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Common Fixed Point Theorem for Two Self-Mappings in Dislocated Quasi-Metric Spaces with Application

Rajesh Patel[1,*], Rakesh Tiwari[2]

[1] *Department of Mathematics, C. L. C. Government Arts and Commerce College Dhamdha, Durg, Chhattisgarh, India*
[2] *Department of Mathematics, Government V. Y. T. Post Graduate Autonomous College, Durg, Chhattisgarh, India*

Abstract

The purpose of this paper, to establish a coupled common fixed point theorem for self-mappings in dislocated quasi metric space. Moreover, we have an illustrative example to support our results. Also, we provide an application for dislocated quasi metric spaces of integral type.

Keywords: Dislocated quasi metric space; coupled common fixed point; coupled coincidence point; weakly compatible maps.
2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction and Preliminaries

In 2001, Hitzler [7] presented the concept of dislocated metric spaces, as a simultaneously generalized famous Banach contraction principle in dislocated metric space. Later, Zeyada [14] expanded his approach, and a large number of works addressing fixed point results for a single and a pair of mappings satisfying various types of contraction criteria are too widely spread, (see, [1,13]). The fundamental idea of coupled fixed point for non-linear contractions in partially ordered metric spaces was developed by Bhaskar and Lakshmikantham [10]. After that, coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in a complete partially ordered metric space were shown by Lakshmikantham and Ciric [10]. Numerous analysts have become interested in this field of research, and a variety of work has been published in various spaces (see, [2,6,8,9]). In the setting of a dislocated quasi-metric space, coupled fixed point has also been discovered by Mohammad et al. [12]. The following definitions and notations are provided initially and will be used in the main results:

Definition 1.1 ([3]). *Let (X, d) be a metric space. A self-map $T : X \to X$ is said to be a contraction mapping if there exists a constant $k \in [0, 1)$ called a contraction factor such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$.*

*Corresponding author (rajesh.patel@bitdurg.ac.in)

Definition 1.2 ([3]). *Let (X, d) be a metric space and $T : X \to X$ be a self-map. We say that x is a fixed point of T if $Tx = x$.*

Definition 1.3 ([3]). *Suppose (X, d) be a complete metric space and $T : X \to X$ be a contraction, then T has a unique fixed point.*

Definition 1.4 ([7]). *Let X be a non-empty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the conditions:*

1. $d(u, v) = d(v, u) = 0 \Rightarrow u = v$,

2. $d(u, v) \le d(u, w) + d(w, v)$ *for all* $u, v, w \in X$.

Then d is known as dislocated quasi-metric on X and the pair (X, d) is called a dislocated quasi-metric space.

Definition 1.5 ([14]). *A sequence $\{x_n\}$ in a dislocated quasi metric space (X, d) is said to converge to a point $x \in X$ if and only if $\lim\limits_{n\to\infty} d(x_n, x) = \lim\limits_{n\to\infty} d(x, x_n) = 0$.*

Definition 1.6 ([14]). *A sequence $\{x_n\}$ in a dislocated quasi metric space (X, d) is called a Cauchy sequence, if for every $\epsilon > 0$, there exists a positive integer n_0 such that for $m, n > n_0$, we have $d(x_n, x_m) < \epsilon$. That is, $\lim_{n\to\infty} d(x_n, x_m) = 0$.*

Definition 1.7 ([14]). *A dislocated quasi metric space is called complete if every Cauchy sequence converges to an element in the same metric space.*

Definition 1.8 ([4]). *An element $(x, y) \in X \times X$, where X is any non-empty set, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Definition 1.9 ([10]). *An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$, and $(g(x), g(y))$ is called coupled point of coincidence.*

Definition 1.10 ([10]). *An element $(x, y) \in X \times X$, where X is any non-empty set, is called a coupled common fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x) = x$ and $F(y, x) = g(y) = y$.*

Definition 1.11 ([10]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are called commutative if $g(F(x, y)) = F(gx, gy)$ for all $x, y \in X$.*

Definition 1.12 ([10]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are called w-Compatible if $g(F(x, y)) = F(gx, gy)$ and $g(F(y, x)) = F(gy, gx)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.*

Lemma 1.13 ([11]). *If x is a limit of some sequence $\{x_n\}$ in a dislocated quasi-metric space (X, d), then $d(x, x) = 0$.*

2. Main Results

In this section, we developed a coupled common fixed theorem for continuous and commutative mappings in dislocated metric space.

