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On Solution of Non-homogeneous Fractional Linear Boundary Value Problem

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Abstract

Analytical and numerical solutions of non-homogeneous linear fractional boundary value problems play a very important role in the interpretation of various model results. In this work, we present the existence and uniqueness results for the solution of non-homogeneous linear fractional boundary value problem using the Banach fixed point theorem in the space of continuous functions. The analytic solution is presented using the Laplace transform approach and the collocation parallel shooting method is employed for the numerical solutions.

Keywords: existence and uniqueness; parallel shooting; collocation; analytic and numerical solutions.

2020 Mathematics Subject Classification: 3408, 63L10.

1. Introduction

The concept of fractional calculus dated well back in the 17 century. The birth of this concept dated 30th September, 1695, L′ Hopital wrote a letter to Liebniz asking him about the meaning of one half derivative. Leibniz response ”an apparent paradox from which one day useful consequences will be drawn” [1]. In recent years, fractional calculus has developed rapidly within the frame work of pure mathematics. Often times in solving real life problems, fractional order derivatives appear naturally [2]. The wide application of fractional calculus in the field of Engineering, Science and social sciences mark a significant development, these include: Fluid mechanics [3], Electromagnetism [4], Electrochemistry [5], Dynamics of viscoelastic materials [6], Nuclear dynamics [7], Bioscience [8] and Mechanical vibrations [9]. Solution of fractional differential equations particular boundary value problems play an in important role in the interpretation of real world problems, to this end various methods for solving fractional differential equations are proposed, these include: Boubaker polynomials [10], Bernoulli wavelet [11], finite element method [12], Jacobi polynomial [13], Chebyshev polynomials [14], Predictor-correction method [15], Adomia decomposition method [16],

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Homotype perturbation method [17], Extrapolation [18] and the Generalized transform method [19]. The fractional boundary value problem is studied by few authors, for example [20] gives the Boundary Value Problem (BVP) for fractional differential equation (FrDE) as follows:

$$\begin{cases} D^{\beta}_{0+}y(t) = g(t), \quad t \in (0,1), \\ y(0) = y^{'}(0) = y^{''}(1) = 0. \end{cases}$$

Here D^{β}_{0+} denote the Reimman Liouville fractional differential operator of order β, $2 < \beta <$. The existence results were obtained via the lower and upper solution method an fixed point theorem. A non-linear fractional boundary value problem is given in the work of [21]

$$\begin{cases} D^{\beta}_{0+}y(t) = g(t, y(t)), \quad t \in (0,1) \\ y(0) = y^{'}(0) = y^{''}(0) = y^{''1}(1) = 0 \end{cases}$$

where D^{β}_{0+} is the Reimman Liouville fractional operator of order β, $3 < \beta \leqslant 4$. The existence of positive solution is investigated via the fixed the point theorem. A boundary value problem of fractional differential equation at resonance was proposed by [22]

$$\begin{cases} D^{\gamma}_{0+}y(x) = g(t, y(x), y^{'}(x), y^{''}(x), y^{'''}(x)), \quad x \in (0,1), \\ y(0) = y^{'}(0) = y^{''}(0) = 0, \quad y^{'''}(1) = y^{'v}(1). \end{cases}$$

where D^{γ}_{0+} is the Caputo fractional derivative of order γ, $3 < \gamma < 4$. The lower and upper solution method combined with the monotone iteration method were utilized for the solution. Blank [23] gave the numerical treatment of fractional order and applied the method to relaxation equation, later [24] applied the method to find the solution of fractional integro-differential equations. Further, [25] proposed the collocation shooting method for solving boundary value problem. In all the works mentioned above non gave the existence and uniqueness of the exact solution, the analytical solution. In this paper we present those gaps identified above, to this end we consider the fractional boundary value problem:

$$[C_1D^2 + C_2D^{\alpha} + C_3D^0]y(t) = g(t), t \in [0, T], 0 < \alpha < 2 \tag{1}$$

$$y(0) = \tau_0, \quad y(T) = \tau_1. \tag{2}$$

where C_1, C_2, C_3, τ_0 and τ_1 are constants with $C_1 \neq 0$ and $y \in L[0, T]$. Here, D^{α} (α is a non- integer) denote the fractional operator of order α and is given by

$$D^{\alpha}y(x) = \frac{1}{\Gamma(i-\alpha)} \int_0^x (x-t)^{i-\alpha-1} y^i(t)dt \tag{3}$$

where $i = N$ and satisfies the relation $i-1 < \alpha < i$. The paper is organized as follows: In section I, we give some basic definitions of fractional differential equation, section II deals with the existence and uniqueness of the fractional boundary value problem and the analytical results. In section III we present the collocation parallel shooting method and section IV several numerical example are considered to demonstrate the effectiveness, applicability and accuracy of the method.

1.1 Preliminaries

Definition 1.1 ([26]). *The Riemann Liouville fractional integral, Let $\omega(y) \in L,(a,b)$.*

$$\left(I^{\beta}_{a+}\right)(y) = \frac{1}{\Gamma(\beta)}\int_a^y \frac{\omega(t)}{(y-t)^{1-\beta}}dt \quad y > a$$
$$\left(I^{\beta}_{b-}\right)(y) = \frac{1}{\Gamma(\beta)}\int_a^y \frac{\omega(t)}{(t-y)^{1-\beta}}dt \quad y < a$$

where $\beta > 0$ are called left and right sided fractional integrals of order β respectively

Definition 1.2 ([26]). *The left and right Riemann Liouville fractional derivatives of a function f(y) defined on the interval [a, b] as*

$$D^{\beta}_{a+}g(y) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dy}\int_a^y \frac{g(t)dt}{(y-t)^{\beta}}, \quad ,0 < \beta < 1$$
$$D^{\beta}_{b-}g(y) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dy}\int_y^b \frac{g(t)dt}{(t-y)^{\beta}}, \quad ,0 < \beta < 1$$

respectively.

Definition 1.3 ([26]). *The left and right Caputo fractional derivative of order β are defined by*

$$\left({}^{C}\mathbf{D}^{\beta}_{a+}y\right)(t) := \left(\mathbf{D}^{\beta}_{a+}\left[y(x) - \sum_{i=0}^{m-1}\frac{y^i(a)}{i!}(x-a)^i\right]\right)(t),$$

and

$$\left({}^{C}\mathbf{D}^{\beta}_{b-}y\right)(t) := \left(\mathbf{D}^{\beta}_{-b}\left[y(x) - \sum_{i=0}^{m-1}\frac{y^i(a)}{i!}(b-x)^i\right]\right)(t),$$

respectively, where $m = \lfloor R(\alpha) \rfloor + 1$ for $\beta \notin \mathbb{N}_0$, $m = \beta$ for $\beta \in \mathbb{N}_0$.

Definition 1.4 ([26]). *The Laplace transform $\omega(x), 0 < x < \infty$ is defined as follows:*

$$\mathcal{L}\omega = \left(\mathcal{L}\omega\right)(q) = \left(\mathcal{L}\left\{\omega(x), q\right\} = \int_0^{\infty} e^{-qx}\omega(x)dx.\right.$$

and its inverse is given by the formula

$$\left(\mathcal{L}^{-1}g\right)(t) = \mathcal{L}^{-1}\left\{g(q), t\right\} = \frac{1}{2\pi i}\int_{2\pi i}^{\gamma + i\infty} e^{qx}g(q)dq.$$

$\gamma = Req > q_0$.

Definition 1.5 ([26]). *The gamma - function $\Gamma(z)$, is defined by the Euler integral of the second kind*

$$\Gamma(z) = \int_0^{\infty} y^{z-1} e^y dy, \quad Rez > 0.$$

Definition 1.6. *The Gauss hypergeometric function is defined as an analytic continuous of the series*

$${}_2F_1(a, b; c; z) = \frac{\Gamma(C)}{\Gamma(b)\Gamma(c-(b)} \int_0^1 x^{b-1}(1-x)^{(c-b-1)}(1-zx)^a dt,$$

$0 < Reb < Rec.\ |arg(1-x)| < \pi.$

Definition 1.7. *The confluent hypergeometric function or the Kummer function is defined as a*

$${}_1F_1(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} z^n = \lim_{b \to \infty} 2F_1(a, b; c; \frac{z}{b}), \quad |Z| < \infty.$$

Definition 1.8 ([27]). *A generalized Mittag Leffler function is defined as*

$$E^{\gamma}_{\alpha,\beta}(Z) = \sum_{i=0}^{\infty} \frac{\gamma i Z^i}{\Gamma(\alpha i + \beta) i!}, \quad Re(\alpha) > 0.$$

Definition 1.9 ([28]). *The caputo fractional derivative ${}^{C}D^{\beta}_{+a}$ of order β of a function $h(t) \in C^n[a, b]$ is*

$${}^{C}D^{\beta}_{a+}h(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} h^{(n)}(t) dt, & if \beta \notin \mathbb{N} \\ h^{(n)}(t), & if \beta = n \in \mathbb{N}, \end{cases}$$

Where $h^{(n)}(t)$= $\frac{d^n}{dt^n} h(t)$, $Re(\beta) \geqslant 0$, $n = [Re(\alpha)] + 1$.

Definition 1.10 ([29]). *Let $H \in \mathcal{R}^n$, [a,b] $\subset \mathcal{R}^n$ and $f : [a, b] \times H \to \mathcal{R}^n$ be a function such that for any $y_1, y_2 \in H$, f satisfies the Lipschitz condition with respect to the second variable if for all $t \in [a, b]$ and any $y_1, y_2 \in H$ one has*

$$|f(t, y_1) - f(t, y_2)| \leq B|y_1 - y_2|, B > 0 \tag{4}$$

2. The Existence and Uniqueness of Solution of Non-homogeneous Fractional Linear Boundary Value Problem

In this section, we investigate the existence and uniqueness of the solution of the non-homogeneous fractional linear boundary value problem. Before this investigation we give some definitions and important lemmas that are useful for the investigation.

Lemma 2.1 ([28]). *Let $\beta, \omega \in \mathbb{C}, Re(\beta) > 0$ and $a \in \mathbb{R}$ then, there holds*

$$\left({}^{C}D^{\beta}_{a+} E_{\beta}(\omega(Z-a)^{\beta}) \right)(t) = \omega E_{\beta}(\omega(Z-a)^{\beta})(t).$$

Lemma 2.2 ([28]). *If $y(t) = C^{m}[a,b]$, then*

$$\left({}^{C}D^{\beta}_{a+} I^{\beta}_{a+} y \right)(t) = y(t)$$
$$\left(I^{\beta}_{a+} {}^{C}D^{\beta}_{a+} y \right)(t) = y(t) - \sum_{i=0}^{n-1} \frac{y^{(i)}(a)}{i!}(t-a)^{i}.$$

Lemma 2.3 ([28]). *The Laplace transform of the Caputo fractional derivative is defined as*

$$\mathcal{L}\left\{ {}^{C}D^{\beta}_{a+} f(t); s \right\} = y(t) - \sum_{i-0}^{n-1} s^{\beta-i-1} f^{(i)}(0).$$

Lemma 2.4 ([28]). *Let $\mathcal{R}(\beta > 0)$, $n = [\mathcal{R}(\beta)] + 1$, and $\mathcal{R}(\beta) > 0$, then*

$$({}^{C}_{a}D^{\beta}(t-a)^{\alpha-1})(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(x-a)^{\alpha-\beta-1}, \mathcal{R}(\alpha) > n \;\; for \;\; i = 0, 1, \cdots, n-1$$
$$({}^{C}_{a}D^{\beta}(t-a)^{i})(x) = 0.$$

Lemma 2.5 ([30]). *A function $f \in C^{n}_{\delta}[a,b]$ if and only if f can be written in the form*

$$g(t) = \frac{1}{(n-1)!}\int_{a}^{t}(t-u)^{n-1}f(u)du + \sum_{i=0}^{n-1}\frac{f^{i}(a)}{i!}(t-a)^{i}. \tag{5}$$

