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A Study on Hub Sets in Hypergraphs

K. S. Shama[1,*], P. M. Divya[1], T. V. Ramakrishnan[1]

[1]*Department of Mathematical Sciences, Kannur University, Kannur, Kerala, India*

Abstract

The concept of hub sets in hypergraphs, previously introduced in the literature, is further explored in this article. Let H be a hypergraph. A subset $S \subseteq V(H)$ is called a *hub set* of H if for every pair of vertices $u, v \in V(H) - S$, either u and v are adjacent in H, or there exists an S-hyperpath connecting them. The *hub number* of H, denoted $h(H)$, is defined as the minimum cardinality of such a hub set. In this work, the hub number is computed for various classes of hypergraphs. Additionally, the notion of vertex contraction in hypergraphs is introduced, and its influence on the hub number is investigated. Bounds on the hub number in terms of other hypergraph parameters are also established.

Keywords: Hypergraph; Hub set; Hub number; vertex contraction.
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1. Introduction

Hypergraphs, which generalize graphs by allowing edges (called hyperedges) to connect more than two vertices, provide a natural and powerful framework for modeling complex relationships in various domains, such as computer science, biology, and social networks. Many classical graph-theoretic concepts have been extended to hypergraphs to capture richer structural properties and interactions. One such concept is the notion of a *hub set* [7], originally introduced in the context of graphs and more recently extended to hypergraphs. A hub set is a subset of vertices that acts as a connective intermediary among the remaining vertices. Specifically, for a hypergraph $H = (V, E)$, a subset $S \subseteq V$ is called a *hub set* if for every pair of vertices $u, v \in V \setminus S$, either u and v are adjacent (i.e., contained in a common hyperedge), or there exists a path from u to v whose internal vertices lie entirely in S. The *hub number* $h(H)$ is the minimum cardinality of such a set. The study of hub sets in hypergraphs is motivated by applications in communication networks, data clustering, and fault-tolerant systems, where identifying minimal sets of intermediary nodes can optimize connectivity and efficiency. In this paper, we investigate the structural and computational aspects of hub sets in hypergraphs. We

*Corresponding author (shamaks1551@gmail.com)

determine the hub number for several classes of hypergraphs, including paths, cycles, and stars. We introduce the concept of vertex contraction in hypergraphs and explore its relationship with the hub number. Furthermore, we derive bounds on the hub number in terms of other hypergraph parameters, shedding light on the interplay between connectivity and structure. This work contributes to the growing body of research on hypergraph connectivity and offers new tools and results that may be useful in both theoretical studies and practical applications.

2. Preliminaries

Matthew Walsh [7] introduced the notion of hub sets and the hub number in graphs, which has since inspired extensive research leading to the development of several variants, including open hub sets, total hub sets, and doubly connected hub sets, among others. The study of hub sets remains an active and evolving area of research. The extension of the hub set concept to hypergraphs was initiated in [9]. For the basic definitions in hypergraph we refer and [4] and [5]. A hypergraph H is a pair (V, E) where V is a non empty finite set of vertices and E is a family of nonempty subsets of V called edges such that their union is V. A hypergraph is called simple if no edge is proper subset of another edge.Two vertices u and v are called adjacent if there exists an edge $e \in E$ containing both u and v. For a vertex v in V the open neighbourhood of v denoted by $N(v)$ is the set of all vertices adjacent to v.The closed neighbourhood of v denoted by $N[v]$ is given by $N(v) \bigcup \{v\}$. A hypergraph is r-uniform if $|e| = r$ for any $e \in E$ Throughout this paper we are considering simple hypergraphs without isolated vertices. The following are some important definitions we used for this paper.

