\Gamma(m-v)} \int_0^x (x-t)^{n-2m-v-1} t^n \times {}_1F_1\left(\alpha;\beta;-\frac{px^{k+\mu}}{t^k(x-t)^\mu}\right) dt\right\}$$

Setting $t = xu$ in above expression, we obtain:

$$D_x^{v,\rho;\kappa,\mu}\{P_n(x)\} = \left(\frac{d^m}{dx^m}(x)^{n-2m-v}\right) \times$$

$$\sum_{m=0}^{\frac{n}{2}} \frac{(-1)^m \left(\frac{1}{2}\right)_{n-m} (2)^{n-2m}}{m!\,(n-2m)!} {}_1F_1 \frac{1}{\Gamma(m-v)} \int_0^1 (1-u)^{n-2m-v-1} u^{n-2m-v-1} \left(\alpha;\beta;-\frac{px^{k+\mu}}{u^k(1-u)^\mu}\right) du,$$

where $\left(\frac{d^m}{dz^m}(x)^{n-2m-v}\right) = \frac{\Gamma(1+n-v+2m)}{\Gamma(n-v+1)} x^{n-2m-v}$. Using (4) & (5), on right hand side of the last equation, we obtain the desired result:

$$D_x^{v,\rho;\kappa,\mu}\{P_n(x)\} = \sum_{m=0}^{\frac{n}{2}} \frac{(-1)^m \left(\frac{1}{2}\right)_{n-m} (2)^{n-2m}}{m!\,(n-2m)!} \frac{\Gamma(n-2m+1)\, \beta_p^{\alpha,\beta;\kappa,\mu}(n-2m+1, m-v)}{\Gamma(n-2m-v+1), \beta(n-2m+1, m-v)} x^{n-2m-v}$$

which is the desired result (17), and thus Theorem 3.1 is established. □

Theorem 3.2. *The following linear generating relation holds:*

$$D_x^{\upsilon,\rho;\kappa,\mu}\left\{P_n\left(x\right)\right\} = 2^{n-2m}\sum_{m=0}^{\frac{n}{2}}\left(\alpha\right)_n\frac{\left(-1\right)^m\left(\frac{1}{2}\right)_{n-m}\left(2x\right)^{n-2m}}{m!\left(n-2m\right)!}\left(x\right)^{n-2m} \tag{18}$$

where, $\{|x| < \min(1,|1-t|)\}$, $\left(\alpha \in C, |x| < 1; |\frac{t}{1-y}| < 1\right)$, *&* $m-1 \leq Re(\beta - x) < m < Re(\beta)$, *for some* $m \in N$, *&* $Re(x) < Re(v)$.

Proof. Applying (8) in (18) to the function $f(z)$ with its series expansion, we obtain

$$D_x^{\upsilon,\rho;\kappa,\mu}\{P_n\left(x\right)\} = \sum_{m=0}^{\frac{n}{2}}\frac{\left(-1\right)^m\left(\frac{1}{2}\right)_{n-m}\left(2\right)^{n-2m}}{m!\left(n-2m\right)!}\frac{d^m}{dx^m}\left\{\frac{1}{\Gamma(m-v)}\int_0^x\left(x-t\right)^{n-2m-v-1}t^n\right.$$
$$\left.\times{}_1F_1\left(\alpha;\beta;-\frac{px^{k+\mu}}{t^k(x-t)^\mu}\right)dt\right\}$$

Setting $t = xu$, in above expression, we obtain:

$$D_x^{\upsilon,\rho;\kappa,\mu}\{P_n\left(x\right)\} = \left(\frac{d^m}{dx^m}\left(x\right)^{n-2m-v}\right)\times$$

$$\sum_{m=0}^{\frac{n}{2}}\frac{\left(-1\right)^m\left(\frac{1}{2}\right)_{n-m}\left(2\right)^{n-2m}}{m!\left(n-2m\right)!}{}_1F_1\frac{1}{\Gamma\left(m-v\right)}\int_0^1\left(1-u\right)^{n-2m-v-1}u^{n-2m-v-1}\left(\alpha;\beta;-\frac{pz^{k+\mu}}{u^k\left(1-u\right)^\mu}\right)du$$

$$D_x^{\upsilon,\rho;\kappa,\mu}\left\{P_n\left(x\right)\right\} = 2^{n-2m}\sum_{m=0}^{\frac{n}{2}}\left(\alpha\right)_n\frac{\left(-1\right)^m\left(\frac{1}{2}\right)_{n-m}\left(2x\right)^{n-2m}}{m!\left(n-2m\right)!}\left(x\right)^{n-2m}$$

which is the desired result (18), and thus Theorem 3.2 is established. □

Theorem 3.3. *The following linear generating relation holds by Using Maclaurin expansion:*