Theorem 2.1. *Let (X,d) be a complete dislocated quasi-metric space and $S,T : X \to X$ are continuous and commutative mappings such that $d(Sx,Ty) \leq \lambda M(x,y)$, where*

$$M(x,y) = max \left\{ \begin{array}{l} 2d(x,y), \frac{2d(x,Sx)d(y,Ty)}{d(x,y)}, \left[d(x,Sx)+d(y,Ty)\right], \\ \frac{\left[d(x,Ty)+d(y,Sx)\right]}{2}, \frac{2\left[d(x,Ty)+d(x,y)\right]}{3}, \\ \left[d(x,Sx)+d(x,y)\right], \left[d(y,Ty)+d(x,y)\right] \end{array} \right\}, \forall \ x, \ y \in X \tag{1}$$

with $d(x,y) \neq 0$ and $\lambda \in \left[0, \frac{1}{2}\right)$. Then S and T has a unique common fixed point in X.

Proof. Assume that S and T verifies (1). Then, we discuss the following different cases.
Case 1 : If $M(x,y) = \frac{2d(x,Sx)d(y,Ty)}{d(x,y)}$, then

$$d(Sx,Ty) \leq \lambda \frac{2d(x,Sx)d(y,Ty)}{d(x,y)}, \forall \ x, \ y \in X.$$

Taking $y = Sx$, we have

$$d(Sx,TSx) \leq 2\lambda \ d(Sx,TSx), \ \forall \ x \in X. \tag{2}$$

Since $(1-2\lambda) > 0$, we have $d(Sx,TSx) = 0$, implies that $TSx = Sx$. Similarly, taking $x = Ty$, we have $d(STy,Ty) = 0$, implies that $STy = Ty$. By taking $x = y$, then we have $d(STx,Tx) = 0$, implies that $STx = Tx$. Since S and T are commuting, i.e. $TSx = STx, \ \forall \ x \in X$. Therfore $Sx = Tx$ for some $x \in X$. Hance x is a coincidence point of S and T. Thus, we conclude that $T^2x = Tx$ and $S^2x = Sx$. Therefore x is a common fixed point of S and T.
Case 2 : If $M(x,y) = 2d(x,y)$, then

$$d(Sx,Ty) \leq 2\lambda d(x,y), \ \forall \ x, \ y \in X.$$

We consider a sequence $\{x_n\}$ in X such that $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ with $x_0 \in X$. We will show that $\{x_n\}$ is a Cauchy sequence in X. For that, let $n \in \mathbb{N}$ and using (2), we get

$$\begin{aligned} d(x_{2n+2}, x_{2n+1}) &= d(Sx_{2n+1}, Tx_{2n}) \\ &\leq 2\lambda d(x_{2n+1}, x_{2n}) \\ &\leq hd(x_{2n+1}, x_{2n}), \quad where \ \ h = 2\lambda. \end{aligned}$$

Continuing this process we have, $d(x_{2n+2}, x_{2n+1}) \leq h^{2n+1} d(x_1, x_0)$ and then, we conclude that

$$\begin{aligned} d(x_{2n}, x_{2m}) &\leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + ... + d(x_{2m-1}, x_{2m}) \\ &\leq (h^{2n} + h^{2n+1} + ... + h^{2m-1}) d(x_1, x_0) \\ &\leq \frac{h^{2n}}{1-h} d(x_1, x_0). \end{aligned} \tag{3}$$