Definition 2.1 (Dominating set [1]). *A set $S \subseteq V$ is said to be a dominating set of H if for every $v \in V - S$ there is $u \in S$ such that u and v are adjacent. The cardinality of minimal dominating set is denoted by $\gamma(H)$.*

Definition 2.2 (Paths in hypergraph [2]). *Let H be a hypergraph and $k \geq 2$ be an integer. A path of length k is a sequence $(x_0, e_1, x_1, e_2, x_2......, e_k, x_k)$ where $x_{i-1}, x_i \in e_i$. Hence a path in a hypergraph is a sequence of edges and vertices where adjacent edges intersects in at least one vertex. The vertices $x_1, x_2, ...x_k$ are called anchors. Any vertex $u \in e_i$ for some $i \in \{1, 2, ...k\}$ that is not a anchor of the path is called a floater of P. A path in a hypergraph is said to be linear if any two adjacent edges intersect in a single vertex.*

Definition 2.3. *The r-uniform linear path P_n^r is the hypergraph $H = (V, E)$, where $V = \{v_i : 1 \leq i \leq n\}$ and $E = \{e_1, e_2, ...e_m\}$ such that $|e_i| = r$ for every i and $|e_i \bigcap e_i{+}1| = 1$ and there are no other intersections.*

Definition 2.4. *The r-uniform linear cycle C_n^r is the hypergraph $H = (V, E)$, where $V = \{v_i : 1 \leq i \leq n\}$ and $E = \{e_1, e_2, ...e_m\}$ such that $|e_i| = r$ for every i and $|e_i \bigcap e_{i+1}| = 1$ for all i and $|e_1 \bigcap e_m| = 1$.*

Definition 2.5 (Weak deletion of vertices). *Let $H = (V, E)$ be a hypergraph and $v \in V$ then $H - v$ obtained by the weak deletion of the vertex v is a hypergraph with vertex set $V - \{v\}$ and edge set is obtained by removing v from each hyper edge of H.*

Definition 2.6 (Separating Vertex [2]). *Let $H = (V, E)$ be a connected hypergraph. A vertex $v \in V$ is a separating vertex if H decomposes in to two nonempty connected hypergraphs with just vertex v in common.ie $H = H_1 \bigoplus H_2$where H_1 and H_2 are two nonempty connected sub hypergraphs of H with $V(H_1) \bigcap V(H_2) = \{v\}$.*

Definition 2.7 (Star hypergraph). *A star hypergarph is a simple hypergraph where all edges intersects at a single vertex.*

3. Main Results

3.1 Hub sets and hub number in hypergraphs

The concept of hub sets in hypergraphs has already been introduced in earlier works [9]. In this paper, we adopt the same definition, but present it using alternative notation and terminology for consistency with our framework and subsequent results.

Definition 3.1. *Let $H = (V, E)$ be a hypergraph. Let $S \subseteq V(H)$ and $x_0, x_k \in V(H)$. An S-hyperpath (simply S-path) between x_0 and x_k is a path $P = (x_0, e_1, x_1, e_2, x_2......, e_k, x_k)$ such that,*

(1) $x_0, x_k \notin S$.

(2) $x_i \in S$ *for* $i = 2, 3, .., k - 1$.

(3) x_i*'s and* e_i*'s are distinct.*

Definition 3.2. *Let $H = (V, E)$ be a hypergraph, then $S \subseteq V$ is a hub set of H if for any $u, v \in V - S$ either u and v are adjacent in H or there exists an S-path from u to v in H. The minimum cardinality of S is the hub number of H, denoted by $h(H)$.*

Theorem 3.3. *Let P_n^r be an r-uniform linear path with k edges. Then the hub number, $h(P_n^r) = k - 1$.*

Proof. Consider an r-uniform linear path P_n^r consisting of k edges $e_1, e_2, \ldots, e_k$, where each e_i is a set of r vertices and any two consecutive edges e_i and e_{i+1} intersect in exactly one vertex. All other edge pairs are disjoint. The total number of vertices in P_n^r is:

$$n = r + (k - 1)(r - 1)$$

The first edge contributes r vertices, and each subsequent edge contributes $r - 1$ new vertices due to the overlap. Define the set:

$$S = \{x_1, x_2, \ldots, x_{k-1}\}$$

where each $x_i = e_i \cap e_{i+1}$ is the shared vertex between consecutive edges e_i and e_{i+1}. Clearly, $|S| = k - 1$.
Claim: S is a hub set of P_n^r.