$$D_x^{\upsilon,\rho;\kappa,\mu}\{x^{\lambda-1}P_n\left(x\right)\} = \sum_{m=0}^{\frac{n}{2}}\frac{\left(-1\right)^m\left(\frac{1}{2}\right)_{n-m}\left(2\right)^{n-2m}}{m!\left(n-2m\right)!}D_x^{\upsilon,\rho,\kappa,\mu}x^{(\lambda+n-2m-1)}$$
$$= 2^{n-2m}\frac{\Gamma n-2m\left(z\right)^{\lambda+n-2m-1}}{\Gamma\left(\lambda-v\right)}\left(\alpha\right)_n$$
$$\times\sum_{m=0}^{\frac{n}{2}}\frac{\left(-1\right)^m\left(\frac{1}{2}\right)_{n-m}\left(2\right)^{n-2m}}{m!(n-2m)!}\frac{\left(\lambda\right)_n}{\left(\lambda-v\right)_n}\frac{\beta_\rho^{\alpha,\beta,\kappa,\mu}\{(\lambda+n-2m),(m-v)\}}{\beta(\lambda+n-2m,m-v)}\left(x\right)^{n-2m} \tag{19}$$

where, $\{|x| < \min(1,|1-t|)\}$, $(\alpha \in C, |x| < 1; |\frac{t}{1-y}| < 1)$ *&* $m-1 \leq Re(\beta - x) < m < Re(\beta)$, *for some* $m \in N$ *&* $Re(x) < Re(v)$.

Proof. Using Maclaurin expansion,

$$f\left(x\right) = \sum_{n=0}^{\infty}(\alpha)_n x^n,$$

we can write:

$$D_x^{v,\rho;\kappa,\mu}\left\{f\left(x\right)\right\}=2^{n-2m}\sum_{m=0}^{\frac{n}{2}}\left(\alpha\right)_n\sum_{m=0}^{\frac{n}{2}}\frac{\left(-1\right)^m\left(\frac{1}{2}\right)_{n-m}\left(2\right)^{n-2m}}{m!\left(n-2m\right)!}x^{n-2m}$$

Applying (17), on the R.H.S of the last equation, we obtain:

$$2^{n-2m}\sum_{m=0}^{\frac{n}{2}}(\alpha)_n D_x^{v,\rho;\kappa,\mu}(x)^{n-2m}=$$

$$\sum_{m=0}^{\frac{n}{2}}(\alpha)_n\frac{\left(-1\right)^m\left(\frac{1}{2}\right)_{n-m}\left(2\right)^{n-2m}}{m!\left(n-2m\right)!}\frac{\Gamma\left(n-2m+1\right)D_z^{v,\rho;\kappa,\mu}(n-2m+1,m-v)}{\Gamma(n-2m-v+1),\beta(n-2m+1,m-v)}(x)^{n-2m}$$

Applying (18), on both sides of the last equation, we obtain:

$$D_x^{\lambda-v,\rho,\kappa,\mu}\left\{x^{\lambda-1}D_x^{v,\rho;\kappa,\mu}2^{n-2m}\sum_{m=0}^{\frac{n}{2}}(\alpha)_n D_x^{v,\rho;\kappa,\mu}(x)^{n-2m}\right\}=$$

$$\sum_{m=0}^{\frac{n}{2}}(\alpha)_n\frac{\left(-1\right)^m\left(\frac{1}{2}\right)_{n-m}\left(2\right)^{n-2m}}{m!\left(n-2m\right)!}\frac{(\lambda)_n}{(\lambda-v)_n}\frac{\beta_\rho^{\alpha,\beta,\kappa,\mu}\{(\lambda+n-2m),(m-v)\}}{\beta(\lambda+n-2m,m-v)}(x)^{n-2m}$$

$$D_z^{v,\rho;\kappa,\mu}\left\{z^{\lambda-1}P_n\left(x\right)\right\}=2^{n-2m}\frac{\Gamma n-2m(z)^{\lambda+n-2m-1}}{\Gamma\left(\lambda-v\right)}(\alpha)_n\times$$

$$\sum_{m=0}^{\frac{n}{2}}\frac{\left(-1\right)^m\left(\frac{1}{2}\right)_{n-m}\left(2\right)^{n-2m}}{m!\left(n-2m\right)!}\frac{(\lambda)_n}{(\lambda-v)_n}\frac{\beta_\rho^{\alpha,\beta,\kappa,\mu}\{(\lambda+n-2m),(m-v)\}}{\beta(\lambda+n-2m,m-v)}(x)^{n-2m}$$

which is the desired result (19), and thus Theorem 3.3 is established. □

Theorem 3.4. *The following linear generating relation holds by Using the generalized binomial expansion:*