Since $h = 2\lambda \in [0, 1)$, it follows from (3) that $\{x_n\}$ is a Cauchy sequence in X and therefore, there exist $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$ and $\lim\limits_{n\to\infty} Tx_n = x^*$, $\lim\limits_{n\to\infty} Sx_n = x^*$. From (3), we deduce that

$$d(Sx_n, Tx^*) \leq 2\lambda d(x_n, x^*) \text{ and } d(Sx^*, Tx_n) \leq 2\lambda d(x^*, x_n), \ \forall \ n \in \mathbb{N}.$$

Since S and T both are continuous, we deduce

$$\lim_{n\to\infty} d(Sx_n, Tx^*) \leq 2\lambda \lim_{n\to\infty} d(x_n, x^*) \text{ and } \lim_{n\to\infty} d(Sx^*, Tx_n) \leq 2\lambda \lim_{n\to\infty} d(x^*, x_n),$$

implies that $d(x^*, Tx^*) \leq 2\lambda d(x^*, x^*)$ and $d(Sx^*, x^*) \leq 2\lambda d(x^*, x^*)$. Thus $d(x^*, Tx^*) = 0$ and $d(Sx^*, x^*) = 0$. Therefore, we conclude that $Tx^* = x^*$ and $Sx^* = x^*$. Hance S and T haav a common fixed point.

Case 3 : If $M(x, y) = [d(x, Sx) + d(y, Ty)]$, then

$$d(Sx, Ty) \leq \lambda[d(x, Sx) + d(y, Ty)], \ \forall \ x, \ y \in X. \tag{4}$$

We consider a sequence $\{x_n\}$ in X such that $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. We will show that $\{x_n\}$ is a Cauchy sequence in X. For that, let $n \in \mathbb{N}$ and using (4), we get

$$\begin{aligned} d(x_{2n+2}, x_{2n+1}) &= d(Sx_{2n+1}, Tx_{2n}) \\ &\leq \lambda[d(x_{2n+1}, Sx_{2n+1}) + d(x_{2n}, Tx_{2n})] \\ &\leq \lambda[d(x_{2n+1}, x_{2n+2}) + d(x_{2n}, x_{2n+1})], \ \forall \ n \in \mathbb{N}. \end{aligned}$$

Which implies $d(x_{2n+2}, x_{2n+1}) \leq \frac{\lambda}{1-\lambda} d(x_{2n}, x_{2n+1})$, $\forall \ n \in \mathbb{N}$. Since $h = \frac{\lambda}{1-\lambda} \in [0, 1)$, then $\{x_n\}$ is a Cauchy sequence in X. Therefore, there exist $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$ and $\lim\limits_{n\to\infty} Tx_n = x^*$, $\lim\limits_{n\to\infty} Sx_n = x^*$. From (4), we deduce that

$$\begin{aligned} d(Sx_n, Tx^*) &\leq \lambda[d(x_n, Sx_n) + d(x^*, Tx^*)] \\ &\leq \lambda[d(x_n, x^*) + d(x^*, Sx_n) + d(x^*, Tx^*)], \ \forall \ n \in \mathbb{N}. \end{aligned}$$

Then, by passing to limit for in the above inequality, we find $d(x^*, Tx^*) \leq \frac{2\lambda}{1-\lambda} d(x^*, x^*)$. Similarly, $d(Sx^*, x^*) \leq \frac{2\lambda}{1-\lambda} d(x^*, x^*)$. Thus $d(x^*, Tx^*) = 0$ and $d(Sx^*, x^*) = 0$. Therefore, we conclude that

$Tx^* = x^*$ and $Sx^* = x^*$. Hance S and T have a common fixed point.