Lemma 2.6 ([31]). *Let $0 < \beta < b < \infty$ and $0 < \delta < 1$, then*

(a). If $0 < \delta < 1$, then ${}_{a}I^{\beta}$ is bounded from $C_{\delta}[a,b]$ into $C_{\delta-\beta}[a,b]$:

$$\|{}_{a}I^{\beta} f\|_{C_{\delta-\beta}} \leqslant (b-a)^{\beta} \frac{\Gamma(1-\delta)}{\Gamma(1-\delta+\beta)} \|f\|_{C_{\delta}}. \tag{6}$$

(b). If $\beta \geqslant \delta$, then ${}_{a}I^{\beta}$ is bounded from $C_{\delta}[a,b]$ into $C[a,b]$:

$$\|{}_{a}I^{\beta} f\|_{C} \leqslant (b-a)^{\beta-\delta} \frac{\Gamma(1-\delta)}{\Gamma(1-\delta+\beta)} \|f\|_{C_{\delta}}. \tag{7}$$

(c). The fractional integral operator ${}_{a}I^{\beta}$ presents a mapping from $C[a,b]$ into $C[a,b]$ and

$$\|{}_{a}I^{\beta} f\|_{C} \leqslant \frac{1}{\Gamma(\beta+1)}(b-a)^{\beta} \|f\|_{C}. \tag{8}$$

Lemma 2.7 ([28]). *Let $\beta, \omega \in \mathbb{C}, Re(\beta) > 0$ and $a \in \mathbb{R}$ then, there holds*

$$\left({}^{C}D^{\beta}_{a+} E_{\beta}(\omega(Z-a)^{\beta}) \right)(t) = \omega E_{\beta}(\omega(Z-a)^{\beta})(t).$$

Lemma 2.8 ([28]). *If $y(t) = C^{n-1}[a,b]$, then*

$$\left({}^{C}D^{\beta}_{a+} I^{\beta}_{a+} y \right)(t) = y(t), \tag{9}$$

$$\left(I^{\beta}_{a+} {}^{C}D^{\beta}_{a+} y \right)(t) = y(t) - \sum_{i=0}^{n-1} \frac{y^{(i)}(a)}{i!}(t-a)^{i}. \tag{10}$$

Lemma 2.9 ([28]). *For a function f defined on the interval $[a,b]$, we defined a semi-group property for the Caputo integral as*

$$({}_{a}I^{\beta}({}_{a}I^{\alpha} f))(t) = ({}_{a}I^{\beta+\alpha} f)(t), \ \ \mathbb{R}(\beta) > 0, \ \ \mathbb{R}(\alpha) > 0.$$

Lemma 2.10 ([28]). *The Laplace transform of the Caputo fractional derivative is defined as*

$$\mathcal{L}\left\{ {}^{C}D^{\beta}_{a+} f(t); s \right\} = y(t) - \sum_{i=0}^{n-1} s^{\beta-i-1} f^{(i)}(a) \tag{11}$$

Theorem 2.11 (Banach fixed point theorem [32]). *Let (Y,d) be a nonempty complete metric space, and let $0 \leqslant \delta < 1$. If $T : Y \to Y$ is mapping such that for every $y_1, y_2 \in Y$, the relation*

$$d(T_{y_1}, T_{y_2}) \leqslant \delta d(y_1, y_2) \tag{12}$$

holds, then the operator T ha a unique fixed point $y^* \in Y$. Further, if $T^*(s \in \mathcal{N})$ is the sequence which is defined by

$$\begin{cases} T^{s} = TT^{s-1}, s \in \mathcal{N} \backslash (1) \\ T^{1} = T, \end{cases}$$

then, for any $y_0 \in Y$, $\left\{ T^{s}_{y_0} \right\}_{s=1}^{\infty}$ converges the above fixed point y^*.

2.1 The existence and uniqueness

In order to investigate the existence and uniqueness of solution of equation (65), we defined a max metric d_{δ} containing y^{α}, and prove that any two solutions of equation (65) are equivalent in the metric space $(C^{n}[a,T], d_{\delta})$. Further we show that a solution sequence $\{y_j\}_{j=1}^{\infty}$ of equation (65) is a Cauchy sequence in the metric space. Setting $B_1 = B_2 = B_3 = 1$, we rewrite equation (65) in the form,

$$y''(t) = g(t) - \left({}^{C}_{a}D^{\alpha}_{t} y(t) \right) - y(t) = h\left(t, y(t), y^{(\alpha)}(t) \right), \quad 0 < \alpha < 2,$$

$$y(a) = \bar{\tau}_0, y'(a) = \bar{\tau}_1. \tag{13}$$

Where $h(t)$ is continuous if $g(t)$ is continuous in the interval $[a, T]$. For the same value of α the following equations are equivalent to equation (65).

Lemma 2.12. *Let* $h\Big(t, y(t), y^{(\alpha)}(t)\Big) = g(t) - \left({}_a^C D_t^\alpha y(t)\right) - y(t)$, *then the initial value problem (65) is equivalent to the following equations.*

(a)

$$y(t) = \int_0^t (t-s) h\Big(s, y(s), y^{(\alpha)}(s)\Big) + \bar{\tau}_1 t + \bar{\tau}_0. \tag{14}$$

(b) For $0 < \alpha < 1$,

$$y^\alpha(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} h\Big(s, y(s), y^{(\alpha)}(s)\Big) ds + \frac{\bar{\tau}_1 t^{1-\alpha}}{\Gamma(2-\alpha)} \quad t \in [a, T]. \tag{15}$$

(c) and for $1 < \alpha < 2$,

$$y^\alpha(t) = I^{2-\alpha} y''(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} h\Big(s, y(s), y^{(\alpha)}(s)\Big) ds \quad t \in [a, T]. \tag{16}$$

Proof.

(a) To proof the above lemma we take the Laplace transform of equation (13), we get

$$\begin{aligned}
\mathcal{L}\left\{y''(t), s\right\} =& \mathcal{L}\left\{h\Big(t, y(t), y^\alpha(t)\Big)\right\} \\
s^2 Y(s) =& H\Big(s, y(s), y^\alpha(s)\Big) + \bar{\tau}_1 + s\bar{\tau}_0 \\
Y(s) =& s^2 H\Big(s, y(s), y^\alpha(s)\Big) + s^{-2}\bar{\tau}_1 + s^{-1}\bar{\tau}_0.
\end{aligned} \tag{17}$$

Applying the Laplace convolution property and inverse Laplace transform on (17) we obtain the following solution

$$y(t) = \int_0^t (t-s) h\Big(s, y(s), y^\alpha(s)\Big) ds + \bar{\tau}_1 t + \bar{\tau}_0, \quad t \in [a, T]. \tag{18}$$

(b) To prove part (b) we first differential both sides of equation (18)

$$y'(t) = \frac{d}{dt} \int_0^t (t-s) h\Big(s, y(s), y^\alpha(s)\Big) ds + \bar{\tau}_1.$$

Interchanging the derivative and the integral and differentiating with respect to t, we have

$$= \int_0^t h\Big(s, y(s), y^{(\alpha)}(s)\Big) ds + \bar{\tau}_1. \tag{19}$$

For $0 < \alpha < 1$ and $t \in [a, T]$, by applying the defined of Caputo derivative in definition (1.9), we get

$$\begin{aligned} y^{(\alpha)}(t) =& I^{1-\alpha} y'(t) = y^{(\alpha)}\Big[\int_0^t h\Big(s, y(s), y^{\alpha}(s)\Big) ds + \bar{\tau}_1\Big] \\ =& \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\Big[\int_0^s h\Big(\varrho, y(\varrho), y^{\alpha}(\varrho)\Big) d\varrho + \bar{\tau}_1\Big] ds \\ y^{\alpha}(t) =& \frac{1}{\Gamma(1-\alpha)} \int_0^t h\Big(\varrho, y(\varrho), y^{\alpha}(\varrho)\Big) d\varrho \int_{\varrho}^t (t-s)^{-\alpha} ds + \frac{1}{\Gamma(1-\alpha)} \int_0^t (s-t)^{-\alpha} \bar{\tau}_1 ds. \end{aligned}$$

We change the order of integration in the iterative integral to get,

$$\begin{aligned} & \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{(t-\varrho)^{1-\alpha}}{(1-\alpha)} h\Big(\varrho, y(\varrho), y^{\alpha}(\varrho)\Big) d\varrho + \frac{1}{\Gamma(1-\alpha)} \Big(\frac{(t-s)^{1-\alpha}}{(1-\alpha)}|_0^t\Big)\bar{\tau}_1 \\ =& \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \int_0^t (t-\varrho)^{1-\alpha} h\Big(\varrho, y(\varrho), y^{\alpha}(\varrho)\Big) d\varrho + \frac{\bar{\tau}_1}{(1-\alpha)\Gamma(1-\alpha)} \Big((t-s)^{1-\alpha}\Big) \\ =& \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\varrho)^{1-\alpha} h\Big(\varrho, y(\varrho), y^{\alpha}(\varrho)\Big) d\varrho + \frac{\bar{\tau}_1}{\Gamma(2-\alpha)} \Big((t-s)^{1-\alpha}\Big). \end{aligned}$$

Setting $t - s = t$ we get,

$$= \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\varrho)^{1-\alpha} h\Big(\varrho, y(\varrho), y^{\alpha}(\varrho)\Big) d\varrho + \frac{\bar{\tau}_1 t^{1-\alpha}}{\Gamma(2-\alpha)}.$$

If $\alpha = 1$. The result is is gotten from equation (19)

$$y'(t) = \int_0^t h\Big(\varrho, y(\varrho), y'(\varrho)\Big) d\varrho + \bar{\tau}_1.$$

(c) For $1 < \alpha < 2$, we use equation (13) and definition (1.9), we obtain

$$y^{\alpha}(t) = I^{2-\alpha} y''(t) = \frac{1}{\Gamma(2-\alpha)} (t-s)^{1-\alpha} \int_0^t h\Big(s, y(s), y^{\alpha}(s)\Big) ds \quad t \in [a, T].$$

The proof is complete. □

Let $V := \Big((t, w, u) \in \mathcal{R}^3 : t \in [a, T], (w, u) \in \mathcal{R}^2\Big)$. Let the real valued function $h : V \to \mathcal{R}$ be Lipschitz continuous with respect to w and u. Let $w > 0$ and $\delta > 0$ and $Y := C^2[a, T]$ be a set of twice continuously differentiable function on $[a, T]$. We consider the metric space (Y, d_δ) coupled with the max metric

$$d_\delta := \max_{t \in [a,T]} \frac{|x(t) - y(t)|}{E_\tau(\delta t^\tau)} + \max_{t \in [a,T]} \frac{|x^{(\alpha)}(t) - y^{(\alpha)}(t)|}{E_\tau(\delta t^\tau)}, \forall x, y \in Y. \tag{20}$$

Theorem 2.13. *If there exist two positive constants $M \geqslant 0$ and $N \geqslant 0$ $\forall$ $(t, w_j, u_j) \in V(j = 1, 2)$ such that*

$$|h(t, w_1, u_1) - h(t, w_2, u_2)| \leqslant M|w_1 - w_2)| + N|u_1 - u_2)|. \tag{21}$$

and max $\{M, N\} \frac{T^2}{2} < 1$, equation (13) has only one solution $y = y(t)$ defined on the interval $[a, T]$.