Let $u, v \in V(P_n^r) - S$. Since the path is linear, any path from u to v must go through the consecutive overlapping edges and thus through the shared vertices $x_i \in S$. That is, any such path has all internal vertices in S. Therefore, S satisfies the hub condition.

$$\Rightarrow h(P_n^r) \leq k - 1$$

Minimality: Suppose there exists a smaller hub set S' with $|S'| < k - 1$. Then at least one vertex $x_j = e_j \cap e_{j+1}$ is not in S'. This means that any path connecting a vertex u in e_j to a vertex v in e_{j+1} cannot include an internal vertex in S', violating the hub path requirement. Thus, S is minimal and

$$h(P_n^r) = k - 1$$

□

Theorem 3.4. *Let C_n^r be an r-uniform linear cycle on n vertices and $k+1$ edges, where each pair of consecutive edges intersects in exactly one vertex and non-consecutive edges are disjoint. Then the hub number $h(C_n^r)$ is given by:*

$$h(C_n^r) = \begin{cases} n-3, & \text{if } r = 2, \\ k, & \text{if } r \geq 3 \text{ and } n = (k+1)(r-1). \end{cases}$$

Proof. **Case 1:** $r = 2$. Then C_n^2 is a simple cycle graph with n vertices. A hub set must ensure that any two non-hub vertices are connected via a path whose internal vertices lie in the hub set. It is known that removing any three consecutive vertices from a cycle disconnects the graph into a path of length $n-3$, and those remaining vertices form a hub set. Thus, the minimal hub set has size $n-3$, and we get:

$$h(C_n^2) = n - 3.$$

Case 2: $r \geq 3$. Let C_n^r be an r-uniform linear cycle with $k+1$ edges. In such a cycle, each edge shares exactly one vertex with the next, and the last edge wraps around to the first to complete the cycle. This implies:

$$n = (k+1)(r-1).$$

We construct a hub set S by selecting one representative vertex from each edge such that consecutive selections respect the overlap structure. For example, choosing the last vertex of each edge (except the final overlap vertex) ensures that any path between non-hub vertices must pass through S. This gives:

$$|S| = k.$$

To show minimality, suppose there exists a smaller hub set S' with $|S'| < k$. Then there exists at least one edge with no representative in S', meaning some pair of vertices in neighboring edges might not be connected via a path through S', violating the hub set condition. Hence, the minimal hub set has cardinality k, and we conclude:

$$h(C_n^r) = k.$$

□

Theorem 3.5. *For the complete r-uniform hypergraph K_n^r, $h(K_n^r) = 0$.*

Definition 3.6. *A hypergraph $H = (V, E)$ is called* pairwise complete hypergraph *if for every pair of distinct vertices $u, v \in V$, there exists a hyperedge $e \in E$ such that $\{u, v\} \subseteq e$. In other words, every pair of vertices is* adjacent, *meaning they are contained together in at least one common hyperedge. Note that a complete r-uniform hypergraph K_n^r is pairwise complete.*

Definition 3.7. *A* pairwise clique *in a hypergraph H is a subset of vertices in which every pair of vertices is adjacent, i.e., each pair lies together in at least one hyperedge of H. The maximum cardinality of such a subset is called the* clique number *of H.*

Theorem 3.8. *A hypergraph H is a pairwise complete hypergraph if and only if $h(H) = 0$.*

Theorem 3.9. *Let $H = (V, E)$ be a hypergraph with n vertices and let k be the pairwise clique number of H then $h(H) \leq n - k$.*

Proof. Let k be the pairwise clique number of H and let Y be the set of all vertices in the clique.Then consider $S = V - Y$, which is obviously a hub set, since any pair of vertices in Y are adjacent. Hence $h(H) \leq n - k$. □

4. Vertex Contraction in hypergraphs

Vertex contraction is a useful tool in analyzing hub sets of hypergraphs. By contracting a vertex and examining the resulting structure, we can understand how local connectivity influences the hub number. In particular, contraction helps to identify key vertices whose neighborhoods significantly impact the existence and size of hub sets.