Case 4 : If $M(x,y) = \frac{d(x,Ty)+d(y,Sx)}{2}$, then

$$d(Sx,Ty) \le \frac{\lambda}{2}[d(x,Ty) + d(y,Sx)], \ \forall \ x, \ y \in X. \tag{5}$$

We consider a sequence $\{x_n\}$ in X such that $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ with $x_0 \in X$. We will show that $\{x_n\}$ is a Cauchy sequence in X. For that, let $n \in \mathbb{N}$ and using (5), we get

$$\begin{aligned} d(x_{2n+2}, x_{2n+1}) &= d(Sx_{2n+1}, Tx_{2n}) \\ &\le \frac{\lambda}{2}[d(x_{2n+1}, Tx_{2n}) + d(x_{2n}, Sx_{2n+1})] \\ &\le \frac{\lambda}{2}[d(x_{2n+1}, x_{2n+1}) + d(x_{2n}, x_{2n+2})] \\ &\le \frac{\lambda}{2}d(x_{2n}, x_{2n+2}), \ \forall \ n \in \mathbb{N}. \end{aligned}$$

We can easily see that $h = \frac{\lambda}{2} \in [0,1)$, then $\{x_n\}$ is a Cauchy sequence in X. Therefore, there exist $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} Tx_n = x^*$, $\lim_{n\to\infty} Sx_n = x^*$. From (5), we deduce that

$$d(Sx_n, Tx^*) \le \frac{\lambda}{2}[d(x_n, Tx^*) + d(x^*, Sx_n)].$$

Since S is continuous, we have

$$\lim_{n\to\infty} d(Sx_n, Tx^*) \le \frac{\lambda}{2} \lim_{n\to\infty}[d(x_n, Tx^*) + d(x^*, Sx_n)].$$

Implies $d(x^*, Tx^*) \le \frac{\lambda}{2}[d(x^*, Tx^*) + d(x^*, x^*)]$. Similarly $d(Sx^*, x^*) \le \frac{\lambda}{2}[d(x^*, x^*) + d(Sx^*, x^*)]$. Thus, $\left(1 - \frac{\lambda}{2}\right)d(x^*, Tx^*) \le 0$ and $\left(1 - \frac{\lambda}{2}\right)dS(x^*, x^*) \le 0$. Implies $d(x^*, Tx^*) = 0$ and $d(Sx^*, x^*) = 0$. Therefore, we conclude that $Tx^* = x^*$ and $Sx^* = x^*$. Hance S and T have a common fixed point.

Case 5 : If $M(x,y) = d(x,Sx) + d(x,y)$, then

$$d(Sx,Ty) \le \lambda[d(x,Sx) + d(x,y)], \ \forall \ x, \ y \in X. \tag{6}$$

We define a sequence $\{x_n\}$ in X such that $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. We will show that $\{x_n\}$ is a Cauchy sequence in X. For that, let $n \in \mathbb{N}$ and using (7), we get

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\le \lambda[d(x_{2n}, Sx_{2n}) + d(x_{2n}, x_{2n+2})] \\ &\le \lambda[d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2})] \\ &\le \lambda d(x_{2n+1}, x_{2n+2}), \ \forall \ n \in \mathbb{N}. \end{aligned}$$

Since $1-\lambda \in [0,1)$, then $\{x_n\}$ is a Cauchy sequence in X. Therefore, there exist $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} Tx_n = x^*$, $\lim_{n\to\infty} Sx_n = x^*$. From (7), we deduce that

$$d(Sx_n, Tx^*) \le \lambda[d(x_n, Sx_n) + d(x_n, x^*)].$$

Since S is continuous, we have

$$\lim_{n\to\infty} d(Sx_n, Tx^*) \le \lambda \lim_{n\to\infty} [d(x_n, Sx_n) + d(x_n, x^*)].$$

Implies $d(x^*, Tx^*) \le \lambda[d(x^*, x^*) + d(x^*, x^*)]$. Similarly $d(Sx^*, x^*) \le \lambda[d(x^*, x^*) + d(x^*, x^*)]$. Therefore, we conclude that $Tx^* = x^*$ and $Sx^* = x^*$. Hance S and T have a common fixed point.