Proof. Define $\lambda := max\,\{M, N\}\,(\frac{T^2}{2} + \frac{1}{\delta})$ and consider the positive constants M and N as defined in equation (35). δ is chosen such that it is sufficiently large such that $\lambda < 1$. For any two solutions x, y the initial value problem (65) is such that $x \equiv y$ in the metric space (Y, d_δ). Using equation (14) and (35) we derive the following,

$$|x(t) - y(t)| = |\int_0^t (t-s)\left[h\left(s, x(s), x^\alpha\right) - h\left(s, y(s), y^\alpha\right)\right]ds|$$
$$|x(t) - y(t)| \leqslant \int_0^t (t-s)|\left[h\left(s, x(s), x^\alpha\right) - h\left(s, y(s), y^\alpha\right)\right]|ds.$$

Multiply both sides by $\frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})}$

$$\begin{aligned}
\frac{|x(t) - y(t)|}{E_{2-\alpha}(\delta t^{2-\alpha})} &\leqslant \frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})} \int_0^t (t-s)|\left[h\left(s, x(s), x^\alpha\right) - h\left(s, y(s), y^\alpha\right)\right]|ds \\
&\leqslant \frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})} \int_0^t (t-s)\left(M|x(s) - y(s)| + N|x^{(\alpha)}(s) - y^{(\alpha)}(s)|\right)ds \\
&\leqslant \left(\frac{(M|x(s) - y(s)|}{E_{2-\alpha}(\delta t^{2-\alpha})} + \frac{N|x^{(\alpha)}(s) - y^{(\alpha)}(s)|}{E_{2-\alpha}(\delta t^{2-\alpha})}\right) \int_0^t (t-s)ds \\
&\leqslant max\,\{M, N\}\left(\max_{t \in [a,T]} \frac{|x(s) - y(s)|}{E_{2-\alpha}(\delta t^{2-\alpha})} + \max_{t \in [a,T]} \frac{|x^{(\alpha)}(s) - y^{(\alpha)}(s)|}{E_{2-\alpha}(\delta t^{2-\alpha})}\right) \int_0^t (t-s)ds \\
&\leqslant max\,\{M, N\}\,\frac{T^2}{2} d_\delta(x, y).
\end{aligned}$$

Additionally, for $0 < \alpha < 2$ from 2.12

$$|x(t) - y(t)| = |\frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha}\left[h\left(s, x(s), x^\alpha\right) - h\left(s, y(s), y^\alpha\right)\right]ds|$$
$$|x(t) - y(t)| \leqslant \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha}|\left[h\left(s, x(s), x^\alpha\right) - h\left(s, y(s), y^\alpha\right)\right]|ds.$$

Multiply both sides by $\frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})}$

$$\begin{aligned}
\frac{|x(t) - y(t)|}{E_{2-\alpha}(\delta t^{2-\alpha})} &\leqslant \frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})} \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha}|\left[h\left(s, x(s), x^\alpha\right) - h\left(s, y(s), y^\alpha\right)\right]|ds \\
&\leqslant \frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})} \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)\Big[E_{2-\alpha}(\delta t^{2-\alpha}) \\
&\times \left(\frac{M|x(s) - y(s)| + N|x^{(\alpha)}(s) - y^{(\alpha)}(s)|}{E_{2-\alpha}(\delta t^{2-\alpha})}\right)ds\Big] \\
&\leqslant \frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})} max\,\{M, N\}\, d_\delta(x, y) \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^(1-\alpha) E_{2-\alpha}(\delta t^{2-\alpha}).
\end{aligned} \tag{22}$$

Applying (1) and (2) to equation (22) we get

$$
\begin{aligned}
&\leqslant \frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})} max\left\{M,N\right\} d_\delta(x,y) \begin{matrix} max \\ t\in[a,T]\end{matrix} \left\{\frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})}\left(I_{0+}^{2-\alpha}{}^{C}D_{0+}^{2-\alpha}\frac{E_{2-\alpha}(\delta t^{2-\alpha})}{\delta}\right)\right\}\\
&\leqslant \frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})} max\left\{M,N\right\} d_\delta(x,y) \begin{matrix} max \\ t\in[a,T]\end{matrix} \left[\frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})}\left(\frac{E_{2-\alpha}(\delta t^{2-\alpha})}{\delta}\right.\right.\\
&\left.\left.-\sum_{k=0}^{n-1}\frac{(E_{2-\alpha}(\delta t^{2-\alpha}))^k(a)}{\delta k!}(t-a)^k\right)\right]\\
&\leqslant \frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})} max\left\{M,N\right\} d_\delta(x,y) \begin{matrix} max \\ t\in[a,T]\end{matrix} \left[\frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})}\left(\frac{E_{2-\alpha}(\delta t^{2-\alpha})}{\delta}-\frac{1}{\delta}\right)\right]\\
&\leqslant \frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})} max\left\{M,N\right\} d_\delta(x,y) \begin{matrix} max \\ t\in[a,T]\end{matrix} \left[1-\frac{1}{E_{2-\alpha}(\delta t^{2-\alpha})}\right)\right].
\end{aligned}
$$

We clearly see that $E_{2-\alpha}(\delta t^{2-\alpha})$ is continuous for $2-\alpha>0$ and strictly increasing on the interval $[a,T]$. Therefore $\frac{|x(t)-y(t)|}{E_{2-\alpha}(\delta t^{2-\alpha})} \leq max\left\{M,N\right\}\frac{1}{\delta}d_\delta(x,y)$. The above result indicated that any two solutions x,y to equation (13) satisfies the relation $x,y\in(Y,d_\delta)$, therefore $d_\delta(x,y)\leq max\left\{M,N\right\}(\frac{T^2}{2}+\frac{1}{\delta})d_\delta(x,y) = \lambda d_\delta(x,y)$, this gives $(1-\lambda)d_\delta(x,y)\leq 0$. δ is chosen such that $\lambda<1$, this implies that $d_\delta(x,y)=0$. In a metric space, distance between two points $d_\delta(x,y)$ can only be zero if $x=y$. Since $d_\delta(x,y)=0$, this implies that $x\equiv y$ indicating that equation (13) has only one solution. □

Let $C^n([a,b],\mathbb{C})$ be the Banach space of all continuously differentiable functions from $C^n[a,b]$ to $\mathbb{C}$. We let the weighted spaces of a function f be $C_\delta[a,b]$ and $C^n_\delta[a,b]$. For $n-1<\mathbb{R}(\beta)\leqslant n$ and $0<\mathbb{R}(\beta)\leqslant 1$, We defined the following

$$C^n[a,b]=\left\{f:[a,b]\to\mathbb{C}\ \ scuh\ \ that\ \ f^{(n)}\in C[a,b]\right\} \tag{23}$$

$$C_\delta[a,b]=\left\{f:(t-a)^\delta f(x)\in C[a,b]\right\} \tag{24}$$

equipped with the norm

$$\|f\|_{C_\delta}=\|(t-a)^\delta f(t)\|_C=\begin{matrix} max \\ t\in[a,b]\end{matrix}|(t-a)^\delta f(t)|. \tag{25}$$

2.2 The equivalent of the Cauchy -problem and the Volterra Integral equation

Consider the solution of the problem (65) in equation (18)

$$u(t)=\int_0^t(t-s)h\Big(s,y(s),y^\alpha(s)\Big)ds+\bar{\tau}_1 t+\bar{\tau}_0,\qquad t\in[a,T]$$

we can write the above equation as

$${}^{C}D^\beta_{a+}u(t)=I^\alpha_{a+}f[t,u(t)],\quad t\in[a,b], \tag{26}$$

$$\begin{matrix} lim \\ t\to a+\end{matrix}\left({}^{C}D^\beta_{a+}u(t)\right)=A_i\quad i=1,2,\ldots,n. \tag{27}$$

Equation (26) and (27) are know as the Cauchy problem. The solution to (26) and (27) are equivalent to the solution of the volterra integral equation

$$u(t) = \sum_{i=1}^{n} A_i \frac{(t-a)^{\beta-i}}{\Gamma(\beta-k+1)} + I_{a+}^{\beta+\alpha} f[x,u(x)], \tag{28}$$

$$u(t) = \sum_{i=1}^{n} A_i \frac{(t-a)^{\beta-i}}{\Gamma(\beta-k+1)} + \frac{1}{\Gamma(\beta)} \int_a^t (t-x)^{\beta-1} I_{a+}^{\alpha} f[x,u(x)]dx,$$

$$u(t) = \sum_{i=1}^{n} A_i \frac{(t-a)^{\beta-i}}{\Gamma(\beta-k+1)} + \frac{1}{\Gamma(\beta)} \int_a^t (t-x)^{\beta-1} h[s,\varpi(s,y)]dx. \tag{29}$$

where we defined $f[x,u(x)] = h[s,\varpi(s,y)]$.

Theorem 2.14. *Let $\beta \in \mathbb{R}, n = [\beta]+1, 0 \leqslant \delta < 1$ such that $\delta \leqslant \beta$ and let $\beta > 0$ satisfies (25), and let H be an open set in $\mathbb{R}$ and a function $h : (a,b) \times H \to \mathbb{R}$ is such that, $h(f,y) \in C_\delta[a,b]$ and Lipschitz condition (4) is satisfies:*

(a) If $n-1 < \beta < n$, then there exists a unique solution y to the Cauchy problem (26) which is equivalent to the solution of Cauchy-Euler equation (13) in the space of $C_\delta^{\beta,n-1}[a,b]$

(b) If $0 < \beta < 1$, then there exists a unique solution $y \in C_\delta^{\beta}[a,b]$ to problem (26) which is equivalent to problem (13) with the condition $y(a) = \bar{\beta}_0 \in \mathbb{R}$.

Proof. To prove Theorem 2.14, we begin by establishing the existence of a unique solution $y \in C_\delta^{\beta,n-1}[a,b]$

(a) Let $t_1 \in [a,b]$, we prove the existence of a unique solution $y \in C_\delta^{\beta,n-1}[a,b]$ satisfies the condition

$$\sum_{k=1}^{n} B(t_1-a)^{Re(\beta)-k} \frac{\Gamma(1-\delta)}{\Gamma(1-\delta+\beta-k)} < 1, \quad \delta \leqslant \beta. \tag{30}$$

We then apply Theorem 2.11 to prove that there exist a unique solution $y \in C_\delta^{\beta,n-1}[a,b]$ to equation (26). We write equation (29) in the form $u(t) = (T_y)(t)$, where

$$(T_y)(t) = \sum_{i=1}^{n} A_i \frac{(t-a)^{\beta-i}}{\Gamma(\beta-k+1)} + \frac{1}{\Gamma(\beta)} \int_a^t (t-x)^{\beta-1} h[s,\varpi(s,y)]dx,$$

$$(T_y)(t) = y_0(t) + \frac{1}{\Gamma(\beta)} \int_a^t (t-x)^{\beta-1} h[s,\varpi(s,y)]dx.$$

We denote $y_0(t) = \sum_{i=1}^{n} A_i \frac{(t-a)^{\beta-i}}{\Gamma(\beta-k+1)}$. It follows that $y_0(t) \in C^{n-1}[a,b]$, since we can write $y_0(t)$ as a finite sum of functions in the space of continuous functions on the interval $[a,t_1]$. Further, $h[s,\varpi(s,y)] \in C_\delta[a,b]$ implies $h[s,\varpi(s,y)] \in C_\delta[a,t_1]$. Applying equation (7), we get

$${}_aI^{\beta} h[s,\varpi(s,y)](t) \in C_\delta^{n-1}[a,t_1] \;\; if \;\; \delta \leqslant \beta$$

where $\beta > 0$ and $0 \leqslant \delta < 1$. Let $y \in C^{n-1}[a,t_1]$, then using equation (8) we see that the integral term in equation (29) on the right hand side belong to $C^{n-1}[a,t_1]$. This shows that ${}_aI^{\beta}h[s,\varpi(s,y)](t) \in C^{n-1}[a,t_1]$. This implies that $T_y \in C^{n-1}[a,t_1]$. Thus we have prove that T is continuous on $C^{n-1}[a,t_1]$. Furthermore, we prove that T is a contraction by showing that, give $y_1, y_2 \in C^{n-1}[a,t_1]$ there exist $B > 0$ such that

$$\|T_{y_1} - T_{y_2}\|_{C^{n-1}_{[a,t_1]}} \leqslant B\|y_1 - y_2\|_{C^{n-1}_{[a,t_1]}}. \tag{31}$$

Applying Lemma 2.8, Lemma 2.9 and equation (4), we obtain

$$\begin{aligned}
\left\|{}_aI^{\beta}(h[s,y_1,{}^C_aD^{\beta}y_1] - (h[s,y_2,{}^C_aD^{\beta}y_2)\right\|_{C^{n-1}_{[a,t_1]}} &\leqslant {}_aI^{\beta}\left(\left\|(h[s,y_1,{}^C_aD^{\beta}y_1] - (h[s,y_2,{}^C_aD^{\beta}y_2])\right\|_{C^{n-1}_{[a,t_1]}}\right) \\
&\leqslant \left(B\left\|({}_aI^{\beta})_aI^{\beta}({}^C_aD^{\beta})(y_1 - y_2)\right\|_{C^{n-1}_{[a,t_1]}}\right), \\
&= \left(B_aI^{\beta}\left\|{}_aI^{\beta}({}^C_aD^{\beta})(y_1 - y_2)\right\|_{C^{n-1}_{[a,t_1]}}\right), \\
&= \left[\left(B_aI^{\beta}\|y_1 - y_2\|_{C^{n-1}_{[a,t_1]}}\right)(s) - \sum_{k=1}^{n}\frac{\frac{d^k}{dt^k}(y_1 - y_2)(a)}{k!}(t-a)^k\right].
\end{aligned}$$