Definition 4.1. *Let $H = (V, E)$ be a hypergraph and v a vertex in H. We define the contraction of v in H as a hypergraph H/v obtained as follows:*

(1) Delete v weakly from H.

(2) Join each pair of edges in $E_v = \{e - \{v\} : v \in e\}$ to form a hyperedge in H/v. Thus, the neighborhood $N(v)$ induces a pairwise complete hypergraph.

(3) If $e \cup f \in E$, where $e, f \in E_v$, do not add it again to avoid parallel edges.

Theorem 4.2. *If H is a star hypergraph, then $h(H/x) = 0$ for some $x \in V(H)$.*

Proof. Let H be a star hypergraph with hub set $S = \{x\}$. Consider the contraction H/x. Since all edges in a star hypergraph contain the hub x, weakly deleting x results in a set $E_x = \{e - \{x\} : x \in e\}$. By the definition of contraction, we add an edge for every union of pairs $e \cup f$, where $e, f \in E_x$. Thus, in H/x, every pair of vertices that were neighbors through x become directly adjacent. As a result, the vertex set of H/x forms a pairwise complete hypergraph. Hence $h(H/x) = 0$, □

Theorem 4.3. *For the r-uniform linear path P_n^r, $h(P_n^r/x) = \lfloor \frac{n-r}{r-1} \rfloor - 1$, if x is a separating vertex.*

Theorem 4.4. *Let H be a hypegraph and $x \in S$. Then S is a hub set of H if and only if $S - x$ is a hub set of H/x.*

Proof. Let S be a hub set of H and $x \in S$. Since S is a hub set there exists $a, b \in V - S$ such that either there is a hyperedge e such that $a, b, \in e$ or there is an S-path P from a to b with x as one internal vertex. For the first case a and b are still adjacent in H/x. For the second case if x is a separating vertex to any two adjacent edges say e and f in P, then in H/x, $(e \cup f) - \{x\}$ is a hyperedge and is a part of an $S - x$ path from a to b in H/x. If x is not a separating vertex then also $P - \{x\}$ is an $S - x$ path from a to b. Hence $S - x$ is a hub set of H/x.
Conversely, if $S - x$ is a hub set of H/x, then for any $a, b \in V - S$ either a and b are adjacent in H/x or there is a path P' from a to b with internal vertices in $S - x$. So if a and b are adjacent, then either a and b are adjacent in H also or there is a two-path with x as a separating vertex in H. If a and b are connected by an S-x path P' in H/x, then either P' is an S- path in H also or there exists an S-path P connecting a and b with x as an internal vertex. Hence S is a hub set of H. □

Corollary 4.5. *If S is a hubset and $x \in S$, then $h(H/x) \leq h(H) - 1$.*

Definition 4.6. *If $S \subseteq V$ then H/S is obtained by contracting vertices successively from S.*

Theorem 4.7. *Let $S \subseteq V$ then S is a hub set of H if and only if H/S is pairwise complete hypergraph.*

Proof. Let S be a hub set of H and $x, y \in V - S$. By definition, either x and y are adjacent in H or there is an S-path from x to y in H with internal vertices in S. If e is a hyperedge containing x and y and there is no path from x to y, then still there exists an edge containing x and y in H/S. Thus x and y are adjacent in H/S. Now if P is a path from x to y with internal vertices in S, then P will be contracted to a hyperedge $f \subseteq N(S)$ containing x and y. Hence x and y are adjacent in H/S. Thus H/S is pairwise complete.
Conversely suppose H/S is pairwise complete. Choose $u, v \in V - S$. Then $u, v \in V(H/S)$ are adjacent in H/S. Suppose they are not adjacent in H. By definition of contraction of vertices there must be an S path from u to v in H, with anchor vertices in S, which is contracted to a hyper edge .Thus S is a hubset of H. □

Theorem 4.8. *For the complete r-partite , r-uniform hypergraph*

$$h(K_{n_1,n_2},\dots,_{n_r}) = \begin{cases} 0 & \text{if } n_i = 1 \text{ for all } i \\ 1 & \text{if and only if } n_i \le 2 \text{ for at least one i.} \\ 2 & \text{if } n_i \ge 3 \text{ for all i.} \end{cases}$$

Proof. Let $H = K_{n_1,n_2},\dots,_{n_r}$ and $V(H) = V_1 \cup V_2 \cup ... \cup V_r$ be the partition of the vertex set of H. By definition any two vertices $v_i \in V_i, v_j \in V_j$ in H are adjacent if $i \neq j$.