$$D_x^{v,\rho;\kappa,\mu}\left\{x^{\lambda-1}\left(1-x\right)^{-\alpha}\right\}=2^{n-2m}\sum_{m=0}^{\frac{n}{2}}\left(\alpha\right)_n D_x^{v,\rho;\kappa,\mu}\left(x\right)^{n-2m}\times$$

$$\frac{\Gamma\lambda(x)^{n-2m+v-1}}{\Gamma\left(v\right)}2^{n-2m}\sum_{m=0}^{\frac{n}{2}}(\alpha)_n\frac{\left(-1\right)^m\left(\frac{1}{2}\right)_{n-m}\left(2\right)^{n-2m}}{m!\left(n-2m\right)!}D_x^{v,\rho;\kappa,\mu}\frac{(\lambda)_n}{(v)_n}\frac{\beta_\rho^{\alpha,\beta,\kappa,\mu}\{(\lambda+n-2m),(m+v-\lambda)\}}{\beta(\lambda+n-2m,m+v-\lambda)}$$

$$=\frac{\Gamma\lambda(z)^{n-2m+v-1}}{\Gamma(v)}\sum_{m=0}^{\frac{n}{2}}(\alpha)_n\frac{\left(-1\right)^m\left(\frac{1}{2}\right)_{n-m}\left(2\right)^{n-2m}}{m!\left(n-2m\right)!}F_{p,k,\mu}(\alpha,\lambda;v;z;m),\tag{20}$$

where, $\{|x|<\min(1,|1-t|)\}$, $(\alpha\in C,|x|<1;|\frac{t}{1-y}|<1)$, *&* $m-1\leq Re(\beta-x)<m<Re(\beta)$, *for some* $m\in N$ *&* $Re(x)<Re(v)$.

Proof. Using the generalized binomial expansion,

$$(1-x)^{-\alpha}=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}x^n,$$

we can write:

$$D_x^{\lambda-v,\rho;\kappa,\mu}\{x^{n-2m+\lambda-1}(1-x)^{-\alpha}\}\sum_{m=0}^{\frac{n}{2}}(\alpha)_n\frac{(-1)^m\left(\frac{1}{2}\right)_{n-m}(2)^{n-2m}}{m!\,(n-2m)!}$$
$$=D_x^{\lambda-v,\rho;\kappa,\mu}\{x^{n-2m+\lambda-1}2^{n-2m}\}\sum_{m=0}^{\frac{n}{2}}(\alpha)_n D_x^{v,\rho;\kappa,\mu}(x)^{n-2m},$$

Applying (17), on both sides of the last equation, we obtain:

$$\sum_{m=0}^{\frac{n}{2}}(\alpha)_n\frac{(a)_n(b)_n}{(c)_n n!}\{D_x^{\lambda-v,\rho;\kappa,\mu}(x^{n-2m+\lambda-1})\}=\frac{\Gamma\lambda+n-2m\,(z)^{n-2m+v-1}}{\Gamma\,(v+n-2m)}$$

$$\sum_{m=0}^{\frac{n}{2}}(\alpha)_n\frac{(-1)^m\left(\frac{1}{2}\right)_{n-m}(2)^{n-2m}}{m!\,(n-2m)!}\frac{\beta_\rho^{\alpha,\beta,\kappa,\mu}\,(\lambda+n-2m)\,,(m+v-\lambda)}{\beta\,(\lambda+n-2m,m+v-\lambda)}(x^{v+n-2m-1})$$
$$=\frac{\Gamma\lambda(z)^{n-2m+v-1}}{\Gamma(v)}\sum_{m=0}^{\frac{n}{2}}(\alpha)_n\frac{(-1)^m\left(\frac{1}{2}\right)_{n-m}(2)^{n-2m}}{m!\,(n-2m)!}\frac{\beta_\rho^{\alpha,\beta,\kappa,\mu}\{(\lambda+n-2m),(m+v-\lambda)\}}{\beta(\lambda+n-2m,m+v-\lambda)}(x^{n-2m}),$$

Applying (18), on both sides of the last equation, we obtain:

$$D_x^{v,\rho;\kappa,\mu}\left\{x^{\lambda-1}\,(1-x)^{-\alpha}\right\}\sum_{m=0}^{\frac{n}{2}}(\alpha)_n\frac{(-1)^m\left(\frac{1}{2}\right)_{n-m}(2)^{n-2m}}{m!\,(n-2m)!}$$
$$=\frac{\Gamma\lambda(x)^{n-2m+v-1}}{\Gamma(v)}\frac{(-1)^m\left(\frac{1}{2}\right)_{n-m}(2)^{n-2m}}{m!\,(n-2m)!}F_{p,k,\mu}\,(\alpha,\lambda;v;x;m)\,.$$

which is the desired result (20), and thus Theorem 3.4 is established. □

4. Concluding Remarks

Linear and bilateral generating relations have been of much interest to various researchers in the recent past. Various mathematicians investigating and introducing certain extended fractional derivatives and integral operators and applying them on various Special functions and obtaining linear and bilateral generating relations involving some Special functions. In the present paper, an attempt has been made to obtain some linear generating relations for ordinary Gauss hypergeometric function, applying the extended Riemann Liouville fractional derivative operator.