We see that $y_1, y_2 \in C^{n-1}[a,t_1]$, this implies that, $\frac{d^k}{dt^k}y_1(a) = \frac{d^k}{dt^k}y_2(a)$ and thus,

$$\left\|{}_aI^{\beta}(h[s,y_1,{}^C_aD^{\beta}y_1] - h[s,y_2,{}^C_aD^{\beta}y_2])\right\|_{C^{n-1}_{[a,t_1]}} \leqslant B\left({}_aI^{\beta}\|y_1 - y_2\|\right). \tag{32}$$

therefore

$$\left\|{}_aI^{\beta}\left(h[s,\varpi(s,y_1)] - (h[s,\varpi(s,y_2)]\right)(t)\right\| \leqslant B\left({}_aI^{\beta}\|y_1 - y_2\|\right)(t). \tag{33}$$

Next, using lemma 2.6 and equation (33) we get and this implies that $\|T_{y_1} - T_{y_2}\|_{C^{n-1}_{[a,t_1]}} \leqslant B\|y_1 - y_2\|_{C^{n-1}_{[a,t_1]}}$ $\forall$ $y_1, y_2 \in C^{n-1}[a,t_1]$. The results indicated that there is a fixed point $y^* \in C^{n-1}[a,t_1]$ which defined the limit of the iterations that mapped T. Thus

$$\lim_{j \to \infty} \|y_j(t) - y^*(t)\|_{C^{n-1}_{[a,t_1]}} = 0, \tag{34}$$

where $y_j(t) = T^j y^*$ and $y^*(t) = y(t)$.

□

Theorem 2.15. *Let $M > 0$ and $N > 0$ be positive real constants $\forall$ $(t, w_j, u_j) \in V$ $(j = 1,2)$ such that*

$$|h(t,w_1,u_1) - h(t,w_2,u_2)| \leqslant M|w_1 - w_2)| + N|u_1 - u_2)|.$$

and max $\{M, N\}\,\frac{T^2}{2} < 1$, equation (13) has a unique solution in $[a, T]$

Proof. We construct a sequence of functions $\{y_j\}_{j=1}^{\infty}$ with $y_0 := \bar{\tau}_1 t + \bar{\tau}_0$ and

$$y_{k+1} := \int_0^t (t-s)h(s, y_k(s), y_k^{\alpha}(s))ds + \bar{\tau}_1 t + \bar{\tau}_0 \quad (k = 1, 2, \dots). \tag{35}$$

We prove that the constructed sequence is Cauchy on the interval $[a, T]$. From Theorem 2.13 we can conveniently write $d_\delta(y_{j+1}, y_j) \leq \lambda d_\delta(y_j, y_{j-1}) \quad (j = 0, 1, \dots)$. Applying mathematical induction $d_\delta(y_{j+1}, y_j) \leq \lambda^2 d_\delta(y_1, y_0) \quad (j = 0, 1, \cdots)$. Here for all $\delta > 0$ is chosen in such way that the definition of our max metric d_δ is such that $\lambda := max\{M, N\}(\frac{T^2}{2}, \frac{1}{\delta}) < 1$. By triangle inequality, $\exists$ a large $N \in \mathcal{N}$ such that $\forall \quad m > n > N$ and for all positive ϵ

$$\begin{aligned} d_\delta(y_m, y_n) &\leqslant d_\delta(y_m, y_{m-1}) + d_\delta(y_{m-1}, y_{m-2}) + \cdots + d_\delta(y_{n+1}, y_n) \\ &\leqslant (\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^n) d_\delta(y_0, y_1) \\ &< \frac{\lambda^n}{1-\lambda} d_\delta(y_0, y_1) < \epsilon. \end{aligned}$$

This shows that the constructed sequence $\{y_j\}_{j=1}^{\infty}$ is Cauchy, this implies that there is a continuously differentiable function $y = y(t)$ such that $\lim_{j=\infty} d_\delta(y_j y) = 0$. Furthermore, we show that limit of the function $y(t)$ satisfies.

$$y(t) = \int_0^t (t-s)h(s, y(s), y^{\alpha}(s))ds + \bar{\tau}_1 + \bar{\tau}_0.$$

This limit of the function $y(t)$ is a solution of equation (13) on the interval $[a, T]$. Theorem 2.13 and Theorem 2.15 put together show that the initial value problem (13) has a unique solution on $[a, T]$. □

2.3 The method

We present the methods for the solution of the non-homogeneous linear boundary value problem as follows.

Consider the boundary value problem:

$$[C_1 D^2 + C_2 D^{\alpha} + C_3 D^0] y(t) = g(t), t \in [0, T], 0 < \alpha < 2$$

subject to,

$$y(0) = \tau_0, \quad y(T) = \tau_1$$

where C_1, C_2, C_3, τ_0 and τ_1 are constants with $C_1 \neq 0$ and $y \in L[0, T]$. Here, D^{α} (α is a non- integer) denote the fractional operator of order α and is given by

$$D^{\alpha} y(x) = \frac{1}{\Gamma(i-\alpha)} \int_0^x (x-t)^{i-\alpha-1} y^i(t) dt$$

where $i = N$ and satisfies the relation $i-1 < \alpha < i$. The existence and uniqueness of the exact solution to problem (65) to the boundary conditions (66) are discuss herein. We recall the following: The Laplace transform for the Caputo's fractional derivative is given by

$$\mathbf{L}[y^p](s) = s^p Y(s) - \sum_{i=0}^{m-1} s^{p-i-1} Y^{(i)}(0), \quad m-1 < p < m. \tag{36}$$

Theorem 2.16. *The fractional boundary value problem*

$$C_1 y''(t) + C_2\left({}_0^C D^\alpha y\right)(t) + C_3 y(t) = g(t), \quad t \in [0,T], \quad 0 < \alpha < 2.$$

has a unique solution give by $y(t) = y_h(t) + \eta y_p(t)$*, where* $\eta = \tau_1 - y_h(t)/y_p(T)$*.*

$$y_h(t) = \mathcal{L}^{-1}\left[\frac{C_1\tau_1 s^{2-\alpha} + C_1\tau_1}{C_1 s^{2-\alpha} + C_2 s^2 + C_3 s^{2-\alpha}}\right].$$

is the solution of the linear homogeneous solution to

$$\begin{aligned} [C_1D^2 + C_2D^\alpha + C_3D^0]y_h(t) &= 0, \ \ 0 < t < T, 0 < \alpha < 2 \\ y_h(0) = 0 \ \ y'(0) &= \tau_1. \end{aligned} \tag{37}$$

and

$$y_p(t) = \mathcal{L}^{-1}\left[\frac{G(s) + C_1 s\tau_0 + C_2 s^{\alpha-1}\tau_0}{C_1 s^2 + C_2 s^\alpha + C_3}\right].$$

is the particular solution of the non- homogeneous linear fractional initial value problem

$$\begin{aligned} [C_1D^2 + C_2D^\alpha + C_3D^0]y_p(t) &= g(t), \ \ 0 < t < T, \ \ 0 < \alpha < 2 \\ y_p(0) = \tau_1; \ \ y_p'(0) &= 0. \end{aligned} \tag{38}$$

Proof.

$$[C_1D^2 + C_2D^\alpha + C_3D^0]y(t) = g(t), \ \ 0 < t < T, 0 < \alpha < 2.$$

for the homogeneous part we have

$$C_1D^2 y_h(t) + C_2D^\alpha y_h(t) + C_3D^0 y_h(t) = 0$$

We take the Laplace transform of both sides

$$\mathcal{L}\left\{C_1D^2 y_h(t) + C_2D^\alpha y_h(t) + C_3D^0 y_h(t)\right\} = 0,$$

$$\mathcal{L}\left\{C_1 D^2 y_h(t)\right\} + \mathcal{L}\left\{C_2 D^\alpha y_h(t)\right\} + \mathcal{L}\left\{C_3 D^0 y_h(t)\right\} = 0,$$
$$C_1\mathcal{L}\left\{D^2 y_h(t)\right\} + C_2\mathcal{L}\left\{D^\alpha y_h(t)\right\} + C_3\mathcal{L}\left\{D^0 y_h(t)\right\} = 0.$$

Using the definition of Caputo derivative in equation (36)

$$C_1\left\{s^2 Y_h(s) - sY_h(0) - Y_h'(0)\right\} + C_2\left\{s^\alpha Y_h(s) - \sum_{i=0}^{m-1} s^{\alpha-i-1} Y_h^{(i)}(0)\right\} + C_3 Y_h(s) = 0,$$
$$C_1\left\{s^2 Y_h(s) - sY_h(0) - Y_h'(0)\right\} + C_2\left\{s^\alpha Y_h(s) - s^{\alpha-1} Y_h(0) - s^{\alpha-2} Y_h'(0)\right\} + C_3 Y_h(s) = 0,$$

substituting the initial conditions we get

$$C_1\left\{s^2 Y_h(s) - sY_h(0) - \tau_1\right\} + C_2\left\{s^\alpha Y_h(s) - s^{\alpha-1} Y_h(0) - s^{\alpha-2}\tau_1\right\} + C_3 Y_h(s) = 0,$$
$$C_1 s^2 Y_h(s) - C_1\tau_1 + C_2 s^\alpha Y_h(s) - C_2 s^{\alpha-2}\tau_1 + C_3 Y_h(s) = 0,$$
$$\left[C_1 s^2 + C_2 s^\alpha + C_3\right] Y_h(s) = C_1\tau_1 + C_2 s^{\alpha-2}\tau_1,$$
$$Y_h(s) = \frac{C_1\tau_1 + C_2 s^{\alpha-2}\tau_1}{C_1 s^2 + C_2 s^\alpha + C_3},$$

multiplying the numerator and denominator by $s^{-(\alpha-2)}$ we have

$$Y_h(s) = \frac{(C_1\tau_1 + C_2 s^{\alpha-2}\tau_1)s^{-(\alpha-2)}}{(C_1 s^2 + C_2 s^\alpha + C_3)s^{-(\alpha-2)}},$$
$$Y_h(s) = \frac{C_1\tau_1 s^{\alpha-2} + C_2\tau_1}{(C_1 s^{4-\alpha} + C_2 s^2 + C_3 s^{-(\alpha-2)}},$$

We take the inverse Laplace transform of both sides

$$\mathcal{L}^{-1}\left\{Y_h(s)\right\} = \mathcal{L}^{-1}\left\{\frac{C_1\tau_1 s^{\alpha-2} + C_2\tau_1}{(C_1 s^{4-\alpha} + C_2 s^2 + C_3 s^{-(\alpha-2)}}\right\},$$
$$y_h(t) = \mathcal{L}^{-1}\left\{\frac{C_1\tau_1 s^{\alpha-2} + C_2\tau_1}{(C_1 s^{4-\alpha} + C_2 s^2 + C_3 s^{-(\alpha-2)}}\right\}, \tag{39}$$

We find the Laplace transform of the non-homogeneous fractional boundary value problem as follows

$$C_1 D^2 y_p(t) + C_2 D^\alpha y_p(t) + C_3 D^0 y_p(t) - g(t) = 0, \quad t \in [0, T], \quad 0 < \alpha < 2,$$

We take the Laplace transform of both sides

$$\mathcal{L}\left\{C_1 D^2 y_p(t) + C_2 D^\alpha y_p(t) + C_3 D^0 y_p(t)\right\} - g(t) = 0,$$
$$\mathcal{L}\left\{C_1 D^2 y_p(t)\right\} + \mathcal{L}\left\{C_2 D^\alpha y_p(t)\right\} + \mathcal{L}\left\{C_3 D^0 y_p(t)\right\} - \mathcal{L}\left\{g(t)\right\} = 0,$$
$$C_1\mathcal{L}\left\{D^2 y_p(t)\right\} + C_2\mathcal{L}\left\{D^\alpha y_p(t)\right\} + C_3\mathcal{L}\left\{D^0 y_p(t)\right\} - \mathcal{L}\left\{g(t)\right\} = 0,$$