Case 1: If $n_i = 1$ for all i, then there is only one edge say $(v_1 v_2 \dots v_r)$. Hence all pairs of vertices are adjacent and thus $S = \phi$ and $h(K_{n_1,n_2},\dots,_{n_r}) = 0$.

Case 2: If $n_i = 1$ for at least one i, let that vertex be v_i, then v_i will be adjacent to all other vertices hence on contraction H/v_i will be complete. Thus the minimum hubset will be $S = \{v_i\}$ and hence $h = 1$. Suppose $n_i = 2$ for at least one i, say $\{u_i, v_i\}$ be such a pair. Then take $S = \{u_i\}$. For any non-adjacent pair $u_j, v_j \in V_j$, there exists an S-path $u_j \to u_i \to v_j$. If $v_j \in V_j$ and $v_k \in V_k$, where $j \neq k$, they are adjacent. Similarly if $j = i$, then v_i and v_k are adjacent. Hence S is a hub set and $h(K_{n_1,n_2},\dots,_{n_r}) = 1$.

Case 3: Suppose $n_i \geq 3$ for all i. Note that we can not contract a single vertex to get a pairwise complete hypergraph. Take $S = \{v_1, v_2 : v_i \in V_i\}$. Then For any pair $u_i, w_i \in V_i$ there is an S path $u_i \to v_1 \to w_i$ if $i \neq 1$ or $u_i \to v_2 \to w_i$ if $i \neq 2$. Thus H/S becomes a pairwise complete hypergraph. Hence $h(K_{n_1,n_2},\dots,_{n_r}) = 2$. □

Theorem 4.9. *Let H be a hypergraph with an edge x and let $L(H)$ denote the line graph of H,then $L(H/x) = L(H)/x$.*

Proof. Let e and f be two edges in H. Then they are adjacent in $L(H/x)$ if either e and f are adjacent in H or there is a path $e - x - f$ in H. For the first case e and f were adjacent in $L(H)$ and hence in $L(H)/x$ also. For the second case if e and f are not adjacent in $L(H)$ and $e - x - f$ is a path in H, then $e - x - f$ is a path in the line graph $L(H)$ also. If x is contracted from $L(H)$ then by definition any two vertices adjacent to x become adjacent. Hence e and f become adjacent in $L(H)/x$. These are the only two ways a pair of vertices in $L(H)/x$ can be adjacent. Hence $L(H/x) = L(H)/x$. □

Theorem 4.10. *Let H be a connected hypergraph and $d(H)$ denote the diameter of H. If x is vertex in H, then $d(H) - 1 \leq d(H/x) \leq d(H)$.*

Proof. Let P be a maximum shortest path in H. Then either P passes through x or it does not. If x is a separating vertex of P, then any two neighbours say u and v of x in P will become adjacent in H/x. So P will have its length reduced by one. Also if x is one of the end vertex of P, then the diameter will be reduced by one if the edge containing x in P has only two vertices. In these cases $d(H/x) = d(H) - 1$. If x is a floater vertex in P then $d(H/x) = d(H)$. If x is an end vertex in P with more than one vertex in the edge containing x, then also $d(H/x) = d(H)$. Now suppose $x \notin P$. If x is adjacent to any two anchor vertices u,v in P, then there exists a path P' connecting u, x and v. If the distance from u to v

in P is greater than the length of P', then P is no more a shortest path. Hence any two neighbours of x contained in P can be at most distance two apart. If the distance is two, then $d(H/x) = d(H) - 1$. From this observation it is clear that contracting $x \notin P$ will leave either P as it is or produces a new path between its end points of length one less than the length of P. Further it is easy to see that any shortest path in H/x is either a shortest path in H or arises from a path in H of length one greater. Hence the proof. □

Theorem 4.11. *Suppose H is a connected hypergraph with n vertices and m edges. Then $h(H) = 1$ if and only if the following two conditions are satisfied:*

(1) H is not a pairwise complete hypergraph.