Case 6 : If $M(x,y) = d(y,Ty) + d(x,y)$, then

$$d(Sx, Ty) \le \lambda[d(y,Ty) + d(x,y)], \ \forall \ x, \ y \in X. \tag{7}$$

We define a sequence $\{x_n\}$ in X such that $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. We will show that $\{x_n\}$ is a Cauchy sequence in X. For that, let $n \in \mathbb{N}$ and using (7), we get

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\le \lambda[d(x_{2n+1}, Tx_{2n+1}) + d(x_{2n}, x_{2n+1})] \\ &\le \lambda[d(x_{2n+1}, x_{2n+2}) + d(x_{2n}, x_{2n+1})] \\ &\le \frac{\lambda}{1-\lambda} d(x_{2n}, x_{2n+1}), \ \forall \ n \in \mathbb{N}. \end{aligned}$$

Since $h = \frac{\lambda}{1-\lambda} \in [0,1)$, we conclude that

$$d(x_{2n+1}, x_{2n+2}) \le hd(x_{2n}, x_{2n+1}), \ \forall \ n \in \mathbb{N}.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X and then $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} Tx_n = x^*$, $\lim_{n\to\infty} Sx_n = x^*$. Furthermore, we have

$$\begin{gathered} d(Sx^*, Tx_n) \le \lambda[d(x_n, Tx_n) + d(x^*, x_n)], \text{ implies} \\ \lim_{n\to\infty} d(Sx^*, Tx_n) \le \lambda \lim_{n\to\infty} [d(x_n, Tx_n) + d(x^*, x_n)]. \end{gathered}$$

Since T is continuous, we have $d(Sx^*, x^*) \le \lambda[d(x^*, x^*) + d(x^*, x^*)]$. Similarly, $d(x^*, Tx^*) \le \lambda[d(x^*, x^*) + d(x^*, x^*)]$. Therefore, $d(Sx^*, x^*) = 0$ and $d(x^*, Tx^*) = 0$. Finally, we conclude that $Tx^* = x^*$ and $Sx^* = x^*$. Hance S and T have a common fixed point.

Case 7 : If $M(x,y) = \frac{2}{3}[d(x,Ty) + d(x,y)]$, then

$$d(Sx,Ty) \leq \frac{2\lambda}{3}[d(x,Ty) + d(x,y)], \ \forall \ x, \ y \in X. \tag{8}$$

We define a sequence $\{x_n\}$ in X such that $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. We will show that $\{x_n\}$ is a Cauchy sequence in X. For that, let $n \in \mathbb{N}$ and use (8), we get

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \frac{2\lambda}{3}[d(x_{2n}, Tx_{2n+1}) + d(x_{2n}, x_{2n+1})] \\ &\leq \frac{2\lambda}{3}[d(x_{2n}, x_{2n+2}) + d(x_{2n}, x_{2n+1})] \\ &\leq \frac{2\lambda}{3}[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(x_{2n}, x_{2n+1})], \ \forall \ n \in \mathbb{N}, \end{aligned}$$

or

$$d(Sx_{2n}, Tx_{2n+1}) \leq \frac{4\lambda}{3-2\lambda} d(x_{2n}, x_{2n+1}), \ \forall \ x, \ y \in X \ and \ \ n \in \mathbb{N}.$$

Since $h = \frac{4\lambda}{3-2\lambda} \in [0,1)$, the above inequality implies that $\{x_n\}$ is a Cauchy sequence in X and then in $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} Tx_n = x^*$, $\lim_{n\to\infty} Sx_n = x^*$. Furthermore, we have

$$\lim_{n\to\infty} d(Sx_n, Tx^*) \leq \frac{2\lambda}{3} \lim_{n\to\infty} [d(x_n, Tx^*) + d(x_n, x^*].$$

Implies, $d(x^*, Tx^*) \leq \frac{2\lambda}{3}[d(x^*, x^*) + d(x^*, x^*]$. Similarly $d(Sx^*, x^*) \leq \frac{2\lambda}{3}[d(x^*, x^*) + d(x^*, x^*]$. Therefore, $d(Sx^*, x^*) = 0$ and $d(x^*, Tx^*) = 0$. Finally, we conclude that $Tx^* = x^*$ and $Sx^* = x^*$. Hance S and T have a common fixed point.