Using the definition of Caputo derivative in equation (36)

$$C_1\left\{s^2Y_p(s)-sY_h(0)-Y_p^{'}(0)\right\}+C_2\left\{s^\alpha Y_p(s)-\sum_{i=0}^{m-1}s^{\alpha-i-1}Y_p^{(i)}(0)\right\}+C_3Y_p(s)-G(s)=0,$$
$$C_1\left\{s^2Y_p(s)-sY_p(0)-Y_p^{'}(0)\right\}+C_2\left\{s^\alpha Y_p(s)-s^{\alpha-1}Y_p(0)-s^{\alpha-2}Y_p^{'}(0)\right\}+C_3Y_p(s)-G(s)=0,$$

Substituting the initial conditions $y_p(0)=\tau_0;\ y_p^{'}(0)=0$

$$C_1\left\{s^2Y_p(s)-s\tau_0-0\right\}+C_2\left\{s^\alpha Y_p(s)-s^{\alpha-1}\tau_0-s^{\alpha-2}(0)\right\}+C_3Y_p(s)-G(s)=0,$$
$$\left[C_1s^2+C_2s^\alpha+C_3\right]Y_p(s)=G(s)+C_1s\tau_0+C_2s^{\alpha-1}\tau_0,$$
$$Y_p(s)=\frac{G(s)+C_1s\tau_0+C_2s^{\alpha-1}\tau_0}{C_1s^2+C_2s^\alpha+C_3},$$

we take the inverse Laplace transform of both sides

$$\mathcal{L}^{-1}\left\{Y_p(s)\right\}=\mathcal{L}^{-1}\left\{\frac{G(s)+C_1s\tau_0+C_2s^{\alpha-1}\tau_0}{C_1s^2+C_2s^\alpha+C_3}\right\},$$
$$y_p(t)=\mathcal{L}^{-1}\left\{\frac{G(s)+C_1s\tau_0+C_2s^{\alpha-1}\tau_0}{C_1s^2+C_2s^\alpha+C_3}\right\} \tag{40}$$

where $Y_h(s)=\mathcal{L}[y(t)]$ and $Y_p(s)=\mathcal{L}[y(t)]$. Applying the properties of the inverse Laplace transform on (39) and (40) we find the homogeneous and particular solutions $y_h(t)$ and $y_p(t)$ respectively as shown above. □

Consider equation (65)

$$C_1D^2y(t)+C_2D^\alpha y(t)+C_3D^0y(t)-g(t)=0,$$

applying the definition of Laplace transform operator on the above equation we have

$$\mathcal{L}\left\{C_1D^2y(t)+C_2D^\alpha y(t)+C_3D^0y(t)\right\}-g(t)=0,$$
$$\mathcal{L}\left\{C_1D^2y(t)\right\}+\mathcal{L}\left\{C_2D^\alpha y(t)\right\}+\mathcal{L}\left\{C_3D^0y(t)\right\}-\mathcal{L}\left\{g(t)\right\}=0,$$
$$C_1\mathcal{L}\left\{D^2y(t)\right\}+C_2\mathcal{L}\left\{D^\alpha y(t)\right\}+C_3\mathcal{L}\left\{D^0y(t)\right\}-\mathcal{L}\left\{g(t)\right\}=0,$$
$$C_1\left\{s^2Y(s)-sY_h(0)-Y^{'}(0)\right\}+C_2\left\{s^\alpha Y(s)-\sum_{i=0}^{m-1}s^{\alpha-i-1}Y^{(i)}(0)\right\}+C_3Y(s)-G(s)=0,$$
$$C_1\left\{s^2Y(s)-sY(0)-Y^{'}(0)\right\}+C_2\left\{s^\alpha Y(s)-s^{\alpha-1}Y(0)-s^{\alpha-2}Y^{'}(0)\right\}+C_3Y(s)-G(s)=0,$$
$$C_1s^2Y(s)-C_1sY(0)-C_1Y^{'}(0)+C_2s^\alpha Y(s)-C_2s^{\alpha-1}Y(0)-C_2s^{\alpha-2}Y^{'}(0)+C_3Y(s)-G(s)=0,$$
$$C_1s^2Y(s)+C_2s^\alpha Y(s)+C_3Y(s)-C_1sY(0)-C_2s^{\alpha-1}Y(0)-C_1Y^{'}(0)-C_2s^{\alpha-2}Y^{'}(0)-G(s)=0,$$
$$\left(C_1s^2+C_2s^\alpha+C_3\right)Y(s)-\left(C_1sY(0)+C_2s^{\alpha-1}\right)Y(0)-\left(C_1+C_2s^{\alpha-2}\right)Y^{'}(0)-G(s)=0,$$

Divide both side by c_1.

$$\left(s^2 + \frac{C_2}{C_1}s^\alpha + \frac{C_3}{C_1}\right)Y(s) = \left(s + \frac{C_2}{C_1}s^{\alpha-1}\right)Y(0) - \left(1 + \frac{C_2}{C_1}s^{\alpha-2}\right)Y^{'}(0) + \frac{1}{C_1}G(s), \tag{41}$$

Let $\lambda = \frac{C_2}{C_1}$ and $\varpi = \frac{C_3}{C_1}$, substituting in equation (41)

$$\left(s^2 + \lambda s^\alpha + \varpi\right)Y(s) = \left(s + \lambda s^{\alpha-1}\right)Y(0) + \left(1 + \lambda s^{\alpha-2}\right)Y^{'}(0) + \frac{1}{C_1}G(s),$$

Divide each term by $s^2 + \varpi$,

$$\begin{aligned}\left(\frac{s^2}{s^2+\varpi} + \frac{\lambda s^\alpha}{s^2+\varpi} + \frac{\varpi}{s^2+\varpi}\right)Y(s) &= (s^2+\varpi)^{-1}\left\{\left(s + \lambda s^{\alpha-1}\right)Y(0)\right.\\ &\left.+\left(1 + \lambda s^{\alpha-2}\right)Y^{'}(0) + \frac{1}{C_1}G(s)\right\},\end{aligned} \tag{42}$$

we take the reciprocal of both sides

$$\begin{aligned}\left[\left(\frac{s^2}{s^2+\varpi} + \frac{\lambda s^\alpha}{s^2+\varpi} + \frac{\varpi}{s^2+\varpi}\right)Y(s)\right]^{-1} &= \left[(s^2+\varpi)^{-1}\left\{\left(s + \lambda s^{\alpha-1}\right)Y(0)\right.\right.\\ &\left.\left.+\left(1 + \lambda s^{\alpha-2}\right)Y^{'}(0) + \frac{1}{C_1}G(s)\right\}\right]^{-1},\\ \left[\left(\frac{s^2}{s^2+\varpi} + \frac{\lambda s^\alpha}{s^2+\varpi} + \frac{\varpi}{s^2+\varpi}\right)\right]^{-1}(Y(s))^{-1} &= \left[(s^2+\varpi)^{-1}\left\{\left(s + \lambda s^{\alpha-1}\right)Y(0)\right.\right.\\ &\left.\left.+\left(1 + \lambda s^{\alpha-2}\right)Y^{'}(0) + \frac{1}{C_1}G(s)\right\}\right]^{-1},\end{aligned} \tag{43}$$

simplifying the left side of equation (43) gives

$$\begin{aligned}\left(1 + \frac{\lambda s^\alpha}{s^2+\varpi}\right)^{-1}(Y(s))^{-1} &= \left[(s^2+\varpi)^{-1}\left\{\left(s + \lambda s^{\alpha-1}\right)Y(0)\right.\right.\\ &\left.\left.+\left(1 + \lambda s^{\alpha-2}\right)Y^{'}(0) + \frac{1}{C_1}G(s)\right\}\right]^{-1},\end{aligned} \tag{44}$$

Using the binomial series expansion for $(1+y)^k$.

$$\begin{aligned}(1+y)^k &= 1^k + ky + \frac{k(k-1)}{2!}y^2 + \frac{k(k-1)(k-2)}{3!}y^3\\ &+ \cdots + \frac{k(k-1)(k-2)\cdots(k-r-1)}{r!}y^r + \cdots,\\ (1+y)^k &= \sum_{k=1}^{\infty}\binom{k}{i}y^i\end{aligned} \tag{45}$$

where

$$\binom{n}{k} = \frac{k(k-1)(k-2)\cdots(k-i-1)}{i}, \quad i = 1, 2, 3, \ldots, k \ \ and \ \binom{k}{0} = 0.$$

we let $k = -1$ and substitute in equation (45)

$$(1+y)^{-1} = 1 - y + \frac{2}{2!}y^2 - \frac{6}{3!}y^3 + \cdots \frac{(-6\cdots - i)}{i!}y^i,$$

$$(1+y)^{-1} = 1 - y + y^2 - y^3 + \cdots + y^i, \tag{46}$$

$$(1+y)^{-1} = \sum_{i=0}^{\infty}(-1)y^i, \tag{47}$$

Let $y = \frac{\lambda s^\alpha}{s^2+\varpi}$, we have = $(1+y)^{-1} = \left(1 + \frac{\lambda s^\alpha}{s^2+\varpi}\right)^{-1}$,

$$(1+y)^{-1} = \sum_{i=0}^{\infty}(-1)^i \frac{\lambda^i s^\alpha}{s^2+\varpi}^i = \sum_{i=0}^{\infty}(-1)^i \frac{\lambda^i s^{\alpha i}}{\left(s^2+\varpi\right)}, \tag{48}$$

substituting (48) into (44) we have

$$\sum_{i=0}^{\infty}(-1)^i \frac{\lambda^i s^{\alpha i}}{\left(s^2+\varpi\right)}(Y(s))^{-1} = \Bigg[(s^2+\varpi)^{-1}\Bigg\{\left(s + \lambda s^{\alpha-1}\right)Y(0) + \left(1 + \lambda s^{\alpha-2}\right)Y'(0) + \frac{1}{C_1}G(s)\Bigg\}\Bigg]^{-1}, \tag{49}$$

$$(Y(s))^{-1} = \left(\sum_{i=0}^{\infty}(-1)^i \frac{\lambda^i s^{\alpha i}}{\left(s^2+\varpi\right)^i}\right)\Bigg[(s^2+\varpi)^{-1}\Bigg\{\left(s + \lambda s^{\alpha-1}\right)Y(0) + \left(1 + \lambda s^{\alpha-2}\right)Y'(0) + \frac{1}{C_1}G(s)\Bigg\}\Bigg]^{-1}, \tag{50}$$

$$(Y(s)) = \left(\sum_{i=0}^{\infty}(-1)^i \frac{\lambda^i s^{\alpha i}}{\left(s^2+\varpi\right)^i}\right)(s^2+\varpi)^{-1}\Bigg\{\left(s + \lambda s^{\alpha-1}\right)Y(0) + \left(1 + \lambda s^{\alpha-2}\right)Y'(0) + \frac{1}{C_1}G(s)\Bigg\}, \tag{51}$$

$$(Y(s)) = \left(\sum_{i=0}^{\infty}(-1)^i \frac{\lambda^i s^{\alpha i}}{\left(s^2+\varpi\right)^{i+1}}\right)\Bigg\{\left(s + \lambda s^{\alpha-1}\right)Y(0) + \left(1 + \lambda s^{\alpha-2}\right)Y'(0) + \frac{1}{C_1}G(s)\Bigg\} \tag{52}$$

in order to find the inverse Laplace transform of Y(s) in equation (52) we rewrite the equation in expanded form as follows

$$
\begin{aligned}
(Y(s)) = Y(0)\sum_{i=0}^{\infty}(-1)^i\lambda^i\frac{s^{\alpha i}}{\left(s^2+\varpi\right)^{i+1}}.s + Y(0)\sum_{i=0}^{\infty}(-1)^i\lambda^i\frac{s^{\alpha i}}{\left(s^2+\varpi\right)^{i+1}}.\lambda s^{\alpha-1}\\
+Y^{'}(0)\sum_{i=0}^{\infty}(-1)^i\lambda^i\frac{s^{\alpha i}}{\left(s^2+\varpi\right)^{i+1}}.s + Y^{'}(0)\sum_{i=0}^{\infty}(-1)^i\lambda^i\frac{s^{\alpha i}}{\left(s^2+\varpi\right)^{i+1}}.\lambda s^{\alpha-2}\\
+\frac{1}{C_1}G(s)\sum_{i=0}^{\infty}(-1)^i\lambda^i\frac{s^{\alpha i}}{\left(s^2+\varpi\right)^{i+1}},
\end{aligned} \tag{53}
$$