(2) Either $\Delta(H) = m$, or there exists a vertex $v \in V(H)$ such that the induced sub hypergraph $H[V - N[v]]$ is pairwise complete, and every vertex in $N(v)$ is adjacent to every vertex in $V - N[v]$.

Proof. Suppose $h(H) = 1$, then it is clear that H is not pairwise complete. Let $S = \{v\}$ be a hub set of H and $d(v) = m$. Then $\Delta(H) = m$ and in this case $V - N[S] = \phi$. Now suppose that $V - N[S] \neq \phi$. Since S is a hub set, H/S is pairwise complete and the induced hypergraph $H[V - N[v]]$ is pairwise complete. Also by hub property every vertex in $N(v)$ is adjacent to every vertex in $V - N[V]$.
Converse part is trivial. □

Corollary 4.12. *Let H be a connected hypergraph with n vertices. Then $h(H) = n - 2$ if and only if H is a 2-uniform path.*

Proof. If H is a 2-uniform path, then clearly $h(H) = n - 2$.
Conversely, suppose $h(H) = n - 2$ and H is a connected hypergraph. Let

$$k = \max\{|e| : e \in E(H)\}$$

be the maximum edge size in H. It is known that for any hypergraph H, we have $h(H) \leq n - k$. Therefore, if $h(H) = n - 2$, it must be that $k = 2$, which implies that H is a 2-uniform hypergraph, i.e., a simple graph. Now suppose H contains a cycle, say $C_k = v_1 v_2 \ldots v_k v_1$ with $k \geq 3$. Consider the set $S = V(H) - \{v_1, v_2, v_3\}$. In this case, and the induced subgraph H/S is complete on these three vertices, i.e., isomorphic to K_3. Hence, S is a hub set of H with cardinality $n - 3$, which implies $h(H) \leq n - 3$, contradicting the assumption that $h(H) = n - 2$. Therefore, H must be acyclic and 2-uniform, which implies that H is a 2-uniform path. □

Definition 4.13 (Star vertex of a hypergraph)**.** *A vertex in a hypergraph H whose open neighbourhood has maximum cardinality in H is called a star vertex. If v is a star vertex then we write $\sigma(H) = |N(v)|$ ie, $\sigma(H) = Max\{|N(x)| : x \in V(H)\}$.*

Theorem 4.14. *If H is a hypergraph, then $h(H) \leq |V(H)| - \sigma(H)$.*

Proof. Note that if H is a star hypergraph $K_{1,n}$ then $h(H) = 1$. If $S = \{v\}$ is a minimum hubset of H, then $\sigma(H) = n - 1$. Hence $h(H) = |V(H)| - \sigma(H)$. For any hypergraph H with v as a star vertex, $S = V - N(v)$ will be a hubset. Hence $h(H) \leq |V(H)| - \sigma(H)$. Let H be a connected hypergraph with minimum hubset S and $v \in S$ be a star vertex in H. We shall show that if $\sigma(H/v) \leq \sigma(H) - 1$, then either $S = \{v\}$ or we can find another minimum hub set S' which does not contain v. Assume that $|N(v)| = \sigma(H)$. Consider H/v. Then any neighbour of v in H will have at least $\sigma(H) - 1$ neighbours in H/v. If any of these vertices say u is adjacent to any other vertex in $V - N[v]$, then $|N(u)| \geq \sigma(H)$ in H/v. Which implies the number of neighbours of u in H/v has not decreased by contracting v. Now suppose that none of the vertices in $N(v)$ is adjacent to any vertex in $V - N[v]$. Then by the connectedness of H, H/v must be complete. Hence $S = \{v\}$ and $h(H) = 1$. Now suppose that $v \in S$ is such that $|N(v)| < \sigma(H)$. If $u \in V(H)$ is such that $|N(u)| = \sigma(H)$ and all vertices in $N[v]$ is adjacent to u then the set $S' = S \cup \{u\} - \{v\}$ is also a minimum weak hubset of H. Hence, we can always find a minimum hub set such that, upon contracting vertices one at a time, the maximum degree of the resulting hypergraph decreases by at most one at each step. □