For Uniqueness: We consider two common fixed points $x^*, y^* \in X$ of self-mappings S and T. Then from (1), we have

$$d(Sx^*, Ty^*) \leq \lambda \max \left\{ \begin{array}{l} 2d(x^*, y^*), \frac{2d(x^*,Sx^*)d(y^*,Ty^*)}{d(x^*,y^*)}, \left[d(x^*, Sx^*) + d(y^*, Ty^*)\right], \\ \frac{\left[d(x^*,Ty^*)+d(y^*,Sx^*)\right]}{2}, \frac{2\left[d(x^*,Ty^*)+d(x^*,y^*)\right]}{3}, \\ \left[d(x^*, Sx^*) + d(x^*, y^*)\right], \left[d(y^*, Ty^*) + d(x^*, y^*)\right] \end{array} \right\},$$

where $\lambda \in [0, \frac{1}{2})$, we deduce the following results for distinguish cases.

Case 1 : We have

$$\begin{aligned} d(x^*, y^*) = d(Sx^*, Ty^*) &\leq 2\lambda \frac{d(x^*, Sx^*)d(y^*, Ty^*)}{d(x^*, y^*)} \\ &\leq 2\lambda \frac{d(x^*, x^*)d(y^*, y^*)}{d(x^*, y^*)}. \end{aligned}$$

Thus $d(x^*,y^*)=0$.

Case 2 : We have

$$d(x^*,y^*)=d(Sx^*,Ty^*)\leq 2\lambda d(x^*,y^*),$$

implies $(1-2\lambda)d(x^*,y^*)\leq 0$. Thus $d(x^*,y^*)=0$.

Case 3 : We have

$$\begin{aligned} d(x^*,y^*)=d(Sx^*,Ty^*) &\leq \lambda\big[d(x^*,Sx^*)+d(y^*,Ty^*)\big] \\ &\leq \lambda\big[d(x^*,x^*)+d(y^*,y^*)\big], \end{aligned}$$

Thus $d(x^*,y^*)=0$.

Case 4 : We have

$$\begin{aligned} d(x^*,y^*)=d(Sx^*,Ty^*) &\leq \lambda\frac{\big[d(x^*,Ty^*)+d(y^*,Sx^*)\big]}{2} \\ &\leq \lambda\frac{\big[d(x^*,y^*)+d(y^*,x^*)\big]}{2}, \end{aligned}$$

or

$$\begin{aligned} d(x^*,y^*) &\leq \frac{\frac{\lambda}{2}}{1-\frac{\lambda}{2}}d(y^*,x^*) \\ &\leq \left(\frac{\frac{\lambda}{2}}{1-\frac{\lambda}{2}}\right)^2 d(x^*,y^*). \end{aligned}$$

Since $\left(\frac{\frac{\lambda}{2}}{1-\frac{\lambda}{2}}\right)\in[0,1)$. Thus $d(x^*,y^*)=0$.

Case 5 : We have

$$\begin{aligned} d(x^*,y^*)=d(Sx^*,Ty^*) &\leq \lambda\big[d(x^*,Sx^*)+d(x^*,y^*)\big] \\ &\leq \lambda\big[d(x^*,x^*)+d(x^*,y^*)\big], \end{aligned}$$

implies $(1-\lambda)d(x^*,y^*)\leq 0$. Thus $d(x^*,y^*)=0$.

Case 6 : We have

$$\begin{aligned} d(x^*,y^*)=d(Sx^*,Ty^*) &\leq \lambda\big[d(y^*,Ty^*)+d(x^*,y^*)\big] \\ &\leq \lambda\big[d(y^*,y^*)+d(x^*,y^*)\big], \end{aligned}$$

implies $(1-\lambda)d(x^*,y^*)\leq 0$. Thus $d(x^*,y^*)=0$.