Simplifying equation (53) we have

$$
\begin{aligned}
(Y(s)) = Y(0)\sum_{i=0}^{\infty}(-1)^i\lambda^i\frac{s^{\alpha i+1}}{\left(s^2+\varpi\right)^{i+1}} + Y(0)\sum_{i=0}^{\infty}(-1)^i\lambda^{i+1}\frac{s^{\alpha(i+1)-1}}{\left(s^2+\varpi\right)^{i+1}}\\
+Y^{'}(0)\sum_{i=0}^{\infty}(-1)^i\lambda^i\frac{s^{\alpha i}}{\left(s^2+\varpi\right)^{i+1}} + Y^{'}(0)\sum_{i=0}^{\infty}(-1)^i\lambda^{i+1}\frac{s^{\alpha(i+1)-2}}{\left(s^2+\varpi\right)^{i+1}}\\
+\frac{1}{C_1}G(s)\sum_{i=0}^{\infty}(-1)^i\lambda^i\frac{s^{\alpha i}}{\left(s^2+\varpi\right)^{i+1}},
\end{aligned} \tag{54}
$$

From the works of [Prabhakar, 1971], we have

$$
E^{\gamma}_{\eta,\beta}(t) = \sum_{i=0}^{\infty}\frac{(\gamma)_i(t^i)}{\Gamma(i\eta+\beta)i!}, \quad (\eta,\beta,\gamma\in\mathcal{R},\mathcal{R}(\eta),\mathcal{R}(\beta),\mathcal{R}(\gamma)>0)
$$

with $u_i = \frac{\Gamma(u+i)}{\Gamma(u)}$ being the Pochhammer symbol. When $\gamma = 1$, these functions are called wiman functions, here, the main fact to be used is related to the inverse of the Laplace transform of the main term that appear in equation (52) is given by

$$
\mathcal{L}^{-1}\left\{\frac{s^{\eta\gamma-\beta}}{(s^2+\lambda)^{\gamma}}\right\} = x^{\beta-1}E^{\gamma}_{\eta,\beta}(-\lambda y^{\eta}).
$$

We now compare the first term in equation (54), $\left(\frac{s^{\alpha i+1-\beta}}{(s^2+\lambda)^{i+1}}\right)$ with $\left(\frac{s^{\eta\gamma-\beta}}{(s^{\eta}+\lambda)^{\gamma}}\right)$, we have

$$
\begin{aligned}
\gamma = i+1,\ \ \eta+2,\ \ \lambda=\varpi\\
\alpha i+1 = \eta\gamma-\beta\\
\alpha i+1 = 2(i+1)-\beta\\
\beta = i(2-\alpha)+1
\end{aligned}
$$

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha i+1}}{(s^2+\lambda)^{i+1}}\right\} = y^{i(2-\alpha)}E^{i+1}_{2.(i(2-\alpha)+1)}(-\varpi y^2), \tag{55}$$

Next we now compare the second, third and forth terms of equation (54), the results are as follows

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha(i+1)-1}}{(s^2+\lambda)^{i+1}}\right\} = y^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2), \tag{56}$$

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha i}}{(s^2+\lambda)^{i+1}}\right\} = y^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2), \tag{57}$$

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha(i+1)-2}}{(s^2+\lambda)^{i+1}}\right\} = y^{i(2-\alpha)-\alpha+3}E^{i+1}_{2.(i(2-\alpha)-\alpha+4)}(-\varpi y^2), \tag{58}$$

the Laplace transform of the fifth term of equation (54) is as follows:

Let

$$z(t) = +\frac{1}{C_1}G(s)\sum_{i=0}^{\infty}(-1)^i\lambda^i\frac{s^{\alpha i}}{\left(s^2+\varpi\right)^{i+1}} \tag{59}$$

and $\phi)(s) = \frac{1}{C_1}G(s)$,

$$z(t) = \phi(s)\sum_{i=0}^{\infty}(-1)^i\lambda^i\frac{s^{\alpha i}}{\left(s^2+\varpi\right)^{i+1}},$$

By the convolution theorem

$$\mathcal{L}^{-1}\{H(s)*G(s)\} = \mathcal{L}^{-1}\{h*g(t)\}$$
$$(h*g)(t) = \mathcal{L}^{-1}\{H(s).G(s)\},$$

from (57), the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha i}}{(s^2+\lambda)^{i+1}}\right\} = \sum_{i=1}^{\infty}(-\lambda)^i y^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2),$$

we let

$$\mathcal{L}^{-1}(G(s)) = \sum_{i=1}^{\infty}(-\lambda)^i y^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2),$$

and

$$\frac{1}{C_1}\mathcal{L}^{-1}\{\phi(s)\} = \frac{1}{C_1}\mathcal{L}^{-1}\{H(s)\} = \frac{1}{C_1}\{\phi(\tau)\},$$
$$z(t) = \int_0^x g(x-\tau)h(\tau)d\tau$$

$$z(t) = \frac{1}{C_1}\sum_{i=1}^{\infty}(-\lambda)^i \int_0^x \phi(x-\tau)\tau^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2)d\tau. \tag{60}$$

Consequently the particular solution of the non homogeneous linear fractional boundary value problem (65) if found by substituting (55), (56), (57 (58), (60) in equation (54)

$$\begin{aligned} y_p(t) &= Y_p(0)\sum_{i=0}^{\infty}(\lambda)^i y^{i(2-\alpha)}E^{i+1}_{2.(i(2-\alpha)+1)}(-\varpi y^2) \\ &+ Y_p(0)\sum_{i=0}^{\infty}(\lambda)^i y^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2) \\ &+ Y_p'(0)\sum_{i=0}^{\infty}(\lambda)^i y^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2) \\ &+ Y_p'(0)\sum_{i=0}^{\infty}(\lambda)^i y^{i(2-\alpha)-\alpha+3}E^{i+1}_{2.(i(2-\alpha)-\alpha+4)}(-\varpi y^2) \\ &+ \frac{1}{C_1}\sum_{i=1}^{\infty}(-\lambda)^i \int_0^x \phi(x-\tau)\tau^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2)d\tau, \end{aligned} \tag{61}$$

substituting the initial conditions, we have, $Y_p(0) = 0$, $Y_p' = \tau_1$

$$\begin{aligned} y_p(t) &= \tau_1\sum_{i=0}^{\infty}(\lambda)^i y^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2) \\ &+ \tau_1\sum_{i=0}^{\infty}(\lambda)^i y^{i(2-\alpha)-\alpha+3}E^{i+1}_{2.(i(2-\alpha)-\alpha+4)}(-\varpi y^2) \\ &+ \frac{1}{C_1}\sum_{i=1}^{\infty}(-\lambda)^i \int_0^x \phi(x-\tau)\tau^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2)d\tau, \end{aligned} \tag{62}$$

similarly the homogeneous solution is

$$\begin{aligned} y_h(t) &= \tau_0\sum_{i=0}^{\infty}(\lambda)^i y^{i(2-\alpha)}E^{i+1}_{2.(i(2-\alpha)+1)}(-\varpi y^2) \\ &+ \tau_0\sum_{i=0}^{\infty}(\lambda)^i y^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2). \end{aligned} \tag{63}$$

Adding equation (63) and (62) gives the general solution of the non homogeneous fractional linear boundary value problem.

$$\begin{aligned} y_h(t) &= \tau_0\sum_{i=0}^{\infty}(\lambda)^i y^{i(2-\alpha)}E^{i+1}_{2.(i(2-\alpha)+1)}(-\varpi y^2) \\ &+ \tau_0\sum_{i=0}^{\infty}(\lambda)^i y^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2) + \tau_1\sum_{i=0}^{\infty}(\lambda)^i y^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2) \\ &+ \tau_1\sum_{i=0}^{\infty}(\lambda)^i y^{i(2-\alpha)-\alpha+3}E^{i+1}_{2.(i(2-\alpha)-\alpha+4)}(-\varpi y^2) \\ &+ \frac{1}{C_1}\sum_{i=1}^{\infty}(-\lambda)^i \int_0^x \phi(x-\tau)\tau^{i(2-\alpha)+1}E^{i+1}_{2.(i(2-\alpha)+2)}(-\varpi y^2)d\tau, \end{aligned} \tag{64}$$

2.4 The Collocation Parallel-shooting Method

Consider the non-homogeneous linear fractional boundary value problem of the form

$$C_1 y''(t) + C_2 \left({}_0^C D^{\alpha} y \right)(t) + C_3 y(t) = g(t), \;\; t \in [0,T], \;\; 0 < \alpha < 2, \tag{65}$$

subject to the boundary conditions

$$y(0) = \tau_0, \quad y(T) = \tau_1, \tag{66}$$

where C_1, C_2, C_3 and τ_0, τ_1 are constants with $C_1 \neq 0$ and $y \in C^2[0,T]$ and $\left({}_0^C D^{\alpha} y \right)(t)$ denote the Caputo derivative of order α. Let $I_h := [0,T]$. The algorithm used to solve the problem (65) subject to (66) presented herein involves collocation-parallel shooting approach for the following IVP.

$$C_1 y''(t) + C_2 \left({}_0^C D^{\alpha} y \right)(t) + C_3 y(t) = g(t), \;\; t \in I, \;\; 0 < \alpha < 2 \tag{67}$$

$$y(0) = \bar{\xi}_0, \;\; y'(0) = \bar{\xi}_1. \tag{68}$$

Here $\bar{\xi}_1$ is unknown constant which shall be found in the solution process. The interval I is subdivided into N uniform sub-intervals $\alpha_n = [t_n, t_{n+1}]$, where $n = 0 \cdots N-1$. Let $I_h = \{t_n = nh : n = 1 \cdots N\}$ with $h = \frac{T}{N}$. Suppose that, the exact solution of problem (5) subject to initial condition (6) can be approximated by an element $u_h \in S^d_{m+d}(I_h)$, where $S^d_{m+d}(I_h) := \{p \in C^d(I)|_{\alpha_n} \in \prod_{m+d}\}$ and $\prod_{m+d}$ denotes the space of all real polynomials of degree not exceeding $m+d$. For known integer $m \geqslant 1$, it should be noted that the integer m present the number of collocation points in each sub-interval $\alpha_n.(n = 0, \cdots, N-1)$ those points are defined as $X_h = \{t = t_{n,i} = t_n + c_i h, \;\; i = 1, \ldots, m, \;\; n = 0, \ldots, N-1\}$, with $0 \leq c_1 < \ldots c_m \leq 1$. The collocation solution u_h satisfies the following initial value problem.

$$C_1 u_h''(t) + C_2 \left({}_0^C D^{\alpha} u_h \right)(t) + C_3 u_h(t) = g(t) \tag{69}$$

$$u_h(0) = \bar{\xi}_0, \quad u_h'(0) = \bar{\xi}_1. \tag{70}$$

On each subinterval, α_n, the spline u_h can be presented as a piecewise polynomials of degree $m+l$ of the form

$$u_h(t) = u_h(t_n + \tau h) = \sum_{q=0}^{l} a_s^{(n)} \tau^q + \sum_{r=1}^{m} b_r^{(n)} \tau^{l+r}, \;\; t \in \alpha_n \;\; and \;\; \tau \in [0,1]. \tag{71}$$

where $\tau \in [0,T]$. The fractional differential operator of order β for the collocation solution at $t = (t_n + c_i h)$ is found by Blank [23] and is given by

$$D^{\beta}\left(\mu_n(t_n + c_i h)\right) = \frac{h^{-\beta}}{\Gamma(1-\beta)} \left[\sum_{j=0}^{n} \sum_{q=0}^{l} W_{i,q}^{(n-j,\beta)} a_q^{(j)} + \sum_{j=0}^{n} \sum_{r=1}^{m} W_{i,r+d}^{(n-j,\beta)} b_r^{(j)} \right], \tag{72}$$

where $\beta \in \mathbb{R}^+, l \in \mathbb{N}$ and in general

$$W_{i,k}^{(j,\beta)} \begin{cases} (j+c_i)^{-\beta} - \delta_{j,0}^{*}(j+c_i-1)^{-\beta}, & k=0 \\ (j+c_i)^{-\beta+k} \prod_{p=1}^{k} \frac{p}{p-\beta} - \delta_{j,0}^{*} \sum_{v=0}^{k} (j+c_i-1)^{v-\beta} S_{v,k}^{\beta}, & k \geq 0 \end{cases}$$

where $S_{v,k}^{\beta} = \prod_{p=1}^{v} \frac{k-v+p}{p-\beta}$ and $\delta_{j,0}^{*} = 0$ if $j=0$ and 1 otherwise

$$D^q\left(\mu(t_n+\tau h)\right) = h^{-q} \sum_{r=q}^{l} r \cdots (r-q+1)\tau^{r-q} a_q^{(n)} + \sum_{r=1}^{m} l+r \cdots (l+r-q+1)\tau^{l+r-q} b_r^{(n)}. \tag{73}$$