References

[1] P. Agarwal, J. Choi and R. B. Paris, *Extended Riemann-Liouville fractional derivative operator and its applications*, Journal of Nonlinear Science and Applications, 8(2015), 451-466.

[2] M. Saigo, *A Remark on integral operations involving the Gauss hypergeometric functions*, Math Rep., XI(2)(1978), 135-143.

[3] A. A. Kilbas, O. I. Marichev and S. G. Samko, *Fractional integrals and derivatives*, Translated from the 1987 Russian original, Gordon & Breach, Y. Verdon, (1993).

[4] E. D. Rainville, *Special Functions*, Macmillan, New York, (1960).

[5] J. B. Seaborn, *Hypergeometric functions & their applications*, Springer New York, NY, (1991).

[6] V. Kiryakova, *All the special functions are fractional different integral of elementary functions*, J. Phys. A., 30(14)(1997), 5085-5103.

[7] K. S Millar, *The Mittag-Leffler and related functions*, Integral Transforms and Special Functions, 1(1)(1993), 41-49.

[8] S. D. Lin, H. M. Srivastava, and P. Y. Wang, *Some fractional calculus Result for the H-function associated with a class of Feymann integrals*, Russ. J. Math. Phyo., 13(1)(2006), 94-100.

[9] A. A. Kilbas and N. Sebastian, *Generalized Fractional integration of Bessel function of the first kind*, Integral Transforms and Special Functions, 19(11-12)(2008), 869-883.

[10] A. A. Kilbas and N. Sebastian, *Fractional Integration of the product of Bessel Function of first kind*, Fract. Calc. Apple. Anal., 13(2)(2010), 159-175.

[11] A. Baricz and J. Sandor, *Extensions of the Generalized Wilker inequality to Bessel functions*, J. Math, Inequal., 2(3)(2008), 397-406.

[12] H. M. Srivastava, *An Introductory overview of Bessel Polynomials, the Generalized Bessel polynomials and q- Bessel polynomials*, Symmetry, 15(2023), 822.

[13] P. Agarwal, J. Choi, S. Mathur and S. D. Purohit, *Certain new integral formula Involving the Generalized Bessel Functions*, Bul. Korean Math. Soc., 51(2014), 995-1003.

[14] G. N. Watson, *A Treatise on the theory of Bessel Function*, Cambridge University Press, Cambridge, (1994).

[15] G. Dattoli and A. Torre, *Theory and Applications of Generalized Bessel Functions via Raffaele Garofalo*, ARACNE, (1996).

[16] E. R. Love, *some integral equations involving hypergeometric functions*, Proceeding of the Edinburgh Mathematical Society, 15(3)(1967), 169-198.

[17] A. C. Mc Bride, *Fractional powers of a class of ordinary differential operators*, Proceeding of the Londan Mathematical Society, 45(1982), 419-546.

[18] S. L. Kalla, *Integral operators involving Fox' H -function*, Acta Mexicana de Ciencia, Technology, 8(3)(1969), 117-122.

[19] S. L. Kalla, *Integral operators involving Fox′ H – function II*, Acta Mexicana de Ciencia, Technology, (1969).

[20] S. L. Kalla and R. K. Saxena, *Integral operators involving hypergeometric functions*, Mathematische Zeitschrift, 108(1969), 231-234.

[21] M. Saigo, *A remark on integral operators involving the Gauss hypergeometric functions*, Mathematical Reports of college of General Education, Kyushu, University, 11(1978), 135-153.

[22] M. Saigo, *A certain boundary value Problem for the Euler-Darboux equation*, Mathematica Japonica, 24(4)(1979), 377-385.

[23] A. A. Kilbas, *Fractional Calculus of the Generalized Wright Function*, Fractional Calculus & Applied Analysis, 2005(2005), 113-126.

[24] V. Kiryakova, *Generalized Fractional calculus & Application*, Vol. 301, Longman Scientific & Technical, Essex, UK, (1994).

[25] K. S. Miller and B. Ross, *Introduction to the Fractional Calculus and Fractional Differential equations*, Wiley-Inter-Science, John Wiley & sons, New York, NY, USA, (1993).

[26] P. Agarwal and S. D. Purohit, *The Unified pathway fractional integral formulae*, Journal of Fractional Calculus and Applications, 4(1)(2023), 105-112.

[27] T. Iryna, I. Podlubny and I. M. Richard, *Historical Survey Niels Hernik Abel & the birth of Fractional Calculus*, An International Journal for Theory and Applications, 20(5)(2017).