Case 7 : We have

$$\begin{aligned} d(x^*,y^*) = d(Sx^*,Ty^*) &\leq \lambda \frac{2\big[d(x^*,Ty^*)+d(x^*,y^*)\big]}{3} \\ &\leq \lambda \frac{2\big[d(x^*,y^*)+d(x^*,y^*)\big]}{3}, \end{aligned}$$

implies $\left(1-\frac{4\lambda}{3}\right)d(x^*,y^*) \leq 0$. Thus $d(x^*,y^*)=0$.

Hence, in all cases $d(x^*,y^*)=0$. Therefore, we conclude that $x^*=y^*$. □

Example 2.2. *Assume the set* $X=\{0,\frac{1}{7},\frac{1}{5},10\}$ *and* $d: X\times X \to \mathbb{R}^+$ *be defined by* $d(x,y)=2x+y$, $\forall\ x,\ y \in X$. *Then* (X,d) *is dislocated quasi-metric space. We define the mappings* $S,T: X\to X$ *by* $T(0)=0$, $T(\frac{1}{7})=0$, $T(\frac{1}{5})=\frac{1}{7}$, $T(10)=\frac{1}{5}$ *and* $S(0)=0$, $S(\frac{1}{7})=\frac{1}{5}$, $S(\frac{1}{5})=0$, $S(10)=\frac{1}{7}$. *For* $\lambda=\frac{1}{3}$, *we can easily see that all the assumptions of Theorem 2.1 are satisfied and* 0 *is the unique common fixed point of* S *and* T.

As a consequence of Theorem 2.1, if $S=T$, we state the following corollary.

Corollary 2.3. *[11] Let* (X,d) *be a complete dislocated quasi-metric space and* $T: X\to X$ *is a self-mappings such that* $d(Tx,Ty)\leq \lambda M(x,y)$, *where*

$$M(x,y) = max \left\{ \begin{array}{l} 2d(x,y), \frac{2d(x,Tx)d(y,Ty)}{d(x,y)}, \big[d(x,Tx)+d(y,Ty)\big], \\ \frac{\big[d(x,Ty)+d(y,Tx)\big]}{2}, \frac{2\big[d(x,Ty)+d(x,y)\big]}{3}, \\ \big[d(x,Tx)+d(x,y)\big], \big[d(y,Ty)+d(x,y)\big] \end{array} \right\}, \ \forall\ x,\ y\in X$$

with $d(x,y)\neq 0$ *and* $\lambda\in[0,\frac{1}{2})$. *Then* T *has a unique fixed point in* X.

Remark 2.4. *In particular corollary 2.3 if* $S=T$ *in Theorm 2.1, we get Theorem* 3.2 *of Mhanna et al. [11].*

3. Application

We provide an application for dislocated quasi-metric spaces of integral type.

3.1 Existence and Uniqueness theorem for the integral type

In this section, we establishing the existence and uniqueness of common fixed point in dislocated quasi-metric spaces of integral type.

Theorem 3.1. *Let* (X,d) *be a complete dislocated quasi-metric space and* $S,T: X\to X$ *be continuous mappings satisfying*

$$\int_0^{d(Sx,Ty)} f(t)\,dt \leq \alpha \int_0^{\frac{d(y,Ty)[1+d(x,Sx)]}{1+d(x,y)}} f(t)\,dt + \beta\int_0^{d(x,y)} f(t)\,dt + \gamma\int_0^{d(y,Ty)} f(t)\,dt \tag{9}$$

for all $x,y\in X$, $\alpha>0$, $\beta>0$, $\gamma>0$ *with* $\alpha+\beta+\gamma<1$. *Then* S *and* T *have a unique common fixed point.*