Applying Blank results (13) on (10) we get

$$C_1 h^{-2}\left[\sum_{q=2}^{l} q(q-1)c_i^{q-2}a_q^{(n)} + \sum_{r=1}^{m} l+r(l+r-1)c_i^{l+r-2}b_r^{(n)}\right] + C_2\frac{h^{-\beta}}{\Gamma(1-\beta)}\left[\sum_{j=0}^{n}\sum_{q=0}^{d} W_{i,q}^{(n-j,\beta)}a_q^{(j)} + \sum_{j=0}^{n}\sum_{r=1}^{m} W_{i,r+l}^{(n-j,\beta)}b_r^{(j)}\right] + C_3\sum_{q=0}^{l} c_i^q a_q^{(n)} + C_3\sum_{r=1}^{m} c_i^{d+r} b_r^{(n)} = g(t_{n,i}),$$

Simplifying the above will get

$$C_1\sum_{q=2}^{d} q(q-1)c_i^{q-2}a_q^{(n)} + C_2\frac{h^{-\beta+2}}{\Gamma(1-\beta)}W_{i,q}^{(n,\beta)}a_q^{(n)} + h^2 C_3\sum_{s=0}^{l} c_i^q a_q^{(n)} + C_1\sum_{r=1}^{m} l+r(l+r-1)c_i^{l+r-2}b_r^{(n)} + C_2\frac{h^{-\beta+2}}{\Gamma(1-\beta)}W_{i,r+l}^{(n,\beta)}b_r^{(n)} + h^2C_3\sum_{r=1}^{m} c_i^{l+r}b_r^{(n)} = h^2 g(t_{n,i}) - \delta_{n,0}^{*}C_2\frac{h^{-\beta+2}}{\Gamma(1-\beta)}\left[\sum_{j=0}^{n-1}\sum_{q=0}^{l} W_{i,q}^{(n-j,\beta)}a_q^{(j)} + \sum_{j=0}^{n-1}\sum_{r=1}^{m} W_{i,r+l}^{(n-j,\beta)}b_r^{(j)}\right],$$

written the above in matrix form we have

$$Aa^{(n)} + Bb^{(n)} = F \tag{74}$$

Here the unknown constants $a^{(n)} = \left[a_0^{(n)}, \cdots, a_d^{(n)}\right]^t$ and $b^{(n)} = \left[b_1^{(n)}, \cdots, a_m^{(n)}\right]^t$ where $[.]^t$ represent the transpose to the vector. At $n=0$ the vector a^0 is known from the initial conditions and a^0 is given by

$$a^0 = \left[\frac{h^q}{q!}\frac{d^q y}{dt^q}\right], \quad q = 0\ldots r \tag{75}$$

$$a^0 = \begin{bmatrix} \ddots & & 0 \\ & \frac{h^q}{q!} & \\ 0 & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \frac{d^q}{dt^q}y(0) \\ \frac{d^q}{dt^q}y(0) \end{bmatrix}, \quad q = 0\ldots r$$

for $n \geq 1$, the smoothness condition at $[t_{n-1}, t_n]$ which gives the relationship between the known vectors $a^{(n)}$, $b^{(n)}$ and the unknown vector $a^{(n+1)}$

$$a^{(n+1)} = M_1 a^{(n)} + M_2 b^{(n)} \tag{76}$$

where $(M_1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ see [24,25] and [35] for details and the references therein. In parallel shooting method, each succeeding approximation is adjusted simultaneously to satisfy the boundary condition and appropriate continuity condition at the interior point $t_n, n = 1, \cdots, N-1$, see for more details in [24]. In this method, the missing (unknown) initial condition at the initial point of the interval is assumed and the differential equation is then approximated as an initial value problem. Let

$$L = C_1 D^2 + C_2 \left({}_0^C D_t^{\beta} \right) + C_3 D^0 \tag{77}$$

we rewrite problem (7) subject the initial condition (8) in the form.

$$L[\omega_n] = g(t), \quad t \in I_h. \tag{78}$$

This implies that, the solution of equation (14) on the time interval I_h can be determined by solving equation (14) on the subintervals $[t_n, t_{n+1}]$ for $n = 0, \ldots, N-1$ so that we have the following set of initial value problems.

$$L[\omega_n] = g(t), \quad t \in [t_n, t_{n+1}] \tag{79}$$

subject to

$$\omega_n(t_n) = \bar{\beta}_{2n}, \quad \omega_n^{'}(t_n) = \bar{\beta}_{2n+1} \tag{80}$$

where $\bar{\beta}_n$ are the initial conditions for the initial value problems defined on each subinterval $[t_n, t_{n+1}]$ for $(n = 0, \cdots, N-1)$ and $\bar{\beta}_0 = \beta_0$. The solution of the initial value problem (15) and (16) will be determined using the method presented in (9) - (12), where the parameters $\bar{\beta}_j; j = 0, \ldots, 2N-1$ shall be determined by solving the system of algebraic equations below.

$$\begin{aligned}
\omega_0[t_1, \bar{\beta}_1] &= \omega_1[t_1, \bar{\beta}_2, \bar{\beta}_3], \\
\omega_0^{'}[t_1, \bar{\beta}_1] &= \omega_1^{'}[t_1, \bar{\beta}_2, \bar{\beta}_3], \\
\omega_1[t_2, \bar{\beta}_2, \bar{\beta}_3] &= \omega_2[t_2, \bar{\beta}_4, \bar{\beta}_5], \\
\omega_2^{'}[t_{2,\bar{\beta}_3}, \bar{\beta}_3] &= \omega_2^{'}[t_2, \bar{\beta}_4, \bar{\beta}_5], \\
\vdots \qquad \vdots \qquad & \vdots \\
\omega_{N-2}^{'}[t_{N-1}, \bar{\beta}_{2j-4}, \bar{\beta}_{2j-3}] &= \omega_{N-1}^{'}[t_{N-1}, \bar{\beta}_{2j-2}, \bar{\beta}_{2j-1}], \\
\omega_{N-1}[t_{N-1}, \bar{\beta}_{2j-2}, \bar{\beta}_{2j-1}] &= \beta_1.
\end{aligned}$$

Matlab and Mathematica are used for the computation of the numerical results, further more details on shooting method see [24] and [35].

3. Numerical Results

In this section we consider some fractional boundary value problems to demonstrate the accuracy of the methods described in the previous section. The notation $AE = |x(t) - \omega(t)|$, $t \in [0,1]$ denote the absolute error and the square L_2 norm error is presented by $E_2 = \int_0^1 |(x(t) - \omega(t))|^2 dt$. We define the collocation parameter c_i, $i = 1\ldots,m$ as $c_i = \frac{x_k+1}{2}$, $i = 1,\ldots,m$, $x_k = 1 - \cos\left[\frac{(k\pi)}{2m}\right]$, $k = 1,\ldots,m$.

Example 3.1. *Consider the fractional boundary value problem* $D^2 y(t) + D^{\frac{1}{2}} y(t) + D^0 y(t) = g(t)$; *Subject to* $y(0) = 0, y(5) = 25$, *where the exact solution* $y(t) = t^2$, $g(t) = t^2 + 2 + \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}$.

Applying the collocation parallel shooting method, the interval $[0,5]$ is subdivided into five subintervals as follows: $t_i = i$, $i = 0,1,2,\ldots,5$. Therefore our problem is divided into the following set of initial value problems.

$$
\begin{aligned}
&L[\omega_0] = f(t),\ \ 0 \le t \le 5,\ \ \omega_0(0,:) = 0,\ \ \omega_0'(0,:) = \bar{\beta}_1 \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
&L[\omega_4] = f(t),\ \ 4 < t < 5,\ \ \omega_4(4,:) = \bar{\beta}_8,\ \ \omega_4'(4,:) = \bar{\beta}_9
\end{aligned}
$$

We clearly see from the forgoing that,

$$\omega_0(t) = \omega_0(t,\beta_1),\ \ \omega_1(t) = \omega_1(t,\beta_2,\beta_3),\ \ \omega_2(t) = \omega_2(t,\beta_4,\beta_5),\ \ \ldots,\ \ \omega_4(t) = \omega_4(t,\beta_8,\beta_9).$$

The parameter values $\bar{\beta}_j$, $j = 1,\ldots,m$ are: $\bar{\beta}_0 = 0.00000000000000$, $\bar{\beta}_1 = -1.902763078556897e - 13$, $\bar{\beta}_2 = 0.999999999999887$, $\bar{\beta}_3 = 1.999999999999986$, $\bar{\beta}_4 = 3.999999999999952$, $\bar{\beta}_5 = 4.000000000000106$, $\bar{\beta}_6 = 9.000000000000039$, $\bar{\beta}_7 = 6.000000000000041$, $\bar{\beta}_8 = 16.000000000000036$, $\bar{\beta}_9 = 7.999999999999958$. Table **??** shows the exact, approximate solution and the absolute error at the mesh points. Furthermore, figure 1 shows the exact $y(t)$ and approximate solution u_h on $[0,5]$. The square L_2 norm E_2 error is computed and found to be $E_2\left(f\right) = 1.7191E - 26$.

t(i)	Eaxct solution	Approximate solution	Absolute
0	0.0000000000000000	0.000000000000000	0
1	1.0000000000000000	0.999999999999887	1.130207039068409e-13
2	4.0000000000000000	3.999999999999995	5.018208071305708e-14
3	9.0000000000000000	9.000000000000039	3.907985046680551e-14
4	16.0000000000000000	16.000000000000036	3.552713678800501e-14
5	25.0000000000000000	25.0000000000000000	0

Table 1: A table shown the Exact, Approximate solution and Absolute for Example 3.1

In Example 3.1, we take the collocation points $m = 3$ and applied the collocation parallel shooting, the results are presented in Table 1. The column 2 of Table 1 present the exact solution, column 3 presents the approximate solution and the last column present the absolute error. We see from the results that the method gives a good approximation of the exact solution. Further, Figure 1 shows the graphs of the approximate solution and exact solution for $m = 10$ and the absolute error graph. We see from the graph that the method approximate exact function with minimal error.