Theorem 4.15. *For any hypergraph H, $\gamma(H) \leq h(H) + 1$.*

Proof. Let S be a hub set of H, then every pair of vertices in $V - N[S]$ is adjacent. Hence a vertex $v \in V - N[S]$ will dominate all other vertices in $V - N[S]$. Let $S' = S \cup \{v\}$. Note that for any $u \in N(S)$, there exists at least one vertex $w \in S$ hence in S'. Thus all vertices in $V - S$ will be dominated by S'. Hence $\gamma(H) \leq h(H) + 1$. □

Theorem 4.16. *For any connected hypergraph H, $h(H) \leq \gamma_c(H)$.*

Proof. Let S be a connected dominating set of H, then for any $x, y \in V - S$ there are vertices u and v in S dominating x and y respectively. Since S is connected there exists a path from u to v in S. Hence there exists a path from x to y with internal vertices in S. Which implies $h(H) \leq \gamma_c(H)$. □

Theorem 4.17. *If H is a hypergraph, then $h(H) \leq \tau_c(H)$.*

Proof. If S is a connected transversal set, then it intersects with every edge, that is if $u \notin S$, then for every edge e such that $u \in e$, $e \cap S \neq \phi$. Hence if $u, v \notin S$ then there exists vertices $u', w' \in S$ and edges e_1 and e_2 such that $u, u' \in e_1$ and $w, w' \in e_2$. Since S is connected there exists a path from u' to v' in S. Hence S is a hub set. Which implies $h(H) \leq \tau_c(H)$. □

Theorem 4.18. *Given any two positive integers k and n, with $k < n$ there exists a connected r-uniform hypergraph H such that $h(H) = k = \gamma(H)$.*

Proof. Let $r = \lceil \frac{n+k}{k+1} \rceil$. Consider P_n^r, an r-uniform linear path with n vertices $\{v_1, v_2, \ldots, v_n\}$ and edges $\{f_1, f_2, \ldots, f_{\frac{n-r}{r-1}+1}\}$. Let H be the r-uniform connected hypergraph with $n + k$ vertices obtained by joining k new vertices $\{u_1, u_2, \ldots, u_k\}$ to P_n^r by adding the following edges:

$$\begin{aligned} e_1 &= \{v_2, v_3, \ldots, v_r, u_1\}, \\ e_2 &= \{v_{r+1}, v_{r+2}, \ldots, v_{2r-1}, u_2\}, \\ e_3 &= \{v_{2r}, v_{2r+1}, \ldots, v_{3r-2}, u_3\}, \\ &\vdots \\ e_k &= \{v_{(k-1)(r-1)+2}, v_{(k-1)(r-1)+3}, \ldots, v_{(k-1)(r-1)+r}, u_k\}. \end{aligned}$$

Let $S = \{v_r, v_{2r-1}, v_{3r-2}, \ldots, v_{(k-1)(r-1)+r}\}$. Clearly, S is a dominating set, so $\gamma(H) = |S| = k$. Rename the vertices in S as $\{w_1, w_2, \ldots, w_k\}$ in the ascending order of their indices. Then for any $u_i, u_j \in V - S$ with $i < j$, there is an S-path

$$u_i \xrightarrow{e_i} w_i \xrightarrow{f_{i+1}} w_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_j} w_j \xrightarrow{e_j} u_j.$$

Similarly, for $u_i, v_j \notin S$ with $i < j$, the path

$$u_i \xrightarrow{e_i} w_i \xrightarrow{f_{i+1}} w_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_j} v_j$$

exists. Thus, S is a hub set. If S' is any minimum hub set of H, then by the construction of H, we must have $w_i \in S'$ for all i, and so $h(H) \geq k+1$. By Theorem 4.15, $h(H) \leq k$. Hence, S is a minimum hub set, and therefore $h(H) = k$. □

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