Proof. Let $\{x_n\}$ be a sequence in X, defined as $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ such that

$$\begin{aligned}
\int_0^{d(x_{2n+1},x_{2n+2})} f(t)\,dt &= \int_0^{d(Sx_{2n},Tx_{2n+1})} f(t)\,dt \\
&\leq \alpha \int_0^{\frac{d(x_{2n+1},Tx_{2n+1})[1+d(x_{2n},Sx_{2n})]}{1+d(x_{2n},x_{2n+1})}} f(t)\,dt + \beta \int_0^{d(x_{2n},x_{2n+1})} f(t)\,dt + \gamma \int_0^{d(x_{2n+1},Tx_{2n+1})} f(t)\,dt \\
&\leq \alpha \int_0^{\frac{d(x_{2n+1},x_{2n+2})[1+d(x_{2n},x_{2n+1})]}{1+d(x_{2n},x_{2n+1})}} f(t)\,dt + \beta \int_0^{d(x_{2n},x_{2n+1})} f(t)\,dt + \gamma \int_0^{d(x_{2n+1},x_{2n+2})} f(t)\,dt \\
&\leq \left(\frac{\beta}{1-\alpha-\gamma}\right) \int_0^{d(x_{2n},x_{2n+1})} f(t)\,dt.
\end{aligned}$$

Let $h = \frac{\beta}{1-\alpha-\gamma}$, $0 < h < 1$. Therefore

$$\int_0^{d(x_{2n+1},x_{2n+2})} f(t)\,dt \leq h \int_0^{d(x_{2n},x_{2n+1})} f(t)\,dt.$$

Similarly,

$$\int_0^{d(x_{2n},x_{2n+1})} f(t)\,dt \leq h \int_0^{d(x_{2n-1},x_{2n})} f(t)\,dt.$$

Continuing this process, we get

$$\int_0^{d(x_{2n+1},x_{2n+2})} f(t)\,dt \leq h^{2n+1} \int_0^{d(x_0,x_1)} f(t)\,dt.$$

Since $0 < h < 1$, so for $n \to \infty$, $h^{2n+1} \to 0$, we have $d(x_{2n+1}, x_{2n+2}) \to 0$. Hence $\{x_n\}$ is a Cauchy sequence in X, so there is a point $x^* \in X$, such that $x_n \to x^*$. Since S and T are continuous,

$$S(x^*) = \lim_{n\to\infty} S(x_{2n}) = \lim_{n\to\infty} x_{2n+1} = x^*$$

and

$$T(x^*) = \lim_{n\to\infty} T(x_{2n+1}) = \lim_{n\to\infty} x_{2n+2} = x^*.$$

Thus $S(x^*) = T(x^*) = x^*$, so S and T have a common fixed point.

Uniqueness: If $x \in X$ a common fixed point of S and T, then by (3.1)

$$\int_0^{d(Sx,Tx)} f(t)\,dt = \int_0^{d(x,x)} f(t)\,dt \leq (\alpha+\beta+\gamma) \int_0^{d(x,x)} f(t)\,dt,$$

which is true only if $d(x, x) = 0$, since $0 < \alpha + \beta + \gamma < 1$ and $d(x, x) \geq 0$. Thus $d(x, x) = 0$ for a common fixed point x of S and T. Let x, y be common fixed point of S and T, then by (3.1),

$$\begin{aligned}
\int_0^{d(x,y)} f(t)\,dt &= \int_0^{d(Sx,Tx)} f(t)\,dt \\
&\leq \alpha \int_0^{d(x,y)} f(t)\,dt + \beta \int_0^{d(x,y)} f(t)\,dt + \gamma \int_0^{d(y,y)} f(t)\,dt
\end{aligned}$$

$$\leq (\alpha + \beta) \int_0^{d(x,y)} f(t)\, dt$$

and from this it follows that $d(x,y) = 0$, since $\alpha + \beta < 1$ and $d(x,y) \geq 0$. Hence $x = y$. Thus S and T have unique common fixed point. □

4. Conclusion

In this paper, we study coupled common fixed point theorems in dislocated quasi-metric spaces. We prove common fixed point theorems for self-mappings in dislocated quasi-metric spaces. We provide application to integral equations, proving existence and uniqueness of solutions. Our result is supported with example to confirm correctness and applicability. Our work aim to advance fixed point theory, offering new tools for nonlinear analysis and applied mathematics. We hope these findings inspire future research to explore more in dislocated metric spaces and contribute to mathematical knowledge.

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