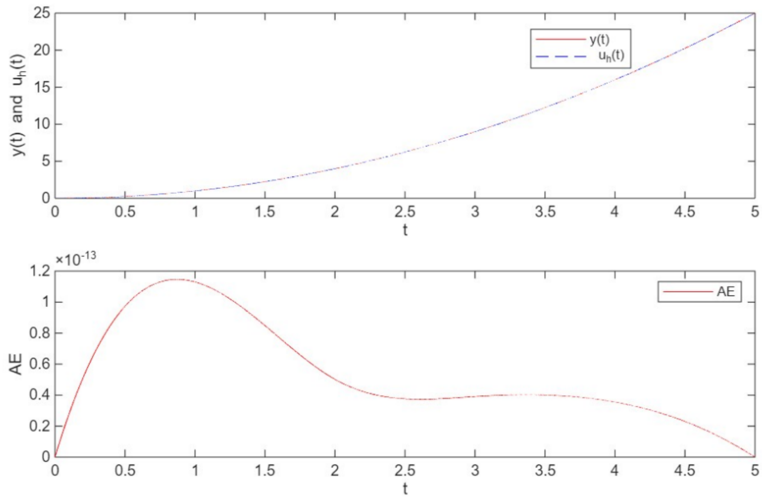

Figure 1: Graph of the exact solution $y(t)$, approximate solution $u_h(t)$ and the absolute error AE for Example 3.1

Example 3.2. *Consider the boundary value problem* $C_1D^2x\,(t)\,C_2D^{\frac{1}{4}}x\,(t) + C_3D^0x\,(t) = g(t)$*; Subject to* $x(0) = 0, x(1) = 3.559752813266941$*, where the exact solution* $x(t) = sin(t) + e^{\lambda t}$*,*

$$g(t) = e^{\lambda t} + \lambda^2 e^{\lambda t} + \left(\frac{5}{\Gamma(\frac{4}{5})}e^t\Gamma\left(\frac{4}{5}\right) - \frac{\Gamma\left(\frac{4}{5},t\right)}{5}\right)$$

$$+\left(\frac{5}{\Gamma\left(\frac{4}{5}\right)}(t^{4/5}cos(t)_1F_1(2/5,[1/2,7/5],-t^{2/4})/4\right)$$

$$+\,t^{9/5}sin(2t)\Gamma(t/\pi+1/1)\Gamma(1/2-t/\pi))_1F_1(9/10,[3/2,19/10],-t^2/4))/(18\pi)))$$

For us to apply the collocation parallel-shooting method, the interval $[0, 1]$ is subdivided into five sub-intervals as follows: $t_i = i$, $i = 0, 1, 2, \ldots, 16$. Therefore our problem is divided into the following set of initial value problems.

$$L[\omega_0] = f(t), \;\; 0 \le t \le 5, \;\; \omega_0(0,:) = 0, \;\; \omega_0^{'}(0,:) = \bar{\beta}_1$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$L[\omega_4] = f(t), \;\; 4 < t < 5, \;\; \omega_4(4,:) = \bar{\beta}_8, \;\; \omega_4^{'}(4,:) = \bar{\beta}_9$$

We clearly see from the forgoing that,

Parameter $(\bar{\beta}_j)$ are: $\bar{\beta} = 1.000000000000000$, $\bar{\beta}_1 = 1.999960107803945$, $\bar{\beta}_2 = 1.257818925076409$, $\bar{\beta}_3 =$

2.125313465116970, $\bar{\beta}_4 = 1.531421871018279$, $\bar{\beta}_5 = 2.252913874643586$, $\bar{\beta}_6 = 1.821254368886025$, $\bar{\beta}_7 = 2.385484806828370$, $\bar{\beta}_8 = 2.128136446611042$, $\bar{\beta}_9 = 2.526300014825962$, $\bar{\beta}_{10} = 2.453333426379004$, $\bar{\beta}_{11} = 2.679216004053744$, $\bar{\beta}_{12} = 2.798630901273591$, $\bar{\beta}_{13} = 2.848706567616902$, $\bar{\beta}_{14} = 3.166414204868768$, $\bar{\beta}_{15} = 3.039900306279105$. Table 1 shows the exact, approximate solution and the absolute error at the mesh points. Furthermore, figure 1 shows the exact $y(t)$ and approximate solution u_h on $[0,5]$. Following the same approach as described in Example 3.1 we have the following η values, where $\bar{\beta}_i$, $(i = 0,\ldots,9)$ are the initial values at the sub interval. The square L_2 norm E_2 error is as follows $E_2\left(f\right) = 4.4664E - 28$.

t(i)	$y(t)$	$u_h(t)$	AE
0.000	1.000000000000000	1.000000000000000	0,00000000000000000
0.125	1.257823186452054	1.257822660864706	5.255873478660078e-07
0.250	1.531429375942264	1.531428450318473	9.256237909482934e-07
0.375	1.821263943704249	1.821262762860220	1.180844029002870e-06
0.500	2.128146809304331	2.128145531421711	1.277882620165371e-06
0.625	2.453343230372685	2.453342021556800	1.208815885167525e-06
0.750	2.798638776636009	2.798637805794381	9.708416279785581e-07
0.875	3.166418796203125	3.166418230330814	5.658723107870856e-07
1.000	3.559752813266941	3.559752813266941	0.0000000000000000

Table 2: A table shown the Exact, Approximate solution and Absolute for Example 3.2

Under this Example 3.2, we take the collocation points $m = 3$ and applied the collocation parallel shooting, the results are shown in Table 2. The column 2 of Table 2 present the exact solution $y(t)$, column 3 shows the approximate solution $u_h(t)$ and column 4 presents the absolute error. We see from the results that the method gives a good approximation of the exact solution.

Further, Figure 2 show the the graphs of the approximate solution and exact solution for m = 10 and the absolute error graph. We see from the graph that the method approximate exact function with minimal error.

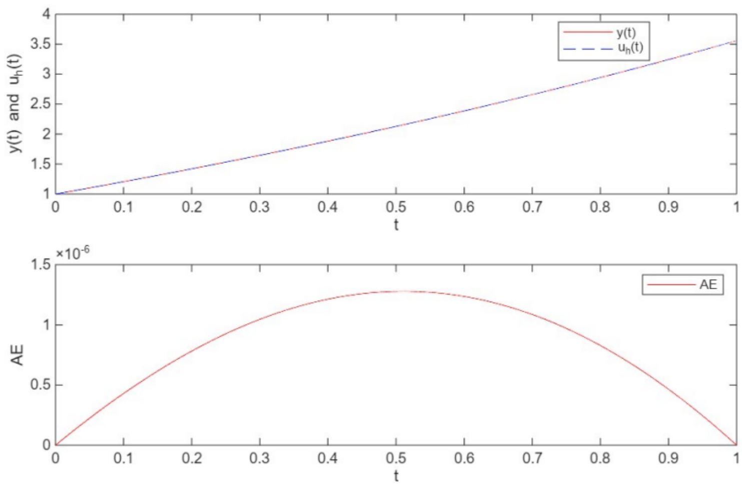

Figure 2: Graph of the exact solution $y(t)$, approximate solution $u_h(t)$ and the absolute error AE for Example 3.2

4. Conclusion

A boundary value problem for non-homogeneous linear fractional differential equation is solved analytically and numerically using the Laplace transform approach and the collocation parallel shooting method, existence and uniqueness of the exact solution are proved via the contraction mapping principle. Numerical examples are constructed to demonstrate the effectiveness and the applicability of method. The results show that the numerical method approximate the exact solution effectively.

References

[1] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Willey, New York, (1993).

[2] P. J. Torvik and R. L. Bagley, *On the appearance of the fractional derivative in the behavior of real materials*, Journal of Applied Mechanics, 51(2)(1984).

[3] V. V. Kulish and J. L. Lage, *Application of fractional calculus to fluid mechanics*, J. Fluids Eng., 124(3)(2002), 803-806.

[4] N. Engheta, *On fractional calculus and fractional multipoles in electromagnetism*, IEEE Transactions on Antennas and Propagation, 44(4)(2002), 554-566.

[5] K. B. Oldham, *Fractional differential equations in electrochemistry*, Advances in Engineering software, 41(1)(2010), 9-12.

[6] C. Lederman, J.-M. Roquejoffre and N. Wolanski, *Mathematical justification of a nonlinear integro-differential equation for the propagation of spherical flames*, Annali di Matematica Pura ed Applicata, 183(2004), 173-239.

[7] S. S. Ray, *Fractional calculus with applications for nuclear reactor dynamics*, CRC press, (2015).

[8] Richard L. Magin, *Fractional calculus models of complex dynamics in biological tissues*, Computers and Mathematics with Applications, 59(5)(2010), 1586-1593.

[9] J. -F. Deu and D. Matignon, *Simulation of fractionally damped mechanical systems by means of a newmark-diffusive scheme*, Computers & Mathematics with Applications, 59(5)(2010), 1745-1753.

[10] K. Rabiei, Y. Ordokhani and E. Babolian, *The boubaker polynomials and their application to solve fractional optimal control problems*, Nonlinear Dynamics, 88(2017), 1013-1026.

[11] P. Rahimkhani and Y. Ordokhani, *A numerical scheme based on bernoulli wavelets and collocation method for solving fractional partial differential equations with dirichlet boundary conditions*, Numerical Methods for Partial Differential Equations, 35(1)(2019), 34-59.

[12] Y. Zhang, *A finite difference method for fractional partial differential equation*, Applied Mathematics and Computation, 215(2)(2009), 524-529.

[13] M. Behroozifar and A. Sazmand, *An approximate solution based on jacobi polynomials for time-fractional convection-diffusion equation*, Applied Mathematics and Computation, 296(2017), 1-17.

[14] A. Baseri, S. Abbasbandy and E. Babolian, *A collocation method for fractional diffusion equation in a long time with chebyshev functions*, Applied Mathematics and Computation, 322(2018), 55-65.

[15] Kai Diethelm, Neville J. Ford and Alan D. Freed, *A predictor-corrector approach for the numerical solution of fractional differential equations*, Nonlinear Dynamics, 29(2)(2002), 3-22.

[16] V. Daftardar-Gejji and H. Jafari, *Solving a multi-order fractional differential equation using adomian decomposition*, Applied Mathematics and Computation, 189(1)(2007), 541-548.

[17] O. Abdulaziz, I. Hashim and S. Momani, *Solving systems of fractional differential equations by homotopy-perturbation method*, Physics Letters A, 372(4)(2008), 451-459.

[18] K. Diethelm and G. Walz, *Numerical solution of fractional order differential equations by extrapolation*, Numerical Algorithms, 16(1997), 231-253.

[19] Z. Odibat, S. Momani and V. S. Erturk, *Generalized differential transform method: Application to differential equations of fractional order*, Applied Mathematics and Computation, 197(2)(2008), 467-477.

[20] Y. Zhao, S. Sun, Z. Han and Q. Li, *The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations*, Communications in Nonlinear Science and Numerical Simulation, 16(4)(2011), 2086-2097.

[21] S. Liang and J. Zhang, *Positive solutions for boundary value problems of nonlinear fractional differential equation*, Nonlinear Analysis: Theory, Methods and Applications, 71(11)(2009), 5545-5550.

[22] T. Jankowski, *Boundary problems for fractional differential equations*, Applied Mathematics Letters, 28(2014), 14-19.

[23] L. Blank, *Numerical treatment of differential equations of fractional order*, (1996).

[24] E.A. Rawashdeh, *Numerical solution of fractional integro-differential equations by collocation method*, Applied Mathematics and Computation, 176(1)(2006), 1-6.

[25] Q. M. Al-Mdallal, M. I. Syam and M. Anwar, *A collocation-shooting method for solving fractional boundary value problems*, Communications in Nonlinear Science and Numerical Simulation, 15(12)(2010), 3814-3822.

[26] S. Samko, A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives*, Theory and applications, Gordon and Breach Science, (1987).

[27] T. R. Prabhakar, *A singular integral equation with a generalized mittag-leffler function in the kernel*, Yokohama Math. J., 19(1)(1971), 7-15.

[28] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Vol. 204, Elsevier, (2006).

[29] E. Kreyszig, *Introductory functional analysis with applications*, Vol. 17, John Wiley & Sons, (1991).

[30] F. Jarad, T. Abdeljawad and D. Baleanu, *On the generalized fractional derivatives and their caputo modification*, J. Nonlinear Sci. Appl., 10(2017), 2607-2619.

[31] Y. Gambo, R. Ameen, F. Jarad and T. Abdeljawad, *Existence and uniqueness of solutions to fractional differential equations in the frame of generalized caputo fractional derivatives*, Advances in Difference Equations, 2018(1)(2018), 1-13.

[32] E. Kreyszig, *Introductory functional analysis with applications*, Vol. 17, John Wiley & Sons, (1991).

[33] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering, Academic press New York, (1999).

[34] E. Rawashdeh, *Numerical solution of semidifferential equations by collocation method*, Applied Mathematics and Computation, 174(2)(2006), 869-876.

[35] S. C. Buranay, J. M. Chin and N. Mahmudov, *A highly accurate numerical method for solving boundary value problem of generalized bagley-torvik equation*, Mathematical Methods in Applied Sciences, 1051(2024).